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Dynamic Portfolio Speculation Via an Informationally More Structured ITO Process

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Abstract

Although theories over portfolio speculation have made remarkable progress so far, the performance of its proposed portfolios depends mainly on the degree of accuracy in predicting future stocks prices dynamics. This study focuses on improving the performance of optimal stock portfolios by modeling unprecedentedly the stocks prices dynamics through a time-inconsistent multivariate diffusion specification with a drift vector. To this end, the share prices are simulated using a semi-martingale process with time-inconsistent (local) martingale and information drift parts over the entire optimization horizon. Then, using the results of price simulation, we have looked into its consequences for constructing the portfolio of assets in the Sharpe ratio maximization method and mean-variance analysis framework. Findings indicate that for the stock market under study (Tehran) within the trading dates spanning the interval 24-Mar-2001 to 19-Sep-2020, return and risk (standard deviation) of the portfolios obtained from applying this simulation scheme for mean-variance analysis and maximization of Sharpe ratio are both respectively higher and lower than those realized by the conventional methods. Additionally, a comparison of the simulation approach with the performance of the actual market portfolios indicates that the Sharpe ratios of the simulation method are higher than those resulting from the market portfolios.

Highlights

- This study proposes a new portfolio construction strategy to enhance the efficiency of the conventional ones in terms of the Sharpe ratio.
- Stock prices dynamics are structured as a Multidimensional Geometric Brownian Motion (MGBM) process.
- The dynamic efficiency gain could be viewed as a reflection of an information filtration enlargement type.
- The resulting Sharpe ratio is higher than the actual market portfolios.

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1. Introduction

The optimal portfolio selection theory has always been one of the main concerns in financial engineering. Many scholars developed various methods to improve portfolio optimization efficiency. One of the core approaches in the field is mean-variance analysis, in which Markowitz made significant progress (Markowitz, 1952, 1959). Markowitz suggests that to constitute an optimal portfolio, one should include stocks' returns by the mean of returns, stocks' risks by the standard deviation of stocks' returns, and the stocks' returns Co-movements in the form of covariance. Based on this, the return and risk of the portfolio can be obtained considering the weights of each stock in the portfolio. Finally, this method considers the relationship between risk and return to acquire the optimal portfolio.

One of the limitations of the mean-variance method is the sensitivity of portfolios. Portfolio sensitivity means that portfolio performance massively depends on the returns and risk of portfolio inputs. If the returns or risks of portfolio stocks change even slightly, the subsequent optimal portfolio will be so different. Therefore, the accuracy of predicting stocks' returns and risks plays a crucial role in portfolio selection and performance. The mean-variance method exploits the ex-post stocks information to construct the optimal portfolio; namely, the average rate of returns as the portfolio expected rate of returns and the actual standard deviation of the rate of returns as the portfolio predicted risks. In other words, the underlying assumption is that the stocks' prices behave precisely as in the past. However, not surprisingly, the real-world experiences show that this method has a significant estimation error. Since introducing this method, several studies have been conducted to overcome this shortcoming and modify and improve the results. The primary purpose of this study is to increase the prediction robustness of stocks' prices dynamics and provide more robust portfolios for different investment horizons (Kim & Fabozzi, 2015).

In addition, one of the fundamental weaknesses of the mean-variance method is that it doesn't include the stochastic part of the stocks' prices behavior. Bachelor first identified stochastic behavior of stock prices, and since then, many researchers have tried to model it (Bachelor, 1900; Muteba Mwamba & Suteni, 2010). Many scholars utilize stochastic differential equations to comprise stochastic behavior in stocks prices simulations. One of the significant studies in this field is the study of Black and Scholes (Black & Scholes, 1973), which used Brownian geometric motion (GBM) to simulate prices. This method assumes that the variables under study are always positive, consistent with stock prices. On the other hand, it considers the stochastic part of stock prices in the simulation. This study initially employs a time-inconsistent Multidimensional Geometric Brownian Motion (MGBM) specification, which considers the relationships between all stocks' prices to simulate each stock price dynamics. This way, we would be able to, one way or another, enlarge the filtration of stock prices dynamics and to make our prime departure from the mainstream literature (e.g., Browne, 1997; Hörfelt, 2003; Pemy et al., 2008; Ladde & Wu, 2009; Zhu, 2009;

Gulisashvili & Stein, 2010; Yam et al., 2012; Mota & Esquível, 2016; Reddy & Clinton, 2016; de Oliveira et al., 2017; Lu et al., 2017; Tie et al., 2018).

The mean-variance method would offer numerous portfolios at any time, ranging from risk-free to maximum risk ones. As a result, a single portfolio is not necessarily obtained as the optimal one. Thus, one needs a criterion to single out among many. The Sharpe ratio (Sharpe, 1966, 1994), defined as the amount of excess return relative to a risk-free asset per unit of risk, has usually served as one. Though market researchers and analysts have tried many alternative models to investigate the performance of stock portfolios, the Sharpe ratio is still one of the most popular ratios in ranking portfolios (Kourtis, 2016).

Another problem is that the mean-variance method for a given period and a certain number of stocks offers countless optimal portfolios defined at different risk levels. Therefore, this method does not provide a single portfolio as the optimal one. We use Sharpe ratio maximization to solve this problem, a prevalent practice in determining the optimal portfolio. The Sharpe ratio is defined as excess return relative to a risk-free asset per unit of risk. We use the constrained quadratic programming method to obtain the maximum Sharpe ratio, a subset of the nonlinear programming method.

