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A NSFD scheme for Lotka–Volterra food web model

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Abstract

A nonstandard finite difference (NSFD) scheme has been constructed and analyzed for a mathematical model that describes Lotka–Volterra food web model. This new discrete system has the same stability properties as the continuous model and,on the whole, it preservesthe same local asymptotic stability properties. Linearized stability theory and Schur–Cohn criteria are used for local asymptoticstability of this discrete time model. Numerical results are given to support the results.

Keywords*:* Lotka–Volterra; nonstandard finite difference scheme; stability

1. Introduction

The study of biological systems has been developed over many years. In these systems, it is common that the state variablesrepresent nonnegative quantities, such as concentrations, physical properties, the size of populations and the amountof chemical compounds (Murray, 2003). These biological models are commonly based on systems of ordinary differential equations (ODEs). Exact solutions of these systems are rarely accessible and usually complicated; hence good approximations are required.

Numerical methodsare often the method of choice and should describe the dynamic behavior of the systems, produce the nonnegativesolutionsand reproduce the real dynamics of the biological systems. The interspecies interaction is among the most intensively explored fields of biology. The increasingamount of realistic mathematical models in that area helps in understanding the population dynamics ofanalyzed biological systems. Mathematical models of predator–prey systems, characterized by decreasinggrowth rate of one of the interacting populations and increasing growth rate of the other, consist of the ODE systems. In most of the modeled interactions, all rates of changes are assumed to be timeindependent, which makes the corresponding systems autonomous. The positivity of the size of bothinteracting populations requires the mathematical models to preserve the invariance of the first quadrant.

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The differential equations in these mathematicalmodels are usually nonlinear autonomous differential equationsystems which have only time-independent parameters. It is not always possible to find the exact solutions of thenonlinear models that consist of at least two ODEs. It is sometimes more useful to find numericalsolutions to these types of systems in order to easily programand visualize the results. By applying a numerical method on a continuous differentialequation system, it becomes a difference equation system, i.e., a discrete time system. While applying thesenumerical methods, it is necessary that the new differenceequation system provides the positivity conditionsand exhibits the same quantitative behaviors of a continuoussystem such as stability, bifurcation and chaos. It is wellknown that some traditional and explicit schemes such as forwardEuler and Runge–Kutta are unsuccessful at generatingoscillation, bifurcations, chaos and false steady states, despite using adaptive step size (Arenas et al., 2008; Mickens, 2005; Moghadas et al., 2003, 2004; Roeger, 2004, 2008). For forward Euler method, if the step size is chosen small enough and the positivityconditions are satisfied, it is seen that local asymptoticstability for a fixed point is saved while in some specialcases Hopf bifurcation cannot be seen. Instead of classicalmethods, NSFD scheme can alternatively be used to obtain more qualitative results andremove numerical instabilities. These schemes are developedfor compensating the weaknesses such as numerical instabilities that may be caused by standarddifference methods. Also, the dynamic consistency can be represented byNSFD scheme (Liao and Ding, 2012). The most important advantages of this scheme isthat by choosing a convenient denominator function instead ofthe step size, better results can be obtained. If the step size is chosen small enough, the obtained results do not changesignificantly but if gets larger this advantage comes intofocus.

This paper is organized as follows: The next sectionprovides a brief overview of the important features of the procedures for constructing NSFD schemes in ODEs. In section 3, we introduce the model thenceforth discretizated in a nonstandard form that provides the positivity conditions. In section 4, we present a lemmaand then a linearized stability theorem is given for the localasymptotic stability of the discrete time systems. Finally in the last section, some numerical experiments are carried out tostudy the solution to this system. Later on some notes are presented on a Hopf bifurcationthat arises at a certain critical value.

2.Nonstandard Finite Difference Schemes for ODEs

The initial foundation of NSFD schemes come from the exact finitedifference schemes. These schemes were well developed by Mickens (1994, 2003, 2005, 2007) in the pastdecades. These schemes are developed for compensating the weaknesses such as numerical instabilities that may be causedby standard finite difference methods .Regardingthe positivity, boundedness and monotonicity of solutions, NSFD schemes have a better performance over the standard finite difference schemes, due to their flexibility to construct a NSFD scheme that can preservecertain properties and structures, which are obeyed by the original equations. Also, thedynamic consistency could be presented well by NSFD schemes.

The advantages of NSFD schemes have been shown in manynumerical applications. Arenaset al . (2010) and González-Parra et al. (2010) developed NSFD schemesto solve population and biological models. Jordan (2003) and Malek (2011) constructed NSFD schemes for heat transfer problems. For symplectic systems, Mickens (2005) derived a NSFD variational integrator for symplectic ODEs.

We now give an outline of the critical points which will allow the construction of NSFD discretizations for ODEs.

Consider the autonomous ODE given by

$$x' = f(x), \qquad x(t_0) = x_0, \qquad t \in [t_0, t_f],$$

where $f(x)$ is, in general, a nonlinear function of x. For a discrete-time grid with step size, $\Delta t = h$, we replace the independent variable t by

$$t \approx t_n = nh, \qquad n = 0, 1, 2, \ldots, N$$

where $h = \frac{t_f - t_0}{N}$ .The dependent variable $x(t)$ is replaced by

$$x(t) \approx x_n$$

where x_n is the approximation of $x(t_n)$.

The first NSFD requirement is that the dependent functions should be modeled nonlocally on the discrete–time computationalgrid. Particular examples of this include the following functions (Mickens, 2005, 1994).

$$\begin{cases} x^2 \approx x_{n+1} x_n, \\ x^2 \approx \left(\dfrac{x_{n+1} + x_n + x_{n-1}}{3}\right) x_n, \\ x^3 \approx \left(\dfrac{x_{n+1} + x_{n-1}}{2}\right) x_n^2. \end{cases}$$

A standard way for representing a discrete first-derivative is given by

$$x' \cong \frac{x_{n+1} - x_n}{h}.$$

However, the NSFD scheme requires that x' has the more general representation

$$x' \cong \frac{x_{n+1} - x_n}{\phi},$$

where the denominator function, i.e. ϕ has the properties:

$$\begin{aligned} &I.\ \ \phi(h) = h + O(h^2), \\ &II.\ \ \phi(h) \text{ is an increasing function of } h, \\ &III.\ \ \phi(h) \text{ may depend on the parameters} \\ &\qquad \text{appearing in the differential equations}. \end{aligned}$$

The paper of Mickens (2007) gives a general procedure for determining $\phi(h)$ for systems of ODEs.

