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On modelling of two-dimensional MHD flow with induced magnetic field: solution of peristaltic flow of a couple stress fluid in a channel

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Abstract

The aim of present paper is to provide mathematical modelling for the two-dimensional MHD flow with induced magnetic field. The flaws in the already existing equations have been pointed out. The results of low magnetic Reynolds number approximation are recovered as a special case from the developed equations. As an example, the peristaltic flow for a couple stress fluid in a channel is considered. For the solution of the problem the governing equations are simplified under the realistic assumption of long wavelength. Exact solution of the problem is presented and some features of peristaltic motion have been discussed. It is observed that the applied magnetic field increases the pressure rise in the pumping region. However, the presence of applied electric field reduced it in that region. It is also found that the stream function is independent of applied electric field.

***Keywords**:* Two-dimensional flow induced magnetic field; mathematical modelling; peristaltic motion; planar channel

1. Introduction

Flow of fluid due to propagation of transverse waves along the flexible walls of the channel/tube, commonly known as peristaltic flow, has promising applications in physiology and industry. In particular, peristaltic flows are involved in transport of urine from the kidney to the bladder, movement of chime in gastro-intestinal tract and transport of corrosive fluids. Due to their wide applications a great deal of literature is available on analysis of these flows. Some recent and interesting studies about these flows can be found in (Siddiqui et al., 1994; El-Shehawey & Mekheimer, 1994; Mekheimer, 2002; Hayat et al., 2002; Hayat et al., 2003; Haroun, 2007; Haroun, 2007). In these attempts researchers have performed analysis by considering both non-Newtonian and electrically conducting nature of fluid. Another important area which has received attention nowadays is heat transfer analysis of peristaltic flows. The works of Vajravelu et al. (2007), Kothandapani and Srinivas (2008), Mekheimer et al. (2008) and Hayat et al. (2009) are important contributions in this area.

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A recent literature survey indicates that the peristaltic motion of magnetohydrodynamic (MHD) electrically conducting fluid is an active area of study nowadays for the researchers in the field. The effects of a constant applied magnetic field on the peristaltic flows of Newtonian and non-Newtonian fluids have been considered by many researchers (Mekheimer (2003); Mekheimer & Al-Arabi, (2003); Mekheimer, (2004); Elshahed & Haroun, (2005); Hayat et al. (2005); El Hakeem et al. (2006); Hayat and Ali, (2006); Hayat et al. (2007); Hayat and Ali, (2007); Hayat et al. (2007); Hayat et al. (2007); Ali et al. (2008); Wang et al. (2008)) and references therein. In the studies (Mekheimer (2003); Mekheimer & Al-Arabi, (2003); Mekheimer, (2004); Elshahed & Haroun, (2005); Hayat et al. (2005); El Hakeem et al. (2006); Hayat and Ali, (2006); Hayat et al. (2007); Hayat and Ali, (2007); Hayat et al. (2007); Hayat et al. (2007); Ali et al. (2008); Wang et al. (2008)), the flow equations are modelled under the assumption that magnetic Reynold number is small. Under this assumption it is assumed that the magnetic field associated with the induced currents is negligible as compared to the applied magnetic field (Davidson, (2001); Vishnyakov & Pavlov, (1972)). Moreover, under the assumption of small magnetic Reynold number the diffusion is dominant and advection is neglected. Thus the equation of motion is simplified considerably under the low magnetic Reynolds number approximation and for a two-dimensional flow it has only one extra term. Some other related studies on MHD flows can be found in references (Parsa et al., (2013); Parsa et al., (2013); Rashidi & Keimanesh, (2010); Rashidi et al. (2013)).

In some recent studies (Eldabe et al., (2007); Mekheimer, (2008); Mekheimer, (2008)) on the peristaltic flows the effects of the induced magnetic field are also incorporated. The aim of incorporating the induced magnetic field is to get rid of the low magnetic Reynolds number approximation and thus look for a MHD flow when magnetic Reynolds number is moderate or high. For this one needs to incorporate not only the induced magnetic field but also the effects of advection of the magnetic field. Moreover, it is also necessary that the reduced form of the Maxwell equations in MHD are also satisfied. The induced magnetic field is caused due to the induced currents in accordance with the Ampere's law. These induced currents come into play because of the relative motion of conducting fluid and a magnetic field that causes an electromotive force. This induced magnetic results the original constant applied magnetic field and interacts with the current density to give a Lorentz force per unit volume. This Lorentz force appears in the equation of motion and thus the effects of various features of magnetic field on the fluid motion can be discussed. However, if the effects of induced magnetic field at any stage are neglected the results for the case of constant applied magnetic field should be recovered. This is not the case with the governing equations provided in refs. (Eldabe et al., (2007); Mekheimer, (2008); Mekheimer, (2008)). This fact shows that there is some flaw in the mathematical modeling of the two-dimensional equations presented in (Eldabe et al., (2007); Mekheimer, (2008); Mekheimer, (2008)). Keeping this fact in mind the purpose of the present study is to provide correct mathematical modeling for two-dimensional flow which is equally valid for small magnetic Reynolds number and beyond. The equations resulting from the presented mathematical modelling are used to discuss the problem analyzed in ref. (Mekheimer, (2008)) for a couple stress non-Newtonian fluid.