In this paper, the results of the conventional method and the simulation method using the time-inconsistent MGBM specification and the Sharpe ratio maximization method for a large number of stocks (808 stock symbols) in a relatively long period (nearly 20 years), as well as different investment horizons and implications of the following enriched information are presented. Therefore, one of the features of our paper is the scale and comprehensiveness of the research. Also, the study period is divided into smaller investment periods, and the stocks' prices are predicted for later periods, and their results are presented in tables and graphs. This enables us to evaluate somewhat the predictive power and performance of the conventional method and our proposed method in different investment horizons and volume of the information.

Although there are many studies in the literature on the mean-variance analysis, Sharpe ratio maximization, GBM, and MGBM methods, to our best knowledge, there is no endeavor towards modeling the stock prices by a time-inconsistent MGBM specification to find the Sharpe-ratio maximizing portfolio. Therefore, we try to present the most relevant and closest part of the existing literature to our study in the following.

The rest of the paper has been structured as follows: Section 2 and 3 respectively provide a review of the related literature and a brief overview of the pursued methodology and its background. Section 4 outlines the employed data and significant characteristics of the Tehran Stock Exchange (TSE). Section 5 presents the results and their distinct features. Section 5 concludes.

2. Literature Review

To provide some relevant literature, one may refer to Merton's portfolio model (Merton, 1971), which selects the optimal intertemporal portfolio of stock(s) and bond (as a riskless asset) by maximizing the expected utility. This model typically uses a GBM or MGBM to simulate risky assets (stocks). In general, in this model, the utility function is converted to a value function that is a continuous version of the HJB equation, which usually needs to be solved numerically. Significantly when several state variables rise, the time required for a numerical solution increases exponentially; in other words, this model has a dimensionality problem (Weiner, 2004; Lakner & Ma Nygren, 2006; Chellathurai & Draviam, 2007; Back, 2010; Buckley et al., 2012; Castellano & Cerqueti, 2012; Tourin & Yan, 2013; Pun & Wong, 2016; Biagini & Pınar, 2017; Mariani et al., 2019). The fundamental difference between the Merton portfolio model and our study is that the households maximize the expected lifetime utility in Merton's model. Therefore, the portfolio is optimized for a lifetime. In our model, the long-term period is divided into shorter periods according to the selected investment horizon, and the optimization is done separately in each period. In addition, the Merton method is performed on a limited number of stocks due to the curse of dimensionality, whereas our approach would allow us to bypass this issue somehow. Another group of studies on the Makowitz mean-variance problem uses the GBM to simulate stock prices (Xie, 2009; Muteba Mwamba & Suteni, 2010; Spinu, 2015; Abensur et al., 2020). The main difference between our study and this group of studies places the use of the time-inconsistent MGBM specification in the Sharpe ratio maximization as the reward function for many stocks.

The other group solves the problem of constituting an optimal stock portfolio using the utility function maximization under the Markowitz mean-variance criterion. This group uses MGBM to model stock prices. But in the numerical solution of the model, they use one stock that converts the model to GBM (Xie et al., 2008; Zeng & Li, 2011; Wei & Wang, 2017). One research does the same thing but uses three stocks in model simulation (Yunita et al., 2015). Pedersen and Peskir (Pedersen & Peskir, 2017) study the dynamics of a nonlinear mean-variance optimal control problem that uses the MGBM model to model risky stock prices. The approach of these studies is mainly Analytical. The two studies tried to improve the performance of portfolios focusing only on different risk measures. In their models, share prices follow MGBM, too (Dmitrašinović & Ware, 2006; Gambrah & Pirvu, 2014). One exciting piece of research that focuses on the shortcoming of the mean-variance model predicts stocks prices by a hybrid model based on machine learning, then selects stocks with higher potential returns for mean-variance portfolio optimization (Chen et al., 2021). None of those mentioned above strands of literature share the same trading strategy offered in this paper.

Starting from the influential study by Beasley et al. (2003), a branch of portfolio optimization is the enhanced indexation approach that tries to find portfolio outperforming a given index (Beasley et al., 2003; Canakgoz & Beasley, 2008; Roman et al., 2013; Xu et al., 2018; Chen et al., 2019; Li et al., 2021).

3. The Methodology & Background

The optimal stock portfolio detection regularly involves using equity data in its raw format. Instead, this paper offers a simulation of share prices by the MGBM process and then uses the result to obtain the Sharpe-ratio maximizing portfolio. Next, the resulting portfolio will be compared with the conventional method.

3.1 Multidimensional Geometric Brownian Motion

To simulate the price of market shares, we use multidimensional geometric Brownian motion, in which case the correlation between the returns of all shares is considered in the simulation. For this purpose, we consider the following stochastic differential equations system.

$$dX_t = \mu_t X_t dt + A(X_t)\Sigma_t dB_t \tag{1}$$

Where dX_t, μ_t, X_t, $A(X_t)$, Σ_t and B_t are respectively $n \times 1$ vector of the differential processes, $n \times n$ matrix of expected returns (the drift parameter), $n \times 1$ state vector of the random process variables, a square matrix of order n whose primary diagonal values are the same as the elements of the X_t and the other values are zero, a square matrix of order n that the primary diagonal values are standard deviation of the random variables and other off-diagonal elements represent the covariance between the variables, and $n \times 1$ vector of one-dimensional independent Brownian motion. To solve system (1), we need Ito's formula. For simplicity, we show the solution of one process; other processes follow the same solution.