An example of the NSFD discretization process is its application to the decay equation

$$x' = -\lambda x,$$

where λ is a constant. The discretization scheme is (Mickens, 2007)

$$\frac{x_{n+1}-x_n}{\phi}=-\lambda x_n, \quad \phi(h,\lambda)=\frac{1-e^{-\lambda h}}{\lambda}.$$

Another elementary example is given by

$$x'=\lambda_1 x-\lambda_2 x^2,$$

where the NSFD scheme is as follows (Mickens, 2007)

$$\frac{x_{n+1}-x_n}{\phi}=\lambda_1 x_n-\lambda_2 x_{n+1}x_n,$$

where the denominator function is

$$\phi(h,\lambda_1)=\frac{e^{\lambda_1 h}-1}{\lambda_1}.$$

It should be noted that the NSFD schemes for both ODEs are exact in the sense that $x_n=x(t_n)$ for all applicable valuesof $h>0$. In general, for an ODE with polynomial terms,

$$x'=ax+(NL), \quad NL\equiv nonlinear\ terms,$$

the NSFD discretization for the linear expressions is given by Mickens (2007)

$$\frac{x_{n+1}-x_n}{\phi}=ax_n+(NL)_n,$$

where the denominator function is

$$\phi(h,a)=\frac{e^{ah}-1}{a}.$$

It follows that if x' is a function of x which does not have a linear term, then the denominator function would be just h, i.e. $\phi(h)=h$.

3. Discretization of the Model

In a food web, a species is called basal if it is prey but is not predatory, intermediate if it is both prey and predator, and top if it is only a predator; the compositionof predator and prey relationships in a food web is referred to as its trophic structure and individual levels as trophic levels. We use the word population to meanabundance or biomass of a species. Let $x(t)$, $y(t)$ and $z(t)$ represent the populations of basal, intermediate, and top species respectively in a food web at time t. A sensible model for the trophic structure of a closed food-web population at time t is a generalized Lotka–Volterra system of the form

$$\begin{aligned} x'&=ax-bx^2-cxy-dxz,\\ y'&=-ey+fxy-gyz,\\ z'&=-hz+ixz+jyz,\\ x(0)&=x_0,\quad y(0)=y_0,\quad z(0)=z_0, \end{aligned} \tag{1}$$

where $a, b, ..., j>0$. In this model, the basal species with population x haveintrinsic growth rate a with environmental carrying capacity a/b and the strengthof the effect of predation form. The other two species are measured by interaction–termcoefficients c and d. As the top species with population z preys on both the basaland intermediate species, its interaction terms xz and yz have positive coefficients,since z increases under interaction with each of the other species. The intermediatespecies with population y grows through interaction with the basal species butdeclines through interaction with the top species.

This system is a special case of the well–known Lotka–Volterra cascade model (Chen and Cohen, 2001) given by

$$x_i'(t)=x_i(t)\left[e_i+\sum_{j=1}^{n}p_{ij}x_j(t)\right],\ i=1,2,...,n \tag{2}$$

where $x_i(t)$ is the population of species i, e_i is the intrinsic growth or decline rateof species i and p_{ij} is the interaction coefficient between species i and j. We can consider here the case $n=3$ and then use the NSFD scheme which applies to predict the populationin the case of only one basal species, so that $p_{11}<0$ and $p_{22}=p_{33}=0$ in (2), and with hierarchal predation, meaning that each successive species preys on thosebelow it. This means that in (2) species j preys on species i if and only if $i<j$, so that $p_{ij}<0$ if $i<j$ and $p_{ij}>0$ if $i>j$.

In order to get a better analysis for the system, we reduce the number of parameters using the nondimensionalization method as in (Murray, 2003) as follows. Letting

$$u(T)=\frac{b}{a}x(t),\ v(T)=\frac{c}{a}y(t),\ w(T)=\frac{d}{a}z(t),$$

where $T=at$, consequently we get

$$x'(t)=\frac{a^2}{b}u'(T), \qquad y'(t)=\frac{a^2}{c}v'(T),$$
$$z'(t)=\frac{a^2}{d}u'(T). \tag{3}$$

Substituting (3) into (1) and renaming T to t, gives

$$\begin{aligned} u' &= u(1-u-v-w),\\ v' &= v(-A+Bu-Cw),\\ w' &= w(-D+Eu+Fv),\\ u(0) &= u_0,\ \ v(0)=v_0,\ \ w(0)=w_0 \end{aligned} \tag{4}$$

where

$$A=\frac{e}{a}, \qquad B=\frac{f}{b}, \qquad C=\frac{g}{d},$$
$$D=\frac{h}{a}, \qquad E=\frac{i}{b}, \qquad F=\frac{j}{c},$$

with

$$u_0=\frac{b}{a}x_0, \qquad v_0=\frac{c}{a}y_0, \qquad w_0=\frac{d}{a}z_0.$$

The system of nonlinear differential (4) will be discretizated as follows

$$\begin{aligned} u(T) &\approx u_n,\\ v(T) &\approx v_{n+1},\\ w(T) &\approx w_{n+1},\\ u^2(T) &\approx u_{n+1}u_n,\\ u(T)v(T) &\approx u_{n+1}v_n,\\ u(T)w(T) &\approx u_{n+1}w_n,\\ v(T)w(T) &\approx v_{n+1}w_n. \end{aligned}$$

If u_{n+1}, v_{n+1} and w_{n+1} explicitly solved (4), the following iterations will be obtained:

$$u_{n+1}=\frac{(1+\phi_1(h))u_n}{1+\phi_1(h)(u_n+v_n+w_n)},$$
$$v_{n+1}=\frac{(1+B\phi_2(h,A)u_{n+1})v_n}{1+\phi_2(h,A)(A+Cw_n)},$$
$$w_{n+1}=\frac{(1+E\phi_3(h,D)u_{n+1}+F\phi_3(h,D)v_{n+1})w_n}{1+D\phi_3(h,D)}, \tag{5}$$

where denominator functions are chosen as by

$$\phi_1(h)=e^h-1,$$
$$\phi_2(h,A)=\frac{e^{Ah}-1}{A},$$
$$\phi_3(h,D)=\frac{e^{Dh}-1}{D}.$$

4. Stability Analysis of the Model

Consider the system of ODEs given by

$$\begin{aligned} X' &= F(x,y,z),\\ Y' &= G(x,y,z),\\ Z' &= H(x,y,z), \end{aligned} \tag{6}$$

where F, G and H are nonlinear functions. Let $\bar{X}$, $\bar{Y}$ and $\bar{Z}$ be the steady–state solution, i.e.,

$$F(\bar{X},\bar{Y},\bar{Z})=G(\bar{X},\bar{Y},\bar{Z})=H(\bar{X},\bar{Y},\bar{Z})=0.$$

Now consider small perturbations to steady–state solutions

$$\begin{aligned} X(t) &= \bar{X}+x(t),\\ Y(t) &= \bar{Y}+y(t),\\ Z(t) &= \bar{Z}+z(t). \end{aligned}$$

Frequently these are called perturbations of the steady–state. Substituting, we arrive at

$$\begin{aligned} (\bar{X}+x)' &= F(\bar{X}+x,\bar{Y}+y,\bar{Z}+z),\\ (\bar{Y}+y)' &= G(\bar{X}+x,\bar{Y}+y,\bar{Z}+z),\\ (\bar{Z}+z)' &= H(\bar{X}+x,\bar{Y}+y,\bar{Z}+z). \end{aligned}$$