2. Governing Equations

The flow of a unsteady, incompressible couple stress fluid in the presence of a magnetic field is governed by the following equations

$$\mathrm{div}\bar{\mathbf{V}} = 0, \tag{1}$$

$$\rho \frac{d\bar{\mathbf{V}}}{d\bar{t}} = -\nabla\bar{p} + \mu\nabla^2\bar{\mathbf{V}} - \bar{\gamma}\nabla^4\bar{\mathbf{V}} + \bar{\mathbf{J}}\times\bar{\mathbf{B}}, \tag{2}$$

where $\bar{\mathbf{V}}$ is the fluid velocity, ρ the fluid density, μ the dynamic viscosity, $\bar{p}$ the hydrostatic pressure, $\bar{\gamma}$ is the couple stress parameter, $\bar{\mathbf{J}} = \bar{\mathbf{J}}_0 + \bar{\mathbf{J}}_1$ is the current density in which $\bar{\mathbf{J}}_1$ is the induced current density, $\bar{\mathbf{B}} = \bar{\mathbf{B}}_0 + \bar{\mathbf{B}}_1$ is the magnetic field ($\bar{\mathbf{B}}_0$ is the constant applied magnetic field and $\bar{\mathbf{B}}_1$ is the induced magnetic field). When $Rm \ll 1$, we take $\bar{\mathbf{J}}_1 \approx \bar{\mathbf{E}}_1 + \sigma\left(\bar{\mathbf{V}}\times\bar{\mathbf{B}}_1\right)$ negligible in comparison to $\bar{\mathbf{J}}_0 \approx \bar{\mathbf{E}}_0 + \sigma\left(\bar{\mathbf{V}}\times\bar{\mathbf{B}}_0\right)$, and therefore, $\bar{\mathbf{E}}_1 = \bar{\mathbf{B}}_1 = \bar{\mathbf{J}}_1 = 0$ and thus the Ampere's and Faraday's laws are identically satisfied. Here σ is the electrical conductivity, $\bar{\mathbf{E}}_0$ and $\bar{\mathbf{E}}_1$ are respectively the constant applied and induced electric fields. And, in the absence of electric field obtains:

$$\bar{\mathbf{J}}\times\bar{\mathbf{B}} = \bar{\mathbf{J}}_0\times\bar{\mathbf{B}}_0 = \sigma\left(\bar{\mathbf{V}}\times\bar{\mathbf{B}}_0\right)\times\bar{\mathbf{B}}_0 \tag{3}$$

Furthermore, the induction equation in this case reduces to

$$\frac{\partial\bar{\mathbf{B}}_0}{\partial\bar{t}} = \lambda_1\nabla^2\bar{\mathbf{B}}_0, \tag{4}$$

which is identically satisfied for a constant applied magnetic field. If we need to incorporate the effects of induced magnetic field which is due to induced currents $\mathbf{J}_1$ then the low magnetic Reynolds number approximation must be dropped and thus $\mathbf{E}_1 = \mathbf{B}_1 = \mathbf{J}_1 \neq 0$ and from Ampere's, Faradays, Gauss's and Ohm's law we may write

$$\nabla\times\bar{\mathbf{B}}_1 = \mu^*\bar{\mathbf{J}}_1, \tag{5}$$

$$\nabla\times\bar{\mathbf{E}}_1 = -\frac{\partial\bar{\mathbf{B}}_1}{\partial\bar{t}}, \tag{6}$$

$$\nabla\cdot\bar{\mathbf{B}} = 0, \tag{7}$$

$$\bar{\mathbf{J}} = \sigma\left[\bar{\mathbf{E}}_0 + \bar{\mathbf{V}}\times\bar{\mathbf{B}}_0 + \bar{\mathbf{E}}_1 + \bar{\mathbf{V}}\times\bar{\mathbf{B}}_1\right], \tag{8}$$

The induction equtation now takes the form

$$\nabla\times\left(\bar{\mathbf{V}}\times\bar{\mathbf{B}}\right)+\lambda_1\nabla^2\bar{\mathbf{B}}=\frac{\partial\bar{\mathbf{B}}}{\partial\bar{t}}, \tag{9}$$

where μ^* is the magnetic permeability, λ_1 is the magnetic difusivity and $d/d\bar{t}$ is the material time derivative defined by

$$\frac{d}{d\bar{t}}=\frac{\partial}{\partial\bar{t}}+\mathbf{V}\cdot\nabla \tag{10}$$

Instead of using $\bar{\mathbf{J}}_1=\bar{\mathbf{E}}_1+\sigma\left(\bar{\mathbf{V}}\times\bar{\mathbf{B}}_1\right)$ from Ohm's law for the effects of induced magnetic field we use the value of $\bar{\mathbf{J}}_1$ from the Ampere's law and therefore, we have

$$\bar{\mathbf{J}}=\sigma\left[\bar{\mathbf{E}}_0+\bar{\mathbf{V}}\times\bar{\mathbf{B}}_0\right]+\frac{1}{\mu^*}\left(\nabla\times\bar{\mathbf{B}}_1\right), \tag{11}$$