3.2 Ito's Formula

If X_{it} is an Ito process such that $dX_{it} = \mu_i X_{it} dt + X_{it} \sum_{j=1}^{n} \sigma_{ij} dB_{jt}$ and $f: R^2 \to R$ is a twice continuously differentiable function, then $Y_{it} = f(X_{it})$ is also an Ito process, and we have

$$dY_{it} = f_x(X_{it})dX_{it} + \frac{1}{2} f_{xx}(X_{it})(dX_{it})^2 \tag{2}$$

If $Y_{it} = ln\, X_{it}$, using the stochastic differential equations system (1) and Ito's formula for $i \in [1, n]$ we have

$$X_{it} = X_{i0} exp\left[\left(\mu_i - \frac{1}{2}\sum_{j=1}^{n} \sigma_{ij}^2\right) t + \sum_{j=1}^{n} \sigma_{ij} B_{jt}\right] \tag{3}$$

Equation (3) defines X_{it} as a geometric Brownian motion process (Duffie, 2001) (Glasserman, 2013).

Where X_{i0}, μ_i, σ_{ij} and B_{jt} are the last observation of i-th stock price (where the future price simulation begins), the drift parameter obtained from the realized (ex-post) mean of the i-th stock price, the standard deviation of the i-th stock price if $i = j$ and the covariance between the two stocks if $i \neq j$ and the j-th Brownian motion at the time t.

Brownian motion is a Gaussian Markov process with stationary independent Increments (Duffie, 2001) and geometric Brownian motion is an exponentiated Brownian motion. Although ordinary Brownian motion can take negative values, which is an undesirable feature for stock prices simulation, as prices cannot be negative, geometric Brownian motion always takes positive values as it is an exponential function. The advantage of the MGBM model over the GBM is that in the MGBM model, the correlation between all stocks prices is taken into account in the simulation of the prices of each stock price. Consequently, due to the large number of stocks studied in this paper, much more information than the GBM model is included to simulate the prices (Glasserman, 2003).

3.3 Portfolio Optimization

This section explains how an investor builds the optimal stock portfolio by maximizing the Sharpe ratio. To form the Sharpe ratio problem, we use Back (2010) assumptions and settings: Our market is composed of a riskless asset and n stocks (risky assets)[1], r_i is the return on stock i at time t and $\bar{r} > 0$ represents the return on riskless asset. R, $E[R]$, Σ and I are respectively n-dimensional vector with r_i as its ith element, vector of the expected returns, $n \times n$ nonsingular covariance matrix, and n-dimensional column vector of ones. Since Σ is nonsingular, all possible portfolios are risky. $\omega' R$ is the portfolio's return, where ω is the portfolio's vector of assets' weights. $\omega'\mu$ and $\omega'\Sigma\omega$ are respectively mean and variance of the portfolio's return.

We further assume $I'\omega = 1$ which guarantees that the portfolio consists of only risky assets. In addition, we assume that we have a possible portfolio ω^* such that $\mu\omega^* > \bar{r}$ (Back, 2010).[2] To obtain the maximum Sharpe ratio at the time t, we solve the following problem:

$$
\begin{aligned}
max \quad & \frac{\omega^T\mu - \bar{r}}{\sqrt{\omega'\Sigma\omega}} \\
s.t. \quad & I'\omega = 1 \\
& \mu'\omega^* > \bar{r}
\end{aligned} \tag{4}
$$

The numerator and denominator of the model represent the portfolio's risk premium (portfolio' return over the risk-free rate) and the square root of the variance (risk) of the portfolio's return, respectively. The higher the value of this criterion, the higher the return per unit of volatility (risk) (Kourtis, 2016).

To solve problem (5), we use the constrained quadratic programming method, a subset of the nonlinear programming method. The quadratic programming method minimizes the objective function, so it must first turn the Sharpe ratio maximization problem into the following minimization problem[3]:

$$
\begin{aligned}
min \quad & \gamma'\Sigma\gamma \\
s.t. \quad & \tau > 0, (\gamma, \tau) \in H^+ \\
& (\mu - \bar{r}I)'\gamma = 1
\end{aligned} \tag{5}
$$

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[1] These risky assets are in fact the stocks and risk-free asset is bank deposits.
[2] To solve the Sharpe ratio maximization problem with a convex quadratic programming problem, we need these two assumptions.
[3] Proof of it and the quadratic programming method to solve this problem are available upon request from the authors.

Where $H^{+} := \left\{ \omega \in \Re^{n}, \tau \in \Re \middle| \tau > 0, \frac{\omega}{\tau} \in H \right\} \bigcup (0,0)$, $\mathcal{H}$ is a set of feasible portfolios with the properties that $I'\omega = 1$, $\forall \omega \in \mathcal{H}$ and $\mu'\widehat{\omega} \geq \bar{r}$, $\exists \widehat{\omega} \in H$, $\tau = \frac{1}{(\mu - \bar{r}I)'\omega}$ and $\gamma = \tau\omega$ (Cornuejols & Tütüncü, 2006).

4. The Data

All available data on the price of active symbols in the Tehran Stock Exchange within the trading dates spanning the interval 24-Mar-2001 to 19-Sep-2020 were used to conduct the study. Some symbols, such as mutual funds and government securities, were removed from the analysis due to differences in risk and return from other symbols. Some other symbols that cannot be invested in, such as market indicators and test symbols, were removed from the list. After this refinement, 808 symbols remained. In addition, some companies have old and new symbols, which increases the number of symbols compared to the actual number of active companies. The method used in this research, which is described in the missing data section, will lead to the use of the original data and the removal of additional symbols.

There are missing data in the time series of stock prices due to two main ins and outs. First, the period under study from 24-Mar-2001 onwards is considered the most extended period available. Many companies have entered the capital market after this date; thus, there is no data before arrival. Second, corporate symbols are often discontinued during their operation for various reasons, including extraordinary general meetings.