On the left–hand side we expand the derivatives and that by definition

$$\bar{X}'=\bar{Y}'=\bar{Z}'=0.$$

On the right–hand side we now expand F G and H in aTaylor series about the point $(\bar{X},\bar{Y},\bar{Z})$. The result is

$$x'=F(\bar{X},\bar{Y},\bar{Z})+F_x(\bar{X},\bar{Y},\bar{Z})x+F_y(\bar{X},\bar{Y},\bar{Z})y$$
$+F_z(\bar{X},\bar{Y},\bar{Z})z +$ ms of order $x^2, y^2, z^2, xy,$ yz, xz, and higher,

$y' = G(\bar{X},\bar{Y},\bar{Z}) + G_x(\bar{X},\bar{Y},\bar{Z})x + G_y(\bar{X},\bar{Y},\bar{Z})y$
$+G_z(\bar{X},\bar{Y},\bar{Z})z +$ ms of order x^2, y^2, z^2, xy,
yz, xz, and higher,
$z' = H(\bar{X},\bar{Y},\bar{Z}) + H_x(\bar{X},\bar{Y},\bar{Z})x + H_y(\bar{X},\bar{Y},\bar{Z})y$
$+H_z(\bar{X},\bar{Y},\bar{Z})z +$ ms of order x^2, y^2, z^2, xy,
yz, xz, and higher.

Again by definition,

$$F(\bar{X}, \bar{Y}, \bar{Z}) = G(\bar{X}, \bar{Y}, \bar{Z}) = H(\bar{X}, \bar{Y}, \bar{Z}) = 0,$$

so we are left with

$$x' = a_{11}x + a_{12}y + a_{13}z,$$
$$y' = a_{21}x + a_{22}y + a_{23}z,$$
$$z' = a_{31}x + a_{32}y + a_{33}z,$$

where the matrix of coefficients

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$
$$= \begin{pmatrix} F_x(\bar{X},\bar{Y},\bar{Z}) & F_y(\bar{X},\bar{Y},\bar{Z}) & F_z(\bar{X},\bar{Y},\bar{Z}) \\ G_x(\bar{X},\bar{Y},\bar{Z}) & G_y(\bar{X},\bar{Y},\bar{Z}) & G_z(\bar{X},\bar{Y},\bar{Z}) \\ H_x(\bar{X},\bar{Y},\bar{Z}) & H_y(\bar{X},\bar{Y},\bar{Z}) & H_z(\bar{X},\bar{Y},\bar{Z}) \end{pmatrix},$$

is the Jacobian of the system of equations (6). Hence the problem has been reduced to a linear system of equations, i.e., $w' = Aw$ with $w = (x, y, z)^T$, for states that are in proximity to the steady state $(\bar{X}, \bar{Y}, \bar{Z})$.

A parallel statement exists for linearity concept systems at difference equations (Elaydi, 1999). Consider the autonomous (time-invariant) linear difference equations given by

$$x_{n+1} = Ax_n, \tag{7}$$

where $x_n = (x_{1n}, x_{2n}, ..., x_{kn})^T \in \lambda^k$ and $A = (a_{ij})$ is a $k \times k$ real nonsingular matrix, in which the values of $A$are all constants and

$$P(\lambda) = \det(A - I\lambda)$$

isthe characteristic polynomial of the matrix A.
The following theorem gives necessary and sufficient conditions for asymptotic stability of the linear autonomous system (7).

Theorem 1. The zero solution of (7) is asymptotically stable if and only if $\rho(A) < 1$.

Proof: (Elaydi, 1999).

Consider the k-th order difference equation

$$x_{n+k} + p_1 x_{n+k-1} + p_2 x_{n+k-2} + ... + p_k x_n = 0, \tag{8}$$

where any p_i for $i = 1, 2, ..., k$ is real number and $p_k \neq 0$. For problem (8) the characteristic equation is given by

$$\lambda^k + p_1\lambda^{k-1} + ... + p_k = 0,$$

where

$$P(\lambda) = \lambda^k + p_1\lambda^{k-1} + ... + p_k,$$

is called the characteristic polynomial of the difference equation (8). One of the main tools that provides necessary and sufficient conditions for the zeros of a k-th degree polynomial, such as $P(\lambda)$, to lie inside the unit disk is the Schur–Cohn criterion (Elaydi, 1999). This is useful for studying the stability of zero solution of (8). By analyzing the Schur–Cohn criterion for $k = 3$, the following result can be gained.

Lemma 1. (Jury conditions, Schur–Cohn criteria, $k = 3$). Suppose the characteristic polynomial $P(\lambda)$ is given by $P(\lambda) = \lambda^3 + p_1\lambda^2 + p_2\lambda + p_3$. The solutions λ_i, $i = 1, 2, 3$ *of* $P(\lambda) = 0$ satisfy $|\lambda_i| < 1$ if the following three conditions are held:

$$I.\ P(1) = 1 + p_1 + p_2 + p_3 > 0,$$
$$II.\ (-1)^3 P(-1) = 1 - p_1 + p_2 - p_3 > 0,$$
$$III.\ 1 - (p_3)^2 > |p_2 - p_3 p_1|.$$

Proof: (Elaydi, 1999).

Theorem 2. (The linearized stability theorem). Let $\bar{x}$ be anequilibrium point of the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, ..., x_{n-k}), \quad k = 0, 1, ...$$

where the function F is a continuously differentiable functiondefined on some open neighborhood of an equilibrium point $\bar{x}$. Then the following statements are true.
I. If all the roots of the characteristic polynomial haveabsolute value less than one, then the equilibrium point $\bar{x}$ is locally asymptotically stable.
II. If at least one root of the characteristic polynomial has absolute value greater than one, then the equilibriumpoint $\bar{x}$ is unstable.

Proof: (Elaydi, 1999).
Equilibrium points of (4) are found as follows:

$$X_1^* = (0,0,0),\ X_2^* = (1,0,0),\ X_3^* = (\frac{D}{E}, 0, 1-\frac{D}{E}),$$
$$X_4^* = (\frac{A}{B}, 1-\frac{A}{B}, 0),\ X_5^* = (h_1, h_2, h_3),\ X_6^* = (0, \frac{D}{F}, -\frac{A}{C}),$$

where

$$h_1 = \frac{AF - CD + CF}{BF - CE + CF}, \tag{9}$$

$$h_2 = \frac{-AE + BD + CD - CE}{BF - CE + CF}, \tag{10}$$

$$h_3 = \frac{AE - AF - BD + BF}{BF - CE + CF}. \tag{11}$$

Only fixed points $X_i^*,\ i = 1,2,...,5$ have real biological meaning.Coordinates of all five steady states are nonnegative if

$$A < B, \quad D < E, \quad \frac{A}{B} < \frac{AF - CD + CF}{BF - CE + CF} < \frac{D}{E}.$$

Equations (5) can be written as follows as

$$f = \frac{(1+\phi_1(h))u_n}{1+\phi_1(h)(u_n + v_n + w_n)},$$
$$g = \frac{(1+B\phi_2(h,A)u_{n+1})v_n}{1+\phi_2(h,A)(A + Cw_n)},$$
$$h = \frac{(1+E\phi_3(h,D)u_{n+1} + F\phi_3(h,D)v_{n+1})w_n}{1+D\phi_3(h,D)}.$$