For the two-dimensional flow with a transverse constant applied magnetic field we have

$$\bar{\mathbf{V}}=\left[\bar{U}\left(\bar{X},\bar{Y},\bar{t}\right),\bar{V}\left(\bar{X},\bar{Y},\bar{t}\right),0\right],\ \bar{\mathbf{B}}_0=\left[0,\bar{B}_0,0\right], \tag{12}$$

$$\bar{\mathbf{B}}_1=\left[\bar{B}_1\left(\bar{X},\bar{Y},\bar{t}\right),\bar{B}_2\left(\bar{X},\bar{Y},\bar{t}\right),0\right],\bar{\mathbf{E}}_0=\left[0,0,\bar{E}\right], \tag{13}$$

$$\bar{\mathbf{E}}_1=\left[0,0,\bar{E}_3(\bar{X},\bar{Y},\bar{t})\right]. \tag{14}$$

and thus

$$\bar{\mathbf{J}}\times\bar{\mathbf{B}}=\begin{bmatrix}\left\{-\sigma\left(\bar{E}+\bar{U}\bar{B}_0\right)+\frac{1}{\mu^*}\left(\frac{\partial\bar{B}_2}{\partial\bar{X}}-\frac{\partial\bar{B}_1}{\partial\bar{Y}}\right)\right\}\times\left(\bar{B}_0+\bar{B}_2\right),\\ \left\{\sigma\left(\bar{E}+\bar{U}\bar{B}_0\right)+\frac{1}{\mu^*}\left(\frac{\partial\bar{B}_2}{\partial\bar{X}}-\frac{\partial\bar{B}_1}{\partial\bar{Y}}\right)\right\}\times\left(\bar{B}_1\right)\end{bmatrix} \tag{15}$$

It is evident from Eq. (15) that in the absence of induced magnetic field the expression of the Lorentz force is valid when there is only applied magnetic field, i.e.

$$\bar{\mathbf{J}}\times\bar{\mathbf{B}}=\left[-\sigma\bar{U}\bar{B}_0{}^2,0\right] \tag{16}$$

However, this is not possible with the model taken in (Eldabe et al., (2007); Mekheimer, (2008); Mekheimer, (2008)).

3. Mathematical Model

Let us Consider a two dimensional channel of half width a filled with homogenous incompressible couple stress fluid. The walls of the channel are flexible with infinite wave train travelling with velocity c along them. A uniform magnetic field $\bar{\mathbf{B}}_0$ is applied in the transverse direction to the flow. Cartesian coordinates are used to analyze the flow with $\bar{X}$ along the flow direction and $\bar{Y}$ normal to it. Let $\bar{U}$ and $\bar{V}$ be longitudinal and transverse velocity components of the fluid velocity, respectively. The equation of the wall surface is

$$\bar{\eta}\left(\bar{X},\bar{t}\right)=a+b\sin\left[\frac{2\pi}{\lambda}\left(\bar{X}-c\bar{t}\right)\right], \tag{17}$$

in which b is the wave amplitude, λ is the wavelength and $\bar{t}$ is the time. The geometry of the problem is illustrated in Fig. 1.

$M=3, F=-0.5,$

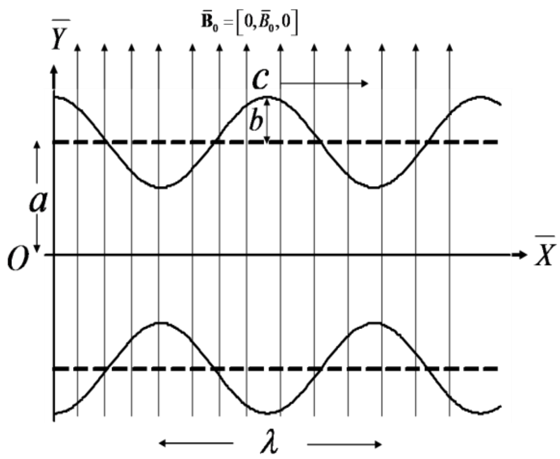

Fig. 1. Schematic diagram of flow geometry

The equations that govern the flow are

$$\frac{\partial\bar{U}}{\partial\bar{X}}+\frac{\partial\bar{V}}{\partial\bar{Y}}=0, \tag{18}$$

$$\begin{aligned}\rho\left(\frac{\partial\bar{U}}{\partial\bar{t}}+\bar{U}\frac{\partial\bar{U}}{\partial\bar{X}}+\bar{V}\frac{\partial\bar{U}}{\partial\bar{Y}}\right)&=-\frac{\partial\bar{p}_m}{\partial\bar{X}}+\mu\left(\frac{\partial^2\bar{U}}{\partial\bar{X}^2}+\frac{\partial^2\bar{U}}{\partial\bar{Y}^2}\right)-\bar{\gamma}\left(\frac{\partial^4\bar{U}}{\partial\bar{X}^4}+\frac{\partial^4\bar{U}}{\partial\bar{Y}^4}+2\frac{\partial^4\bar{U}}{\partial\bar{X}^2\partial\bar{Y}^2}\right)\\&-\sigma\left(\bar{E}+\bar{U}\bar{B}_0\right)\left(\bar{B}_0+\bar{B}_2\right)\\&+\frac{\bar{B}_1}{\mu^*}\frac{\partial\bar{B}_1}{\partial\bar{X}}-\frac{\bar{B}_0}{\mu^*}\left(\frac{\partial\bar{B}_2}{\partial\bar{X}}-\frac{\partial\bar{B}_1}{\partial\bar{Y}}\right)+\frac{\bar{B}_2}{\mu^*}\frac{\partial\bar{B}_1}{\partial\bar{Y}},\end{aligned} \tag{19}$$