There are several ways to manage missing data, including removing and estimating data. In the removal method, at each time, if the prices of all shares are available, we preserve data. Using this method causes much-lost information and extracts an unrealistic portfolio. Another standard method estimates missing data's mean and covariance of stock returns. This method is based on two strong assumptions. According to the first assumption, each observation is generated based on a multivariate normal distribution, and according to the second assumption, the data are lost randomly. We should use this method when it is impossible to estimate the mean and covariance of stock returns, and it is necessary to test the above assumptions beforehand. We employ another method that allows the estimation of mean and covariance; thus, do not need to utilize the estimation method.

Since we are looking to evaluate the model's predictive power, we assume that the investor stands at the beginning of the period t and wants to use the information of the last L days to obtain an optimal portfolio, then buy these shares at the period t. As a result, the investor's criterion is the availability of price data for the shares in a significant number of last L days and at the beginning of the period (t). Shares that have lost more than 25 percent of their data in the last L few days or are not available at the time t (have missing data) will be removed from the list. For each share, initially, we fill missing prices with the last non-missing value, then with the next non-missing value. We consider zero returns for trading suspensions corresponding to reality using this method. In this way, we will not have two other methods, i.e., losing information and estimation of unrealistic mean and covariance. There are five price data per share (open, last, close, highest, and lowest). Because the closing price determines the return on a stock at the end of the trading day, and over a period, we use the closing price as the criterion for calculating the return. Given the number of available periods is 4240 trading days for 19.5 years accordingly, every 18 trading days is considered a month. The investor's strategy can be summarized as follows: the investor uses the price data of, say, the last 216 market days (one calendar year) to estimate and purchase an optimal stock portfolio and holds the purchased shares for 18 market days (1 calendar month), then at the end of the month, again uses the price data of the previous year to estimate and purchase the portfolio and repeats this process until the last period.

We assume that at the beginning of each period, investors sell the previous portfolio and purchase a new one, so to calculate the profit of each method, the stock trading commission will be deducted from it. Before implementing the new law on July 22, 2020, the transaction fee was equal to 1.43 percent, equal to 1.25 percent after that.

We consider the annual interest rate on the one-year investment deposits as the riskless asset's rate of return. We got the data from the Central Bank of Iran. Then we convert this rate to the daily rate using the following formula:

$$DR = \left(1 + \frac{AR}{100}\right)^{1/216} - 1 \tag{6}$$

AR and DR are the annual and daily interest rates on the investment deposits, respectively. Moreover, we have 216 trading days a year.

5. Empirical Results

As mentioned earlier, the conventional method of extracting the optimal stock portfolio uses raw price data to estimate the mean and covariance of returns, estimate the efficient boundary, and maximize the Sharpe ratio. This study proposes to simulate the stock price data using a MBGM model; then, we utilize the simulated results to determine the optimal portfolio through the Sharpe ratio maximization. To evaluate the performance of this method compared to the conventional method, the predictive power of the two methods in terms of mean and variance has been compared with each other. From now on, we call the method of this paper (using the simulated data) as the simulation method and the standard method (using the raw data) as the conventional method.

5.1 Comparing the Efficient Frontier of the Two Methods

To clarify the results of the two methods, Figure (1) plots the efficient frontier of the conventional method (with black mesh lines) and simulation method (without black mesh lines) for about 18 years (from December 2002 to

September 2020) at 18-days intervals.[4] The outputs are based on the investors' beginning-period price information set for the last 378 days. Also, the results reflect one implementation of the method at each period. We have ended up with 20 optimal investment portfolios for each method at any run.

As Figure (1) shows, the graph's height represents the years studied. At a certain height (for example, 2010), we can examine how much each of the two methods has yielded at each level of the portfolio standard deviation. Suppose the stocks portfolio in one method has a higher yield at each standard deviation level, or similarly, it has a lower standard deviation at each level of return. In that case, its graph must be to the left and behind the graph of the other method. According to Figure (1), in all years, the graph of the simulation method is to the left and behind the graph of the conventional method, which is due to the superiority of the simulation method over the conventional one.

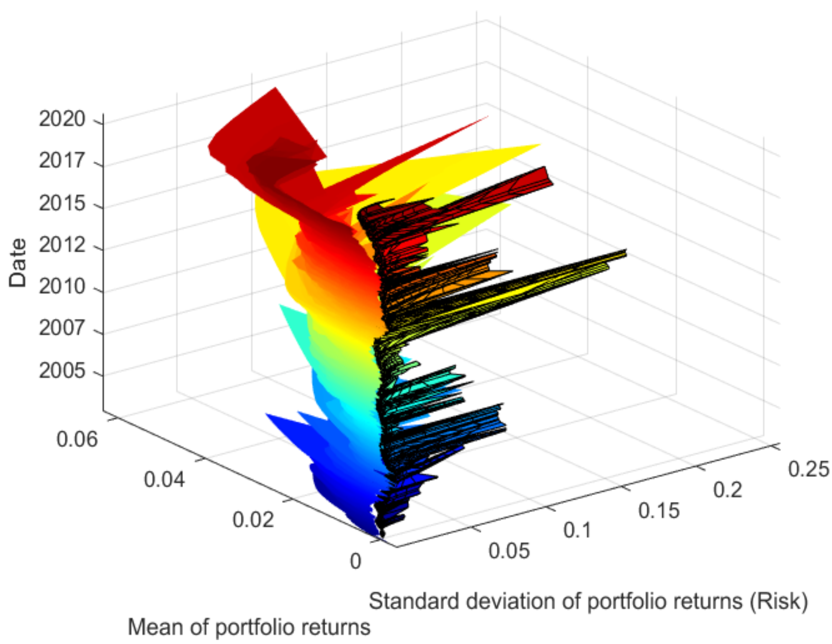

Figure 1. Mean-variance efficient frontier of conventional method (with black mesh lines) and simulation method (without black mesh lines) for 18-year period (December 2002 to September 2020) at 1-month intervals (18 trading days).