By using these equations, Jacobian matrix will be found as:

$$J(u_n, v_n, w_n) = \begin{pmatrix} f_{u_n} & f_{v_n} & f_{w_n} \\ g_{u_n} & g_{v_n} & g_{w_n} \\ h_{u_n} & h_{v_n} & h_{w_n} \end{pmatrix},$$

where

$$f_{u_n} = \frac{\theta\zeta}{\eta^2}, \qquad f_{v_n} = f_{w_n} = -\frac{\theta\phi_1 u_n}{\eta^2},$$
$$g_{u_n} = \frac{B\theta\zeta\phi_2 v_n}{\eta^2\mu},$$
$$g_{v_n} = \frac{\eta^2 + B\eta\theta\phi_2 u_n - B\theta\phi_1\phi_2 u_n v_n}{\eta^2\mu},$$
$$g_{w_n} = -\frac{(C\eta^2 + B\theta\mu\phi_1 u_n + BC\eta\theta\phi_2 u_n)\phi_2 v_n}{\eta^2\mu^2},$$
$$h_{u_n} = \frac{(E\eta\mu - E\mu\phi_1 u_n + BF\eta\phi_2 v_n - BF\phi_1\phi_2 u_n v_n)\theta\phi_3 w_n}{\eta^2\mu(1+D\phi_3)},$$
$$h_{v_n} = \frac{(F\eta^2 - E\theta\mu\phi_1 u_n + BF\eta\theta\phi_2 u_n - BF\theta\phi_1\phi_2 u_n v_n)\phi_3 w_n}{\eta^2\mu(1+D\phi_3)},$$
$$h_{w_n} = -\frac{\alpha - \beta}{\eta^2\mu^2(1+D\phi_3)},$$

with

$$\eta = 1 + (u_n + v_n + w_n)\phi_1,$$
$$\mu = 1 + (A + Cw_n)\phi_2,$$
$$\zeta = 1 + (v_n + w_n)\phi_1,$$
$$\theta = 1 + \phi_1,$$
$$\alpha = (E\mu^2\theta\phi_1 u_n + CF\eta^2\phi_2 v_n)\phi_3 w_n + BF(\mu\phi_1 + C\eta\phi_2)\theta\phi_2\phi_3 u_n v_n w_n,$$
$$\beta = \eta^2\mu^2 + (E\mu\theta u_n + F\eta v_n + BF\theta\phi_2 u_n v_n)\eta\mu\phi_3.$$

We determine stability of each steady state $X_i^*,\ i = 1, 2, ..., 5$ by considering, where possible, the eigenvalues $\lambda_1^{(i)}, \lambda_2^{(i)}$ and $\lambda_3^{(i)}$ for each matrix $J(X_i^*)$.

$$I.\ X_1^* = (0,0,0):$$

$$J(0,0,0) = \begin{pmatrix} \frac{\theta\zeta}{\eta^2} & 0 & 0 \\ 0 & \frac{1}{\mu} & 0 \\ 0 & 0 & \frac{1}{1+D\phi_3} \end{pmatrix},$$

has eigenvalues

$$\lambda_1^{(1)} = e^h, \quad \lambda_2^{(1)} = \frac{1}{e^{Ah}}, \quad \lambda_3^{(1)} = \frac{1}{e^{Dh}}.$$

Now by theorem 2, we conclude that X_1^* is an unstable point.

$$II.\ X_2^* = (1,0,0):$$

$$J(1,0,0)=\begin{pmatrix} \frac{\theta\zeta}{\eta^2} & -\frac{\theta\phi_1}{\eta^2} & -\frac{\theta\phi_1}{\eta^2} \\ 0 & \frac{\eta+B\theta\phi_2}{\eta\mu} & 0 \\ 0 & 0 & \frac{\eta+E\theta\phi_3}{\eta(1+D\phi_3)} \end{pmatrix},$$

has eigenvalues

$$\lambda_1^{(2)}=\frac{1}{e^h},$$
$$\lambda_2^{(2)}=\frac{A-B(1-e^{Ah})}{Ae^{Ah}},$$
$$\lambda_3^{(2)}=\frac{D-E(1-e^{Dh})}{De^{Dh}},$$

so by theorem 2, X_2^* is stable if $A>B$ and $E<D$ and unstable if $A<B$ or $E>D$.

$III.\ X_3^*=(\frac{D}{E},0,1-\frac{D}{E})$:

$$J(\frac{D}{E},0,1-\frac{D}{E})$$
$$=\begin{pmatrix} \frac{\theta\zeta}{\eta^2} & -\frac{D\theta\phi_1}{E\eta^2} & -\frac{D\theta\phi_1}{E\eta^2} \\ 0 & \frac{E\eta+BD\theta\phi_2}{E\eta\mu} & 0 \\ \frac{(E-D)(E\eta-D\phi_1)\theta\phi_3}{E\eta^2(1+D\phi_3)} & \frac{(E-D)\chi\phi_3}{E^2\eta^2\mu(1+D\phi_3)} & \frac{E\eta^2+\rho}{E\eta^2(1+D\phi_3)} \end{pmatrix},$$

where

$$\chi=FE\eta^2-ED\mu\theta\phi_1+FBD\eta\theta\phi_2,$$
$$\rho=ED\eta\theta\phi_3+D(D-E\theta)\theta\phi_1\phi_3,$$

has eigenvalues

$$\lambda_1^{(3)}=\frac{AE-BD(1-e^{Ah})}{CD-CE+(AE+CE-CD)e^{Ah}},$$
$$\lambda_2^{(3)}=\frac{e^{-(1+D)h}}{2E}\Big((D-E)(1-e^h)+Ee^{Dh}(1+e^h)+\sqrt{(1-e^h)\tau}\Big),$$
$$\lambda_3^{(3)}=\frac{e^{-(1+D)h}}{2E}\Big((D-E)(1-e^h)+Ee^{Dh}(1+e^h)-\sqrt{(1-e^h)\tau}\Big),$$

with

$$\tau=(E-D)^2-(E+D)^2e^h+(2DE-2E^2)e^{Dh}(1+e^h)+E^2e^{2Dh}+(3E^2-4ED)e^{(1+2D)h}.$$

So by theorem 2, X_3^* is stable if $AE+CE>BD+CD$ and $E>D$. It is unstable if $AE+CE<BD+CD$.