$$\begin{aligned}\rho\left(\frac{\partial\bar{V}}{\partial\bar{t}}+\bar{U}\frac{\partial\bar{V}}{\partial\bar{X}}+\bar{V}\frac{\partial\bar{V}}{\partial\bar{Y}}\right)&=-\frac{\partial\bar{p}_m}{\partial\bar{Y}}+\mu\left(\frac{\partial^2\bar{V}}{\partial\bar{X}^2}+\frac{\partial^2\bar{V}}{\partial\bar{Y}^2}\right)-\bar{\gamma}\left(\frac{\partial^4\bar{V}}{\partial\bar{X}^4}+\frac{\partial^4\bar{V}}{\partial\bar{Y}^4}+2\frac{\partial^4\bar{V}}{\partial\bar{X}^2\partial\bar{Y}^2}\right)\\&-\sigma\left(\bar{E}+\bar{U}\bar{B}_0\right)\bar{B}_1+\frac{\bar{B}_1}{\mu^*}\frac{\partial\bar{B}_2}{\partial\bar{X}}+\frac{\bar{B}_2}{\mu^*}\frac{\partial\bar{B}_2}{\partial\bar{Y}},\end{aligned} \tag{20}$$

$$\frac{\partial \bar{B}_1}{\partial \bar{X}}+\frac{\partial \bar{B}_2}{\partial \bar{Y}}=0, \tag{21}$$

$$\frac{\partial}{\partial \bar{Y}}\left(\bar{U}\bar{B}_2+\bar{U}\bar{B}_0-\bar{V}\,\bar{B}_1\right)+\lambda_1\left(\frac{\partial^2 \bar{B}_1}{\partial \bar{X}^2}+\frac{\partial^2 \bar{B}_1}{\partial \bar{Y}^2}\right)=\frac{\partial \bar{B}_1}{\partial \bar{t}}, \tag{22}$$

$$-\frac{\partial}{\partial \bar{X}}\left(\bar{U}\bar{B}_2+\bar{U}\bar{B}_0-\bar{V}\,\bar{B}_1\right)+\lambda_1\left(\frac{\partial^2 \bar{B}_2}{\partial \bar{X}^2}+\frac{\partial^2 \bar{B}_2}{\partial \bar{Y}^2}\right)=\frac{\partial \bar{B}_2}{\partial \bar{t}}, \tag{23}$$

$$\frac{\partial \bar{E}_3}{\partial \bar{Y}}=-\frac{\partial \bar{B}_1}{\partial \bar{t}}, \tag{24}$$

$$\frac{\partial \bar{E}_3}{\partial \bar{X}}=\frac{\partial \bar{B}_2}{\partial \bar{t}}, \tag{25}$$

in which $\bar{p}_m=\bar{p}+\left(\bar{B}_1^2+\bar{B}_2^2\right)/2\mu^*$.

In the fixed frame of reference the flow phenomenon is unsteady. To carry out the steady analysis we switch to the coordinate system moving with the speed of the wave called the wave frame. The two frames are related through the following transformations

$$\bar{x}=\bar{X}-c\bar{t},\ \bar{y}=\bar{Y}, \qquad \bar{u}=\bar{U}-c,\ \ \bar{v}=\bar{V}. \tag{26}$$

With the help of Eq. (26) we write Eqs. (18)-(25) as

$$\frac{\partial \bar{u}}{\partial \bar{x}}+\frac{\partial \bar{v}}{\partial \bar{y}}=0, \tag{27}$$

$$\begin{aligned}\rho\left(\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}}+\bar{v}\frac{\partial \bar{u}}{\partial \bar{y}}\right)=&-\frac{\partial \bar{p}_m}{\partial \bar{x}}+\mu\left(\frac{\partial^2 \bar{u}}{\partial \bar{x}^2}+\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right)-\bar{\gamma}\left(\frac{\partial^4 \bar{u}}{\partial \bar{x}^4}+\frac{\partial^4 \bar{u}}{\partial \bar{y}^4}+2\frac{\partial^4 \bar{u}}{\partial \bar{x}^2\partial \bar{y}^2}\right)\\&-\sigma\left(\bar{E}+\left(\bar{u}+c\right)\bar{B}_0\right)\left(\bar{B}_0+\bar{B}_2\right)+\\&\frac{\bar{B}_1}{\mu^*}\frac{\partial \bar{B}_1}{\partial \bar{x}}-\frac{\bar{B}_0}{\mu^*}\left(\frac{\partial \bar{B}_2}{\partial \bar{x}}-\frac{\partial \bar{B}_1}{\partial \bar{y}}\right)+\frac{\bar{B}_2}{\mu^*}\frac{\partial \bar{B}_1}{\partial \bar{y}},\end{aligned} \tag{28}$$