Source*: Research findings*

Figure (2) shows a bivariate histogram of conventional (green) and

[4] Note that for greater clarity these charts are provided for a random lot only. The efficient boundary of the two methods separately are available upon request.

simulation (blue) methods.[5] This diagram illustrates the frequency of efficient frontier points. That is, we have looked into 18 years (from December 2002 to September 2020) with one-month intervals (18 trading days), and at the beginning of each period, price information of the last 378 days has been used. Each period took account of 200 optimal portfolios. Overall, we kept in check 43,000 optimal portfolios for each method. Each point on the diagram shows how many of the 43,000 portfolios have their returns and the standard deviations in the same range. As the diagram reports, most conventional (green) method portfolios are left of the simulation (blue) ones. At each standard deviation level, portfolios' returns of the conventional method are less than those of the simulation method. Diagram also depicts the points with the highest frequency that for the conventional method, we have 699 portfolios with the returns and the standard deviations in the range [0.001335 0.001504] and [0.00378 0.00567] respectively and for the simulation method 1418 portfolios with the returns and the standard deviations in the range [0.0039 0.004575] and [0 0.00257] respectively.

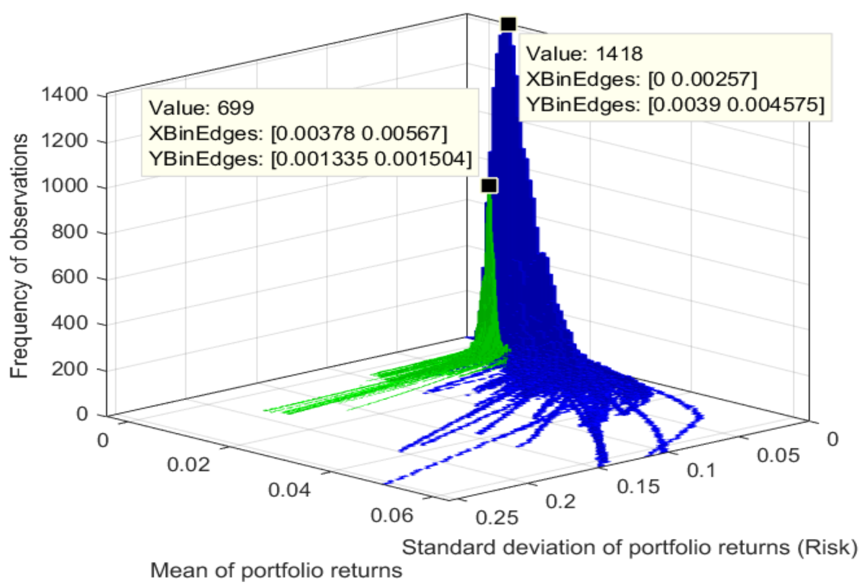

Figure 2. Bivariate histogram of conventional (green) and simulation (blue) methods for 18-year period (December 2002 to September 2020) at 1-month intervals (18 trading days).

***Source**: Research findings*

5.2 Comparing Performance of the Two Methods and the Actual Market Portfolio

This section compares two methods based on return, standard deviation

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[5] The histogram of the two methods separately are available upon request.

(risk) and Sharpe ratio. The higher Sharpe ratio, which measures the rate of return on a portfolio per unit of taken risk, means that the investment strategy is superior in terms of return at each level of taken risk. In other words, it means that the investment risk is lower at each level of return. For this purpose and our results to be empirically robust, we have considered three different lengths of investment: quarterly, semi-annual, and one year. In the quarterly case, the investor constructs the optimal stocks portfolio once every three months during the investment period of approximately twenty years, particularly at the end of each quarter; he revises his constructed portfolio and possibly invests in a new portfolio. In addition, we have obtained the results for the various volumes of data used to form the portfolio. For this purpose, we have reported the results for 2.5-year, 5-year, and 10-year data volumes. For example, in the case of 5 years, the investor uses the information of the last five years of all stocks understudy to construct the optimal portfolio. So in total, we have compared the results for nine different cases.

In addition, we have compared the results of the conventional and our proposed simulation methods with the overall market performance denoted by the market portfolio. This comparison helps us get a more concrete sense of how far the conventional and the simulation approaches are to the performance of market players and to find out how optimally market agents have acted. In what follows, we will explain this in more detail.

To evaluate the performance of the two methods, we assume that the investor stands at the beginning of time t and, using price information of last L days, extracts two optimal portfolios by conventional and simulation methods. Then, buys these portfolios and keeps them for the next f days. On the day $t+f$, we survey the performance of these portfolios. In the next period, which is the beginning of the $t+f$, again, it re-extracts the optimal portfolio using price information of last L days and keeps it for next f days. In the end, we survey the performance of these portfolios. This is repeated until the last possible period. Here, to show the robustness of the results, we report the model outputs for different values of L and f. For this purpose, we chose three values, 54 days (3 months), 108 days (6 months) and 216 days (1 year) for f and three values, 540 days (2.5 years), 1080 days (5 years) and 2160 days (10 years) for L.

First, we consider f equal to 216 market days (1 calendar year) and report the results of the two methods for different values of L. Since the simulation method is based on multidimensional geometric Brownian motion, a vector of random variables is generated to produce the Brownian motion vector in each simulation, so the simulation results depend on the generated random vector. For this reason, to evaluate the robustness of the results, for each L and f, we simulated 1000 times the prices by the MGBM method then extracted the optimal stock portfolio using Sharpe ratio maximization.