$IV.\ X_4^*=(\frac{A}{B},1-\frac{A}{B},0)$:

$$J(\frac{A}{B},1-\frac{A}{B},0)$$
$$=\begin{pmatrix} \frac{\theta\zeta}{\eta^2} & -\frac{A\theta\phi_1}{B\eta^2} & -\frac{A\theta\phi_1}{B\eta^2} \\ -\frac{(A-B)\theta\zeta\phi_2}{\eta^2\mu} & \frac{B\eta^2+A(B\eta+(A-B)\phi_1)\theta\phi_2}{\eta^2\mu} & \frac{(A-B)\nu\phi_2}{B\eta^2\mu^2} \\ 0 & 0 & -\frac{-B\eta\mu+\kappa}{B\eta\mu(1+D\phi_3)} \end{pmatrix},$$

where

$$\nu=C\eta^2+A\theta(\mu\phi_1+C\eta\phi_2),$$
$$\kappa=(AF\eta-AE\mu\theta-BF\eta)\phi_3+(A-B)AF\theta\phi_2\phi_3,$$

has eigenvalues

$$\lambda_1^{(4)}=\frac{BD-(AE+BF-AF)(1-e^{Dh})}{BDe^{Dh}},$$
$$\lambda_2^{(4)}=\frac{e^{-(1+A)h}}{-2B}\Big((B-A)(1-e^h)-B(1+e^h)e^{Ah}+\sqrt{(e^h-1)\omega}\Big),$$
$$\lambda_3^{(4)}=\frac{e^{-(1+A)h}}{-2B}\Big((B-A)(1-e^h)-B(1+e^h)e^{Ah}-\sqrt{(e^h-1)\omega}\Big),$$

with

$$\omega=(A-B)^2(e^h-1)+2B(B-A)e^{Ah}-B^2e^{2Ah}+2B(B-A)e^{(1+A)h}+B(4A-3B)e^{(1+2A)h}.$$

So by theorem 2, X_4^* is stable if $AF+DB>AE+BF$ and $B>A$. It is unstable if $AF+DB<AE+BF$.

$V.\ X_5^*=(h_1,h_2,h_3)$ where h_1, h_2 and h_3 are defined in (9)-(11): $J(X_5^*)$ is extremely complicated, making a general investigation of the stability of X_5^* infeasible.

5. Numerical Results and Conclusion

For the numerical scheme (5), wedivide the experiments into two parts: In Figs. 1-4, we allow at least one zeroinitial condition, while in Figs. 5-9, we consider the case where all initial data arepositive.

Figures 1-3.We choose

$$A=1,\ B=4,\ C=1,\ D=1,\ E=2,\ F=1$$

and $h=0.01$. By thepreceding discussion of steady states, the equilibrium solutions for (4) are given by

$$(0,0,0),\ (1,0,0),\ (\frac{1}{4},\frac{3}{4},0),$$
$$(\frac{1}{2},0,\frac{1}{2}),(\frac{1}{3},\frac{1}{3},\frac{1}{3}). \qquad (12)$$

Although the only stable steady state of these is the last one, all nonzero initialconditions would be necessary for an orbit to approach it, sincethe coordinate planes are invariant under the flow of (4). Wedescribe each figure in further detail as follows.

Figure 1: We choose initial data $u(0)=1/2$, $v(0)=0$ and $w(0)=0$. Thefigure shows that $v_n=0$ and $w_n=0$ for all $n=0,1,2,\ldots$ while $u_n\to 1$ as $n\to\infty$. The second equilibrium in (12) is approached, which means thatthe population of the basal species approaches carrying capacity in absenceof the intermediate and top species as expected.

Figure 2: Initial conditions are $u(0)=0$, $v(0)=2$ and $w(0)=2$. Thefigure shows that as $v_n\to 0$, so does $w_n\to 0$ as $n\to\infty$. The top specieswill remain as long as there is an intermediate species to prey on, while theintermediate species dies off exponentially in absence of a basal species. Thesteady state (0, 0, 0) in (12) is approached with the given initial data.

Figure 3 :Initial conditions in this figure are $u(0)=2$, $v(0)=0$ and $w(0)=2$. Wesee that $u_n\to 1/2$ while $w_n\to 1/2$ as $n\to\infty$. Populations of species oscillate as food web populations progress towardequilibrium .The fourth steady state in (9) is approached

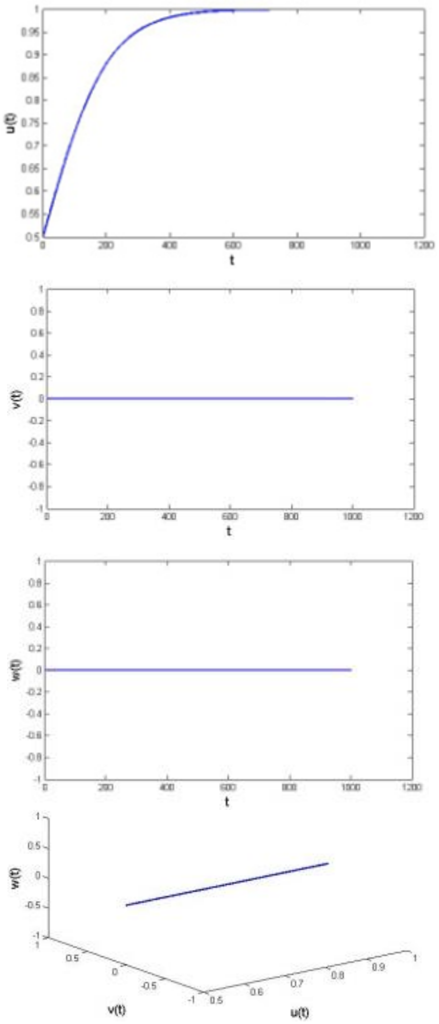
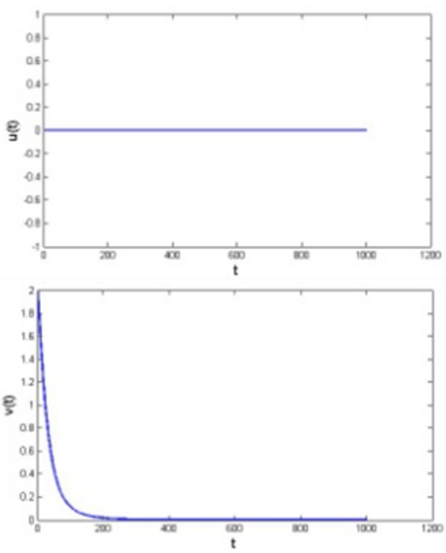


Fig. 1. Solutions for $u(0)=0.5, v(0)=0$ and $w(0)=0$ with $h=0.01$



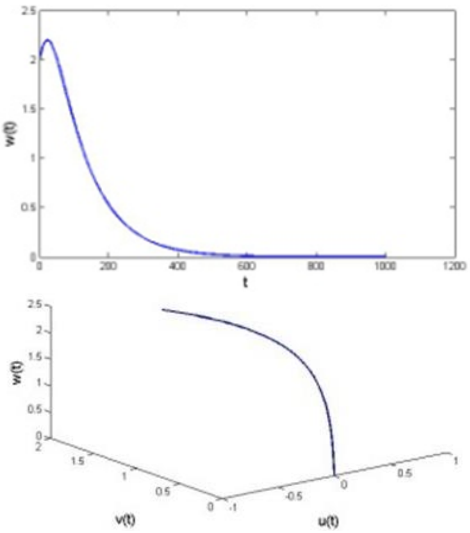
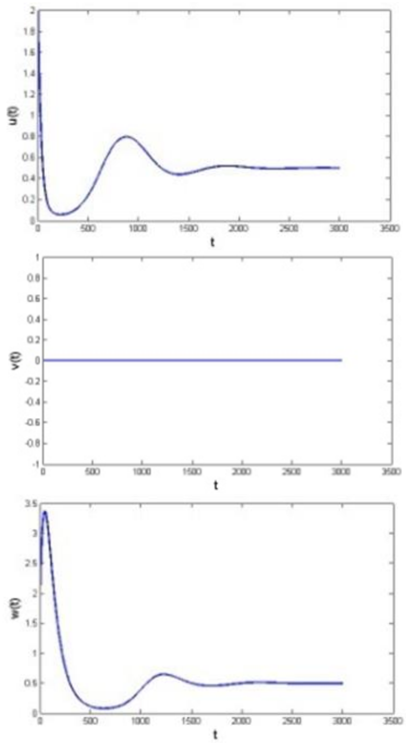