$$\begin{aligned}\rho\left(\bar{u}\frac{\partial \bar{v}}{\partial \bar{x}}+\bar{v}\frac{\partial \bar{v}}{\partial \bar{y}}\right)=&-\frac{\partial \bar{p}_m}{\partial \bar{y}}+\mu\left(\frac{\partial^2 \bar{v}}{\partial \bar{x}^2}+\frac{\partial^2 \bar{v}}{\partial \bar{y}^2}\right)-\bar{\gamma}\left(\frac{\partial^4 \bar{v}}{\partial \bar{x}^4}+\frac{\partial^4 \bar{v}}{\partial \bar{y}^4}+2\frac{\partial^4 \bar{v}}{\partial \bar{x}^2\partial \bar{y}^2}\right)\\&-\sigma\left(\bar{E}+\left(\bar{u}+c\right)\bar{B}_0\right)\bar{B}_1+\frac{\bar{B}_1}{\mu^*}\frac{\partial \bar{B}_2}{\partial \bar{x}}+\\&\frac{\bar{B}_2}{\mu^*}\frac{\partial \bar{B}_2}{\partial \bar{y}},\end{aligned} \tag{29}$$

$$\frac{\partial \bar{B}_1}{\partial \bar{x}}+\frac{\partial \bar{B}_2}{\partial \bar{y}}=0, \tag{30}$$

$$\frac{\partial}{\partial \bar{y}}\left(\left(\bar{u}+c\right)\left(\bar{B}_2+\bar{B}_0\right)_1-\bar{v}\,\bar{B}_1\right)+\lambda_1\left(\frac{\partial^2 \bar{B}_1}{\partial \bar{x}^2}+\frac{\partial^2 \bar{B}_1}{\partial \bar{y}^2}\right)=-c\frac{\partial \bar{B}_1}{\partial \bar{x}}, \tag{31}$$

$$-\frac{\partial}{\partial \bar{x}}\left(\left(\bar{u}+c\right)\left(\bar{B}_2+\bar{B}_0\right)_1-\bar{v}\,\bar{B}_1\right)+\lambda_1\left(\frac{\partial^2 \bar{B}_2}{\partial \bar{x}^2}+\frac{\partial^2 \bar{B}_2}{\partial \bar{y}^2}\right)=-c\frac{\partial \bar{B}_2}{\partial \bar{x}}, \tag{32}$$

$$\frac{\partial \bar{E}_3}{\partial \bar{y}}=c\frac{\partial \bar{B}_1}{\partial \bar{x}}, \tag{33}$$

$$\frac{\partial \bar{E}_3}{\partial \bar{x}}=-c\frac{\partial \bar{B}_2}{\partial \bar{x}}, \tag{34}$$

where $\left(\bar{u},\bar{v}\right)$ are the longitudinal and transverse velocity components in the wave frame. The relevant boundary conditions are

$$\begin{aligned}&\frac{\partial \bar{U}}{\partial \bar{Y}}=0,\ \frac{\partial^3 \bar{U}}{\partial \bar{Y}^3}=0,\ \bar{B}_1=0\ \ \text{at}\ \ \bar{Y}=0,\\&\bar{U}=0,\ \frac{\partial^2 \bar{U}}{\partial \bar{Y}^2}=0,\ \bar{B}_1=0\ \ \text{at}\ \ y=\bar{\eta}.\end{aligned} \tag{35}$$

In order to non-dimensionalize the set of Eqs. (27)-(35) we use the following

$$\begin{aligned}&x=\frac{\bar{x}}{\lambda},\ y=\frac{\bar{y}}{a},\ u=\frac{\bar{u}}{c},\ v=\frac{\bar{v}}{c},\ E_3=\frac{\bar{E}_3}{\bar{B}_0 c},\ \gamma=\sqrt{\frac{\mu}{\bar{\gamma}}}a\\&p_m=\frac{a^2\bar{p}_m}{\lambda\mu c},\ h_x=\frac{\bar{B}_1}{\bar{B}_0},\ h_y=\frac{\bar{B}_2}{\bar{B}_0},\ \ E=\frac{\bar{E}}{\bar{B}_0 c}\end{aligned} \tag{36}$$

Utilizing the relations (12), (13), (26), (27), (30) and (36) and defining the stream function $\psi\left(x,y\right)$ and the magnetic force function $\phi\left(x,y\right)$ by

$$u=\frac{\partial \psi}{\partial y},\ w=-\delta\frac{\partial \psi}{\partial x},\ h_x=\frac{\partial \phi}{\partial y},\ h_y=-\delta\frac{\partial \phi}{\partial x} \tag{37}$$

with $\delta=a/\lambda$, the Eqs. (27) and (30) are satisfied identically and Eqs. (28)-(34) under long wavelength and low Reynolds number assumption yield

$$\frac{\partial p_m}{\partial x}=\frac{\partial^3 \psi}{\partial y^3}-\frac{1}{\gamma^2}\frac{\partial^5 \psi}{\partial y^5}-M^2\left\{\frac{\partial \psi}{\partial y}+1+E-\frac{1}{Rm}\frac{\partial^2 \phi}{\partial y^2}\right\}, \tag{38}$$

$$\frac{\partial p_m}{\partial y}=0, \tag{39}$$

$$\frac{\partial^3\phi}{\partial y^3}+Rm\frac{\partial^2\psi}{\partial y^2}=0 \tag{40}$$

where $M^2=\sigma\bar{B}_0^2a^2/\mu$, $\gamma=\sqrt{\mu/\bar{\gamma}}\,a$ and $Rm=\sigma\mu^* ac$ are the non-dimensional Hartmann number, couple stress fluid parameter and magnetic Reynolds number, respectively.