In addition to these two methods, we examine the performance of the market. The weight of each symbol in the market portfolio is the market capitalization of each symbol to the total market capitalization of the whole market (808 symbols studied).[6] We assume that the investor obtains the average weights of last L and f days as well as the weights of the investment day to constitute the market portfolio and buys these three portfolios at the time t and maintains them for the next f days. This process will be repeated until the last possible period. In this way, the performance of the market portfolios can be obtained, which is the outcome of the activity of the whole market.

Table 1 reports the investment results of two methods and portfolios offered by the market. For different values L, some of the portfolios' returns of the simulation method are significantly higher than the conventional method (results of the simulation methods are the average of 1000 times simulations). Also, the sum of the standard deviation of the portfolios' returns of the simulation method for different values L is less than the conventional method, which means investing with the simulation method is less risky. The best result of the conventional method in terms of return is 30% for $L = 1080$ and the simulation method 185% for $L = 2160$. In addition, the best result of the conventional method in terms of the sum of standard deviation is 3957% for $L = 1080$ and the simulation method 2310% for $L = 2160$.

Also, the average Sharpe ratios[7] of the simulation method for all L values are higher than the Sharpe ratios of other methods and the market portfolios, which shows that the simulation method is superior to the conventional method and the market portfolios in terms of return per unit of risk. According to the Sharpe ratio, the best performance of the conventional method is 0.0108 for $L = 2160$ the simulation method, 0.0557 for $L = 2160$ and market portfolios 0.0446. To evaluate the performance of conventional and simulation methods in all investment periods, the number of periods in which the Sharpe ratio of the simulation method has been higher than the conventional method is reported as a percentage of the total number of investment periods in Table I. This percentage is always more than 68% for different values L, which means the Sharpe ratio of the simulation method is better than the conventional method.

[6] We define the market capitalization of a company as the total company's outstanding shares times the current close price of a single share.

[7] To derive this criterion, first, in each f-days investment period we have calculated the optimal portfolio's Sharpe ratio, then, the average ratio of all periods has been obtained. Using this method, we have only one Sharpe ratio for the conventional method, but 1000 ratios for the simulation method. The tables show the latter's average.

Table 1. Performance of the two methods for f =216

Last L days used to form the portfolio	540 (2.5 years)	1080 (5 years)	2160 (10 years)
Sum of the portfolios' returns (%)			
the Conventional method	-209.98	30.31	3.4
the Simulation method – Average of 1000 times simulations	30.39	165.81	185.95
Sum of standard deviation of the portfolios' returns (%)			
the Conventional method	4757.4	3957.5	4238.4
the Simulation method – Average of 1000 times simulations	3645.6	2949.1	2310.4
Sharpe ratios			
the Conventional method	-0.0371	-0.0123	0.0108
the Simulation method – Average of 1000 times simulations	0.0105	0.0244	0.0557
The Market Portfolio – Using last L Days info.	-0.0307	0.0072	0.0446
The Market Portfolio – Using last f Days info.	-0.0354	-0.0004	0.0189
The Market Portfolio – Using investment day info.	-0.0324	0.011	0.0216
Outperformed periods for the simulation method (%)	68.47	72.83	71.91

Source: *Research findings*

Given that the simulation method depends on random variables and changes in each model run, we need to examine the results in multiple model runs. For this purpose, we produce the results of the simulation method 1000 times and compare the results with the conventional one. Figure (3) compares the simulation results (MGBM) and the conventional methods for all 1000 simulation trials $f = 216$. In all 1000 times simulations, the sum of portfolios' returns, the sum of standard deviations, and the average of Sharpe ratios of the simulation method are more, less, and more than the conventional method, respectively, which indicates the superiority of the simulation method over the conventional method. In addition, in each simulation, the number of periods that the simulation method outperformed concerning Sharpe ratios is shown as a percentage of the total investment periods, which is 998 out of 1000 simulation periods; this measure is above 50%, which indicates the simulation method is more efficient than the conventional method. The simulation method also outperforms the other L s and f s.[8]

[8] Detailed simulations results for other Ls and fs are available upon request.

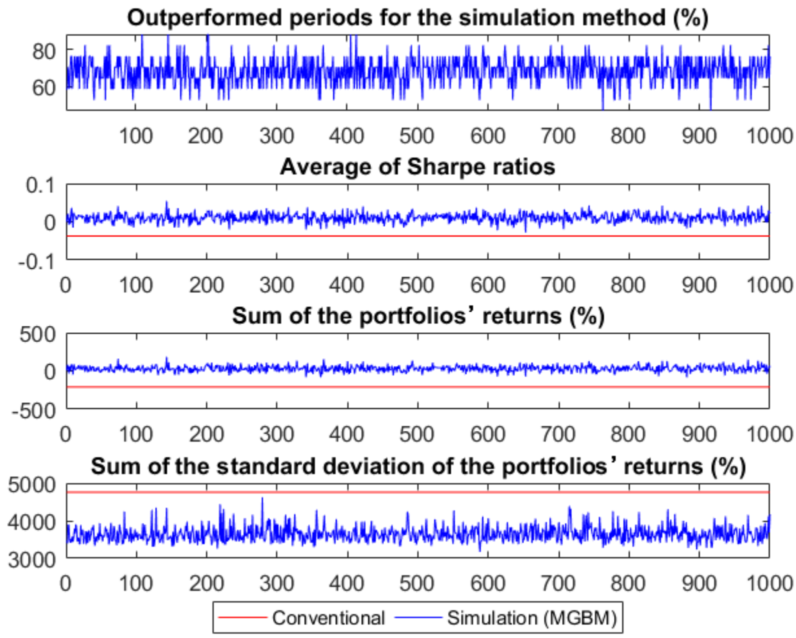

Figure 3. The performance of simulation (MGBM) and conventional methods for all 1000 simulation trials, L = 540 and f = 216.