Fig. 2. Solutions for $u(0)=0, v(0)=2$ and $w(0)=2$ with $\mathrm{h}=0.01$

Figure 4: Here we choose

$$A=1, B=4, C=1, D=2, E=1, F=1$$

and $h=0.01$. This gives rise to (0, 0, 0), (1, 0, 0) and $(1/4,\ 3/4,\ 0)$ as equilibria of (4). We choose initial conditions $u(0)=2,\ v(0)=0$ and $w(0)=2$. By invarianceof the coordinate planes, as $n\longrightarrow\infty$ the steady state (1, 0, 0) is approached by (u_n, v_n, w_n).



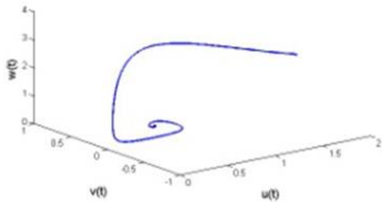


Fig. 3. Solutions for $u(0)=2, v(0)=0$ and $w(0)=2$ with $\mathrm{h}=0.01$

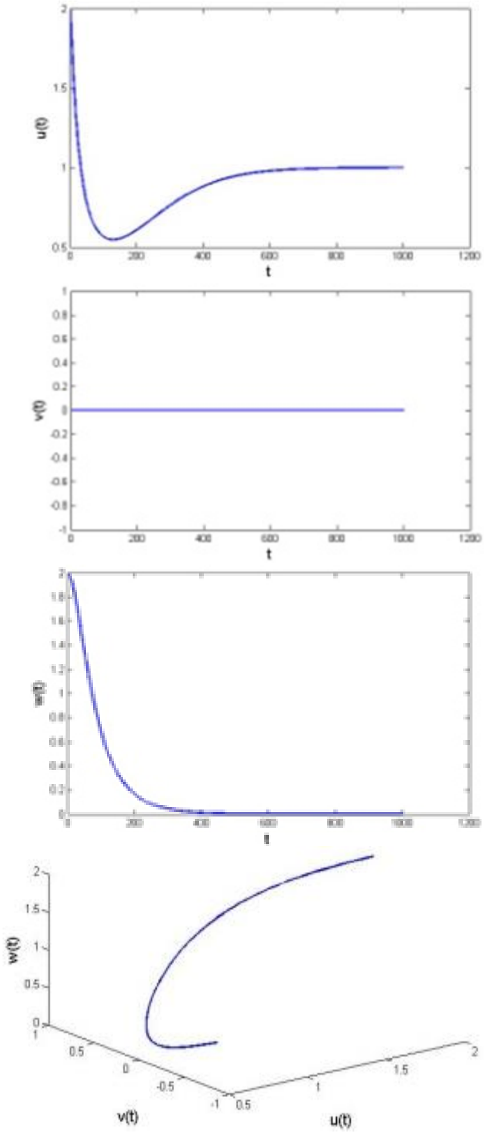


Fig. 4. Solutions for $u(0)=2, v(0)=0$ and $w(0)=2$ with $\mathrm{h}=0.01$

In contrast to Fig. 3, Fig. 4 shows that if the death rate ofthe top species is too large, this species will face extinction, at least compared tothe system with relatively large interaction rate with the basal species comparedto death rate for the top species.

Figure 5: We choose

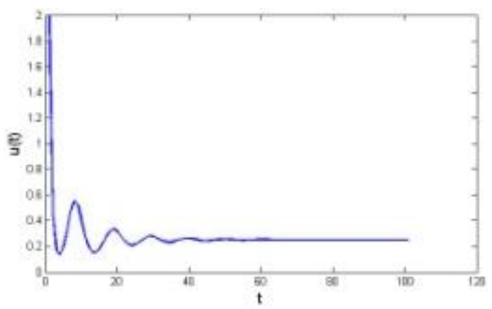
$$A=1, B=4, C=1, D=2, E=1, F=1$$

and $h=1.1$, giving riseto equilibria for (4) of (0, 0, 0), (1, 0, 0) and $(1/4, 3/4, 0)$ as those with allcoordinates nonnegative. We choose initial data $u(0)=2, v(0)=2$ and $w(0)=2$. The figure suggests that since the death rate D of the top species is relativelylarger than the interaction rate E between the basal and top species, and if thedeath rate A of the intermediate species is relatively smaller than the interactionrate B between the basal and intermediate species, then the population of thebasal species will approach $A/B=1/4$ while the intermediate species approaches $1-A/B$. The top species becomes extinct.

Figure 6: This figure has

$$A=2, B=1, C=1, D=1, E=2, F=1$$

and $h=1.1$ so equilibria of (4) are (0, 0, 0), (1, 0, 0), and $(1/2, 0, 1/2)$. With initial conditions chosen as $u(0)=2, v(0)=2$ and $w(0)=2, (u_n, v_n, w_n) \to (1/2, 0, 1/2)$. In general, if the death rate D of the top species is smaller than the interactionrate E between the basal and top species and the death rate A of the intermediatespecies is larger than the interaction rate B between the basal and intermediatespecies, then the population of the basal species will approach D/E, the population of the top species will approach $1-D/E$ and the intermediate speciesdeclines to extinction.



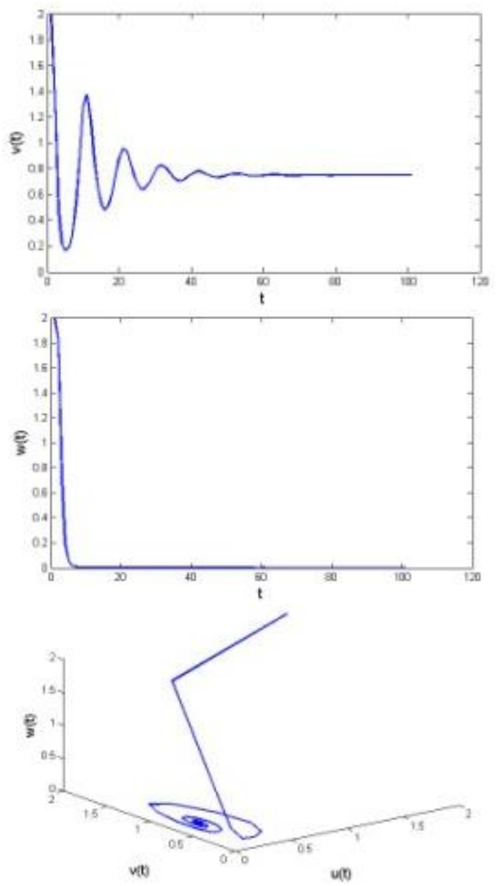


Fig. 5. Solutions for $u(0)=2, v(0)=2$ and $w(0)=2$ with $h=1.1$

Figure 7: Here

$$A=2, B=1, C=1, D=2, E=1, F=1$$

and $h=1.5$, so that the onlypossible equilibria of (4) are (0, 0, 0) and (1, 0, 0). With initial conditionsof $u(0)=2, v(0)=2$ and $w(0)=2$ and given that only (1, 0, 0) is stable, thisis the one approached over time. The figure confirms that if the death rate A of the intermediate species is larger than the interaction rate B between thebasal and top species, and the death rate D of the top species is larger than theinteraction rate E between the basal and top species, then the population of thebasal species will approach the carrying capacity while the intermediate and topspecies become extinct.