Elimination of pressure from (38) and (39) yields the following determining equation for the stream function and magnetic force function

$$\frac{\partial^4\psi}{\partial y^4}-\frac{1}{\gamma^2}\frac{\partial^6\psi}{\partial y^6}-M^2\left\{\frac{\partial^2\psi}{\partial y^2}-\frac{1}{Rm}\frac{\partial^3\phi}{\partial y^3}\right\}=0, \tag{41}$$

$$\frac{\partial^3\phi}{\partial y^3}+Rm\frac{\partial^2\psi}{\partial y^2}=0 \tag{42}$$

It is important to mention that neglecting the effects of induced magnetic field form Eqs. (41) and (42) do not eliminate the effects of applied magnetic field. However, the equations derived in [32] do not delineate this fact.

The dimensionless pressure rise over one wavelength can be calculated by

$$\Delta p=\int_0^{2\pi}\frac{dp_m}{dx}dx, \tag{43}$$

where dp_m/dx is given by Eq. (38). The non-dimensional expression for current density is given by

$$J_z=Rm\left\{E+1+\frac{\partial\psi}{\partial y}-\frac{\partial^2\phi}{\partial y^2}\right\} \tag{44}$$

The appropriate boundary conditions in terms of stream function are

$$\psi=0,\ \frac{\partial^2\psi}{\partial y^2}=0,\ \frac{\partial^4\psi}{\partial y^4}=0,\ \frac{\partial\phi}{\partial y}=0 \text{ at } y=0,$$

$$\left.\begin{array}{l}\psi=F,\ \dfrac{\partial\psi}{\partial y}=-1,\\ \dfrac{\partial^3\psi}{\partial y^3}=0,\ \phi=0,\ \dfrac{\partial\phi}{\partial y}=0\end{array}\right\} \text{ at } y=\eta=1+\alpha^*\sin 2\pi x, \tag{45}$$

where $\alpha^*=b/a$ is the amplitude ratio. The boundary conditions $\psi(0)=0$ and $\psi(\eta)=F$ are consequence of the definition of flow rate F in wave frame. According to this definition

$$F=\int_0^{\eta}\frac{\partial\psi}{\partial y}dy=\psi(\eta)-\psi(0).$$

The above expression furnish the conditions $\psi(0)=0$ and $\psi(\eta)=F$. The dimensionless mean flow rates Θ in the fixed frame and F in the wave frame are related through the following expression

$$\Theta=F+1. \tag{46}$$

4. Exact Solution

Substituting the value of $\partial^3\phi/\partial y^3$ from Eq. (42) into Eq. (41) yields

$$\frac{\partial^4\psi}{\partial y^4}-\frac{1}{\gamma^2}\frac{\partial^6\psi}{\partial y^6}-2M^2\frac{\partial^2\psi}{\partial y^2}=0. \tag{47}$$

Integration of the above equation gives

$$\psi=C_1\sinh(m_1y)+C_2\cosh(m_1y)+C_3\sinh(m_2y)+C_4\cosh(m_2y)+\frac{1}{2M^2\gamma^2}(C_5y+C_6), \tag{48}$$

where $m_{1,2}=\sqrt{\dfrac{\gamma^2\mp\sqrt{\gamma^4-8M^2\gamma^2}}{2}}$.

The boundary conditions on ψ give $C_2=C_4=C_6=0$, while the values of other constants are

$$C_1=\frac{(F+h)m_2^3\cosh(m_2h)}{N},$$

$$C_2=-\frac{(F+h)m_1^3\cosh(m_1h)}{N},$$

$$C_3=2M^2\gamma^2\begin{bmatrix}Fm_1m_2\left(m_1^2-m_2^2\right)\cosh(m_1h)\cosh(m_2h)\\ -m_2^3\cosh(m_2h)\sinh(m_1h)+\\ m_1^3\sinh(m_2h)\cosh(m_1h)\end{bmatrix}/N,$$

$$N=m_2^3\cosh(m_2h)\sinh(m_1h)-m_1^3\cosh(m_1h)\sinh(m_2h)+m_1m_2h\left(m_1^2-m_2^2\right)\cosh(m_1h)\cosh(m_2h).$$

Thus we have the following expression of ψ

$$\psi=C_1\sinh(m_1y)+C_2\sinh(m_2y)+\frac{C_3}{2\gamma^2M^2}y. \tag{49}$$

Utilizing the above value of ψ in Eq. (42) and using the corresponding boundary conditions on ϕ in (45), we get

$$\phi = Rm\begin{bmatrix}2C_1m_2h\left\{\cosh(m_1h)-\cosh(m_1y)\right\}\\ +2C_3m_1h\left\{\cosh(m_2h)-\cosh(m_2y)\right\}\end{bmatrix} + C_1m_1m_2\left(y^2-h^2\right)\sinh(m_1h)+ C_3m_1m_2\left(y^2-h^2\right)\sinh(m_2h)/2m_1m_2h. \tag{50}$$