***Source**: Research findings*

Table 2 reports the results for $f = 108$ and different values of L. According to these results, the sum of portfolios' returns and the standard deviation of portfolios' returns of the simulation method are more and less than those of the conventional method. The best result in terms of returns for the conventional method is 19.4% with $L = 2160$ and the simulation method (on average) 298.5% with $L = 2160$. Moreover, the best result in terms of standard deviation for the conventional method is 4108% with $L = 2160$ and the simulation method (on average) 2404% with $L = 2160$. As to the Sharpe ratio, the simulation method performed better in all cases. The best performance of the conventional Sharpe ratio is 0.0308 for $L = 2160$ the simulation method 0.0898 for $L = 2160$ and market portfolios 0.076. Also, the percentage of periods for which the simulation method has performed better in terms of Sharpe ratio for all values is above 70%. This outperformance is best achieved $L = 2160$ in 75% of periods.

Table 2. Performance of the two methods for f = 108

Last L days used to form the portfolio	540 (2.5 years)	1080 (5 years)	2160 (10 years)
Sum of the portfolios' returns (%)			
the Conventional method	-22.32	6.25	19.48
the Simulation method – Average of 1000 times simulations	238.69	271.5	298.58
Sum of standard deviation of the portfolios' returns (%)			
the Conventional method	4880.8	3359	4108.5
the Simulation method – Average of 1000 times simulations	3709.9	3015.2	2404.6
Sharpe ratios			
the Conventional method	-0.0102	-0.0123	0.0308
the Simulation method – Average of 1000 times simulations	0.0476	0.0557	0.0898
The Market Portfolio – Using last L Days info.	0.0019	0.0347	0.076
The Market Portfolio – Using last f Days info.	0.0033	0.0331	0.0511
The Market Portfolio – Using investment day info.	0.0063	0.0368	0.051
Outperformed periods for the simulation method (%)	70.49	75.12	71.02

Source: *Research findings*

Table 3 reports outputs for $f = 54$ and different values of L. The sum of portfolios' returns, the standard deviation of portfolios' returns, and the Sharpe ratios of the simulation method are respectively more, less, and more than those of the conventional method. In addition, increasing the last days' data to estimate the portfolio, the sum of returns and the sum of standard deviation in both methods increased and decreased, respectively. The best efficiency for the conventional method is the total return of 45% with $L = 2160$ and for the simulation method 290% $L = 2160$. Furthermore, the best result in terms of standard deviation for the conventional method is 3688% with $L = 2160$ and for the simulation method, 2297% with $L = 2160$. The best Sharpe ratio for the conventional method is 0.0458 with $L = 2160$, for the simulation method 0.0921 with $L = 2160$ and for market portfolios 0.0595. Sharpe ratio of the simulation method is superior in all cases.

Table 3. The performance of the two methods for f = 54

Last L days used to form the portfolio	540 (2.5 years)	1080 (5 years)	2160 (10 years)
Sum of the portfolios' returns (%)			
the Conventional method	-82.87	-16.2	45.15
the Simulation method – Average of 1000 times simulations	237.28	254.91	290.48
Sum of standard deviation of the portfolios' returns (%)			
the Conventional method	4439.44	4222.57	3688.17
the Simulation method – Average of 1000 times simulations	3420.96	2852.91	2297.1
Sharpe ratios			
the Conventional method	-0.005	0.0065	0.0458
the Simulation method – Average of 1000 times simulations	0.0688	0.0669	0.0921
The Market Portfolio – Using last L Days info.	-0.0079	0.0178	0.0595
The Market Portfolio – Using last f Days info.	-0.0016	0.0288	0.0422
The Market Portfolio – Using investment day info.	-0.0013	0.0278	0.0387
Outperformed periods for the simulation method (%)	63.64	63.8	59.29

Source: *Research findings*

Consequently, all of the results for different investment horizons (quarterly, six-month and one-year) and different data volumes (2.5, 5, and 10 years) showed that the sum of portfolios' returns, the sum of the standard deviation of portfolios' returns, and the Sharpe ratios of the simulation method are respectively more, less and more than those of the conventional method. Also, in all cases, the simulation method has a higher Sharpe ratio than the market portfolio. Therefore, it can be said that the simulation method has performed better than market players. In other words, market players have not been acted optimally through the investment period.

To study the difference between the two methods over time in terms of returns' mean, risk, and Sharpe ratio, we obtained optimization results for $L = 540$ and $f = 18$. The selection of these two numbers makes it possible to evaluate the performance of the two methods with monthly accuracy in a long-term period. For this purpose, we first calculate the difference in returns of the two methods in each one month, then accumulate these differences.[9] Thus, the more this measure increases over time, the greater the advantage of the simulation method over the conventional one. As presented in Figure (4), in 205 months (from October 2003 to September 2020), this measure has always been positive, which indicates the advantage of the simulation method over the conventional

[9] In other words, the accumulated return per month is equal to the sum of the returns of all months up to the month under review, which includes the month under review. Considering that 1000 times simulation has been done for the simulation method in each 18-days period, for each period the average return of this 1000 times has been calculated as the return of the simulation method.

method during this period. In addition, this criterion has had an upward trend despite various fluctuations throughout the period, which indicates an increase in the advantage of the simulation method.