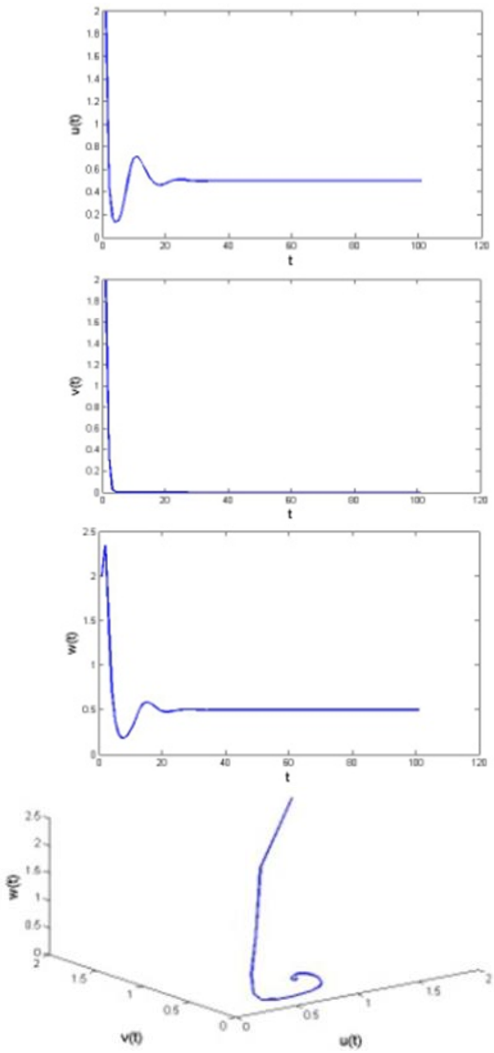
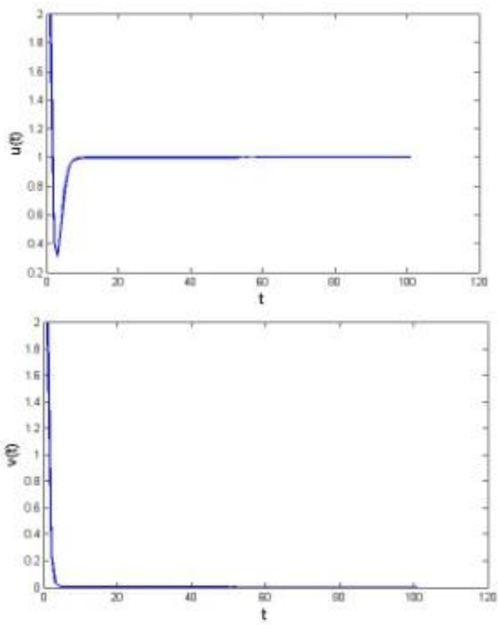


Fig. 6. Solutions for $u(0)=2, v(0)=2$ and $w(0)=2$ with $h=1.1$



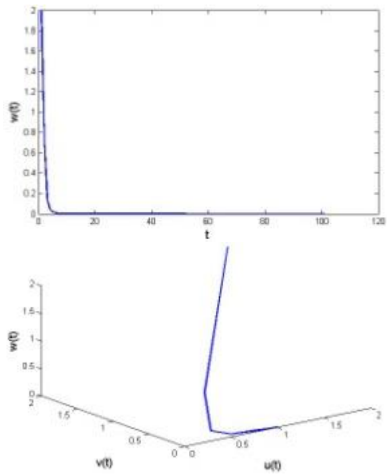


Fig. 7. Solutions for $u(0)=2, v(0)=2$ and $w(0)=2$ with $h=1.5$

In Tables 1 and 2, for different step sizes h, the qualitative stability results, obtained by NSFD scheme, of the fixed point X_2^* and X_3^* are respectively compared to classical methods such as forward Euler and Runge–Kutta. If step size h is chosen small enough, the results of the proposed NSFD scheme are similar with the results of the other two numerical methods. But if the step size h is chosen larger, the efficiency of NSFD scheme is clearly seen.

Table 1. Qualitative results of the fixed point X_2^* for different time step sizes, t = 0–100

h	Euler	Runge-Kutta	NSFD
0.001	Convergence	Convergence	Convergence
0.01	Convergence	Convergence	Convergence
0.1	Convergence	Convergence	Convergence
0.2	Divergence	Convergence	Convergence
0.5	Divergence	Convergence	Convergence
1	Divergence	Divergence	Convergence
10	Divergence	Divergence	Convergence

Table 2. Qualitative results of the fixed point X_3^* for different time step sizes, t = 0–500

h	Euler	Runge-Kutta	NSFD
0.001	Convergence	Convergence	Convergence
0.01	Convergence	Convergence	Convergence
0.1	Convergence	Convergence	Convergence
0.4	Divergence	Convergence	Convergence
0.5	Divergence	Convergence	Convergence
1	Divergence	Divergence	Convergence
10	Divergence	Divergence	Convergence

In Figs. 8 and 9, the NSFD solutions of u, v and w converge to fixed points X_2^* and X_3^* as simulated and also Runge–Kutta, forward Euler and proposed NSFD scheme are compared graphically. All the numerical calculations and simulations are performed by using Matlab programme. In conclusion, the efficiency of the proposed NSFD scheme is investigated and compared with other numerical methods.