With the help of Eq. (38), the axial pressure gradient turns out to be

$$\frac{dp_m}{dx} = -M^2(E+1)-\frac{C_5}{2\gamma^2}+C_1m_1\begin{pmatrix}-2M^2+\\ m_1^2-\dfrac{m_1^4}{\gamma^2}\end{pmatrix}\cosh(m_1y) + C_3m_2\begin{pmatrix}-2M^2+\\ m_2^2-\dfrac{m_2^4}{\gamma^2}\end{pmatrix}\cosh(m_2y)+\frac{M^2}{h}\begin{pmatrix}C_1\sinh(m_1h)\\ +C_3\sinh(m_2h)\end{pmatrix}. \tag{51}$$

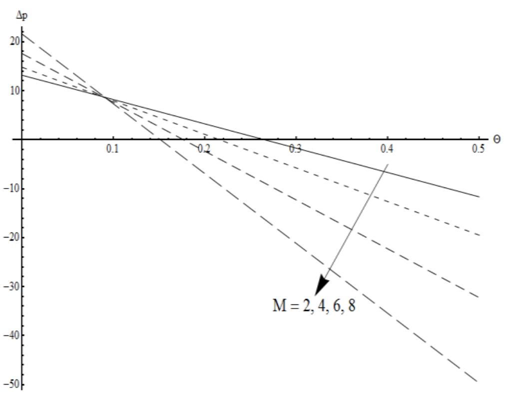

Fig. 2. Pressure rise per wavelength Δp versus flow rate Θ for various values of Hartman number M. The other parameters chosen are $\gamma = 0.8, E = 0$ and $\alpha^* = 0.4.$

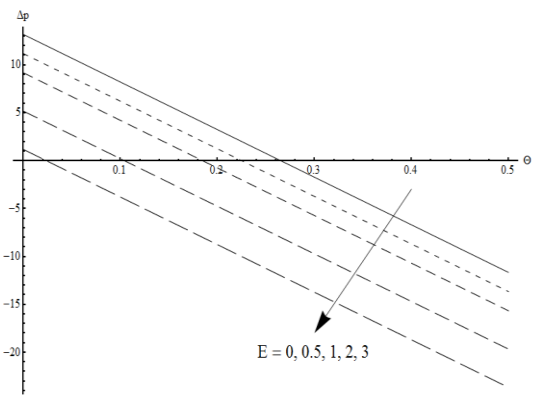

Fig. 3. Pressure rise per wavelength Δp versus flow rate Θ for various values of E. The other parameters chosen are $\gamma = 0.8,\ M = 2$ and $\alpha^* = 0.4.$

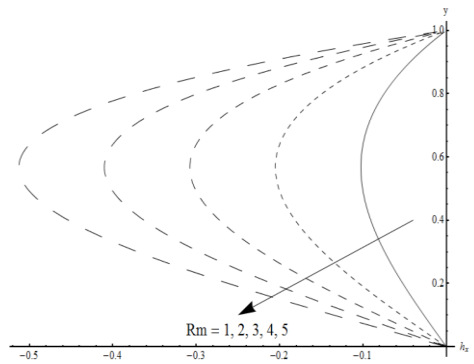

Fig. 4. Variation of axial induced magnetic field h_x as a function of y for different values of magnetic Reynolds number Rm. The other parameters chosen are $\gamma = 0.8, F = -0.5, E = 1,\ x = 0,\ M = 3$ and $\alpha^* = 0.4.$

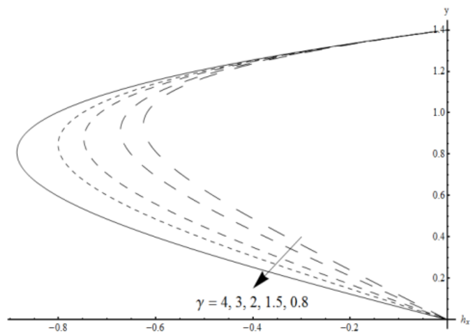

Fig. 5. Variation of axial induced magnetic field h_x as a function of y for different values of couple stress fluid parameter γ. The other parameters chosen are $Rm = 5,\ F = -0.5,\ E = 1,\ x = 1/4, M = 3$ and $\alpha^* = 0.4.$

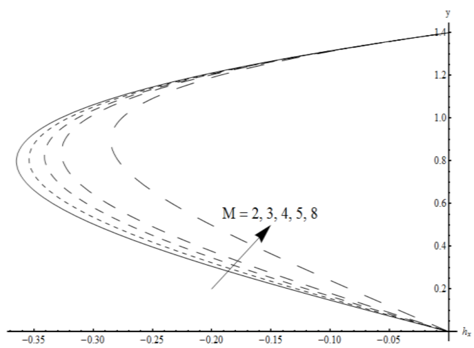

Fig. 6. Variation of axial induced magnetic field h_x as a function of y for different values of Hartman number M. The other parameters chosen are $Rm = 2, F = -0.5, E = 1, x = 1/4,\ \gamma = 0.8$ and $\alpha^* = 0.4.$