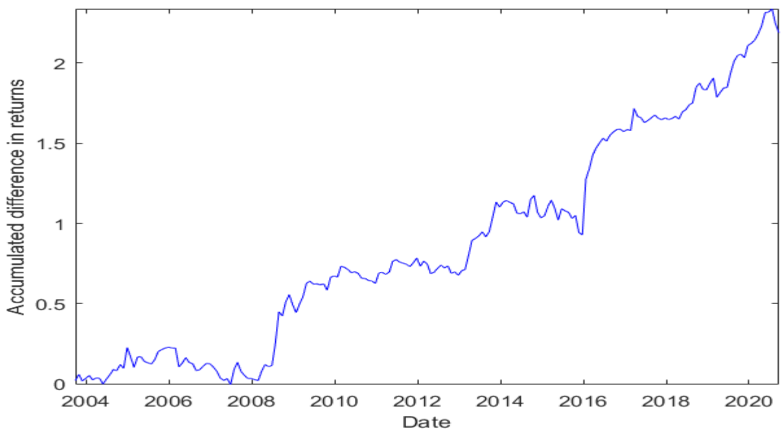

Figure 4. Accumulation of the difference between the returns of simulation method and conventional method (October 2003 - September 2020).
Source*: Research findings*

We have taken the same approach to compare the two methods in terms of standard deviation $L = 540$ and $f = 18$. We have accumulated differences in the standard deviations of the two methods.[10] The more this measure decreases over time, the greater the advantage of the simulation method over the conventional one in terms of standard deviation (risk). As presented in Figure (5), from October 2003 to September 2020, the criterion is always negative, which shows the advantage of the simulation method over the conventional one. Moreover, from October 2003 to April 2013, this criterion has always had a downward trend, which shows an increase in the advantage of the simulation method. Although, from April 2013 to September 2020, this criterion has had an upward trend but remained negative, which indicates the persistence of the simulation method outperformance during the entire study period.

[10] Considering that 1000 runs have been made for the simulation method in each 18-days period, for each period the average standard deviation of these 1000 times has been calculated as the standard deviation of the simulation method.

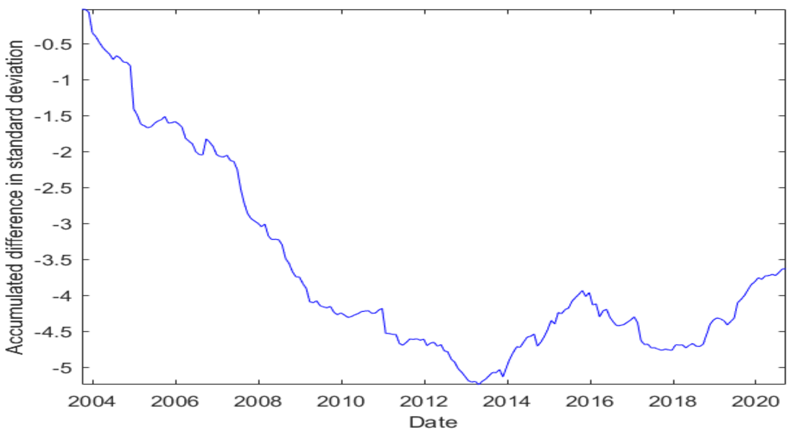

Figure 5. Accumulation of the difference between the standard deviation of simulation method and conventional method (October 2003 - September 2020).

***Source**: Research findings*

Figure (6) plots the performance of two methods in terms of accumulated differences in Sharpe ratios during the same period L and f the previous assessments. Enhancement of this criterion over time indicates an increase in the advantage of the simulation method concerning both returns and standard deviation (Risk) criteria. As Fig. 6 reports, from October 2003 to September 2020, this criterion has always been positive, showing the simulation method's advantage during the entire study period. In addition, despite the various fluctuations, this criterion has always had an upward trend, which indicates an increase in the advantage of the simulation method in terms of the Sharpe ratio during the period under review.

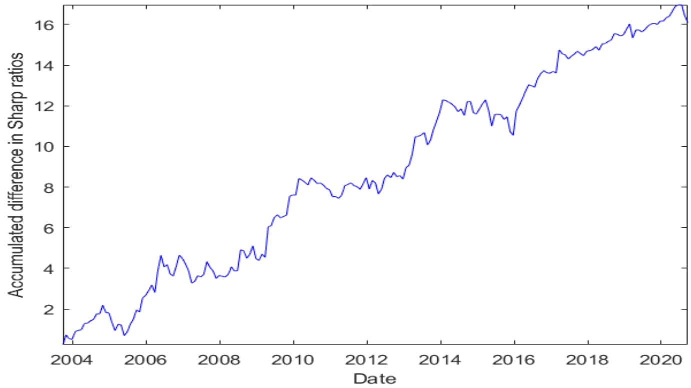

Figure 6. Accumulation of the difference between the sharpe ratios of simulation method and conventional method (October 2003 - September 2020).

***Source**: Research findings*

6. Concluding Remarks

In this paper, we utilized a novel method to perk up the performance of the conventional method, informing the optimal investment portfolio. In the conventional method, raw data is tapped to form the optimal portfolio. In our proposed simulation method, a system of MGBM equations plays a central role in optimal portfolio speculation. To assess the robustness of our original findings, the actual market portfolios have been thoroughly investigated for the corresponding periods too. In doing so, we identified the market portfolios based on the market capitalization of each symbol relative to the total market capitalization using the information of last L and f days, along with information of the investment day (time t). As far as Sharpe ratio maximization is concerned, the results show that the proposed simulation method performs better than the conventional method and actual market portfolios in all cases.

Further research needs to be undertaken to see if our findings are related to the market's fundamental characteristics of the market? And if they can be extended to several other selected stock exchange markets in, for instance, emerging and developed economies. The latter concern is the subject of our next endeavor.

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The authors declare no conflict of interest.

Data Availability Statement:

The data used in the study were taken from http://tsetmc.com/ and https://www.cbi.ir/.

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