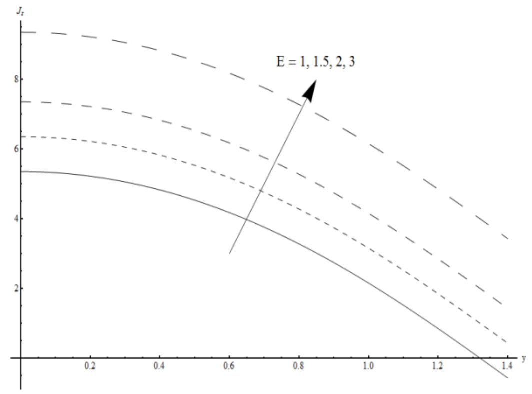

Fig. 7. Variation of current density J_z as a function of y for different values of E. The other parameters chosen are $M=3, F=-0.5,\ Rm=2, x=1/4, \gamma=0.8$ and $\alpha^*=0.4.$

5. Results and Discussion

The main feature of peristaltic motion is the pumping against the pressure rise i.e., when $\Delta p>0$ the positive values of Θ are only due to peristaltic wave. We have prepared Figs. 2 and 3 to see the effects of applied magnetic (M) and electric field (E) respectively on the pressure rise Δp It is observed that Δp increases in the pumping region by increasing M. This increase is perhaps due to the fact that magnetic force acts as a resistance to the flow due to peristalsis and hence causes an increase in Δp. However, an increase in E reduces the pressure rise. This is because of the fact that applied electric field acts perpendicular to the magnetic field and hence reduces the resistance offered to the peristaltic flow by the magnetic field. Further for large values of $E, \Delta p$ become negative for all positive values of Θ. Thus we infer that applied magnetic field enhances the magnitude of Δp but at the same time applied electric field minimizes the effects of applied magnetic field and makes Δp negative. This is a very interesting observation and highlights the importance of imposed electric field which is usually neglected in MHD peristaltic flows. The variation of axial induced magnetic field h_x for different values of Rm, γ and M is shown in Figs. 4-6. We note that magnitude of axial induced magnetic field decreases by increasing γ and M. An increase in both these parameters suppresses the bulk motion of the fluid which results in the reduction of induced currents and hence the induced magnetic field. On the other hand, its magnitude increases for large values of Rm. Since increase in magnetic Reynolds number means low diffusion of magnetic field into the fluid, this results in an increase in the induced current which in turn increases the induced magnetic field. Also, the axial induced magnetic field is independent of E, therefore it is not affected by it. The influence of E on current density J_z can be seen through Figure 7. This figure reveals that J_z increases for large values of $E.$

6. Concluding Remarks

The two-dimensional equations incorporating the effects of the induced magnetic field are presented in this paper. The results for the small magnetic Reynolds number can be retrieved as a special case. The developed equations are used to discuss the peristaltic flow of a couple stress fluid in a planar channel. The exact solution of the problem is provided. These equations are different from those used in the articles (Eldabe et al., (2007); Mekheimer, (2008); Mekheimer, (2008)) for similar kinds of problem.

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Nomenclature

Generally quantities with over bars indicate that they are dimensional.

$\bar{\mathbf{V}}$	Velocity vector
$\bar{t}$	Time
ρ	Density
μ	Viscosity
$\bar{p}$	Pressure
$\bar{\gamma}$	Couple stress parameter
$\bar{\mathbf{J}}$	Total current density
$\bar{\mathbf{J}}_0$	Applied current density
$\bar{\mathbf{J}}_1$	Induced current density
$\bar{\mathbf{B}}$	Total magnetic field
$\bar{\mathbf{B}}_0$	Applied magnetic field
$\bar{\mathbf{B}}_1$	Induced magnetic field
$\bar{\mathbf{E}}_0$	Applied electric field
$\bar{\mathbf{E}}_1$	Induced electric field

σ Electrical conductivity
λ_1 Magnetic diffusivity
μ^* Permeability of free space
$(\bar{X}, \bar{Y})$ Cartesian Coordinates in fixed frame
$(\bar{U}, \bar{V})$ Longitudinal and transverse velocity components in fixed frame
$(\bar{B}_1, \bar{B}_2)$ Longitudinal and transverse components of magnetic field in fixed frame
$\bar{B}_0$ Transverse component of magnetic field in fixed frame
$\bar{E}$ $\bar{Z}$ -component of applied electric field in fixed frame
$\bar{E}_3$ $\bar{Z}$ -component of induced electric field in fixed frame
$\bar{\eta}$ Instantaneous height of peristaltic wall
$\bar{p}_m$ Modified pressure
$(\bar{x}, \bar{y})$ Cartesian Coordinates in wave frame
$(\bar{u}, \bar{v})$ Longitudinal and transverse velocity components in wave frame
δ Wave number
h_x Dimensionless axial magnetic field in wave frame
h_y Dimensionless transverse magnetic field in wave frame
ϕ Magnetic force function
ψ Stream function
R_m Magnetic Reynolds number
M Hartmann number
α^* Amplitude ratio
Θ Mean flow rate in fixed frame
F Mean flow rate in wave frame

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