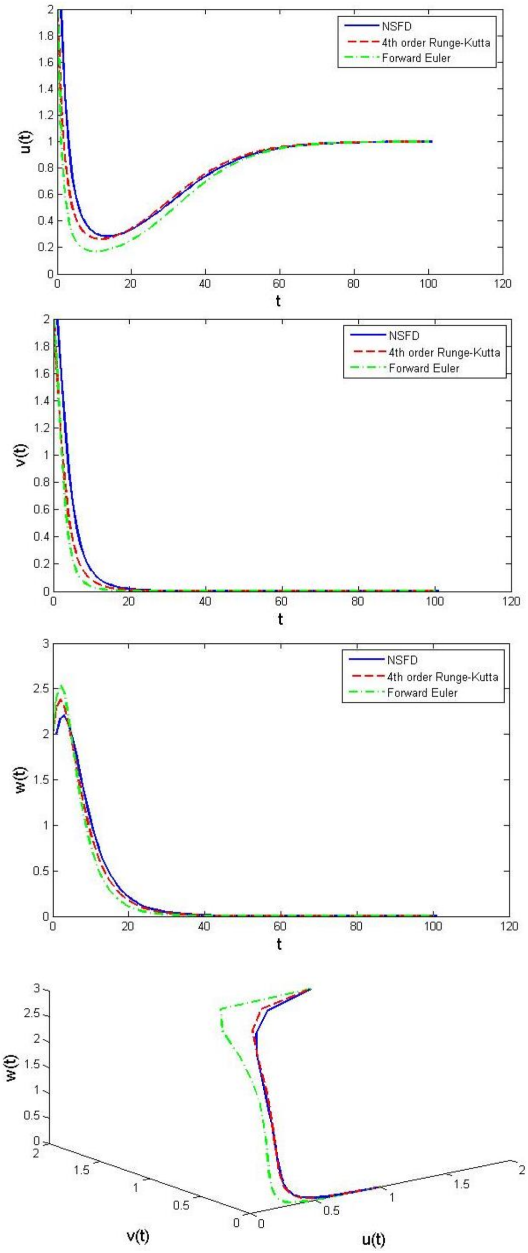


Fig. 8. Comparison with NSFD scheme, 4th order Runge–Kutta and forward Euler solutions of u, v and w converges to fixed point X_2^* with $h = 0.1$

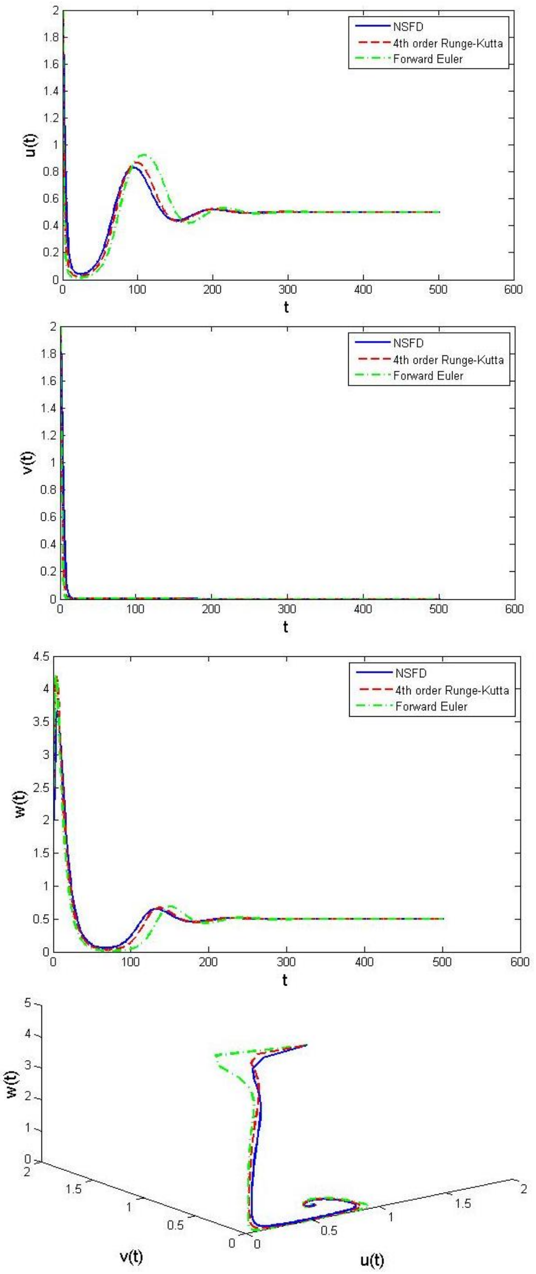


Fig. 9. Comparison with NSFD scheme, 4th order Runge–Kutta and forward Euler solutions of u, v and w converges to fixed point X_3^* with $h = 0.1$

In Tables 3 and 4, for different step sizes h, the qualitative stability results of the fixed point X_4^* and X_5^* obtained by NSFD scheme are respectively compared to classical methods such as forward Euler and Runge–Kutta. If step size h is chosen small enough, the results of the proposed NSFD scheme are similar with the results of the other two numerical methods. But if the step size h is chosen larger, the efficiency of NSFD scheme is clearly seen.

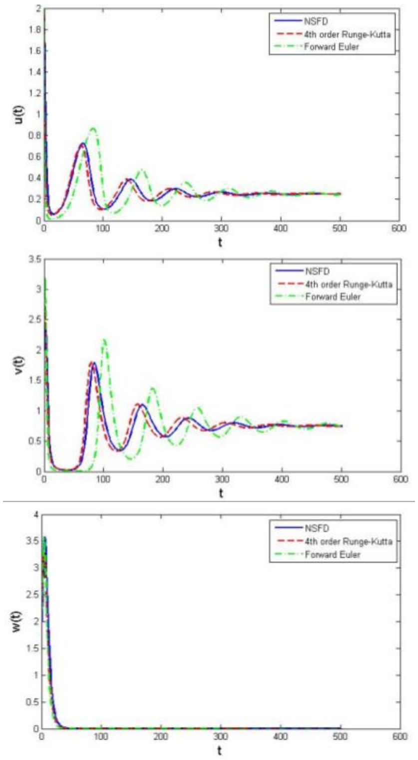
Table 3. Qualitative results of the fixed point X_4^* for different time step sizes, t = 0–500

h	Euler	Runge-Kutta	NSFD
0.001	Convergence	Convergence	Convergence
0.01	Convergence	Convergence	Convergence
0.1	Convergence	Convergence	Convergence
0.4	Divergence	Convergence	Convergence
0.5	Divergence	Divergence	Convergence
1	Divergence	Divergence	Convergence
10	Divergence	Divergence	Convergence

Table 4. Qualitative results of the fixed point X_5^* for different time step sizes, t = 0–5000

h	Euler	Runge-Kutta	NSFD
0.001	Convergence	Convergence	Convergence
0.01	Convergence	Convergence	Convergence
0.1	Divergence	Convergence	Convergence
0.4	Divergence	Divergence	Convergence
0.5	Divergence	Divergence	Convergence
1	Divergence	Divergence	Convergence
10	Divergence	Divergence	Convergence

In Figs 10 and 11, the NSFD solutions of u, v and w converge to fixed points X_4^* and X_5^* as simulated and also Runge–Kutta, forward Euler and proposed NSFD scheme are compared graphically.



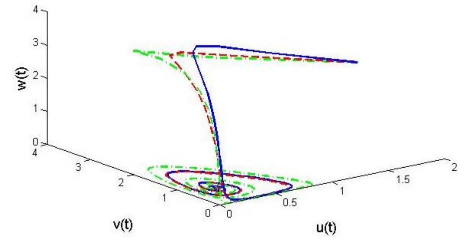


Fig. 10. Comparison with NSFD scheme and 4th order Runge–Kutta and forward Euler solutions of u, v and w converges to fixed point X_4^* with $h = 0.1$

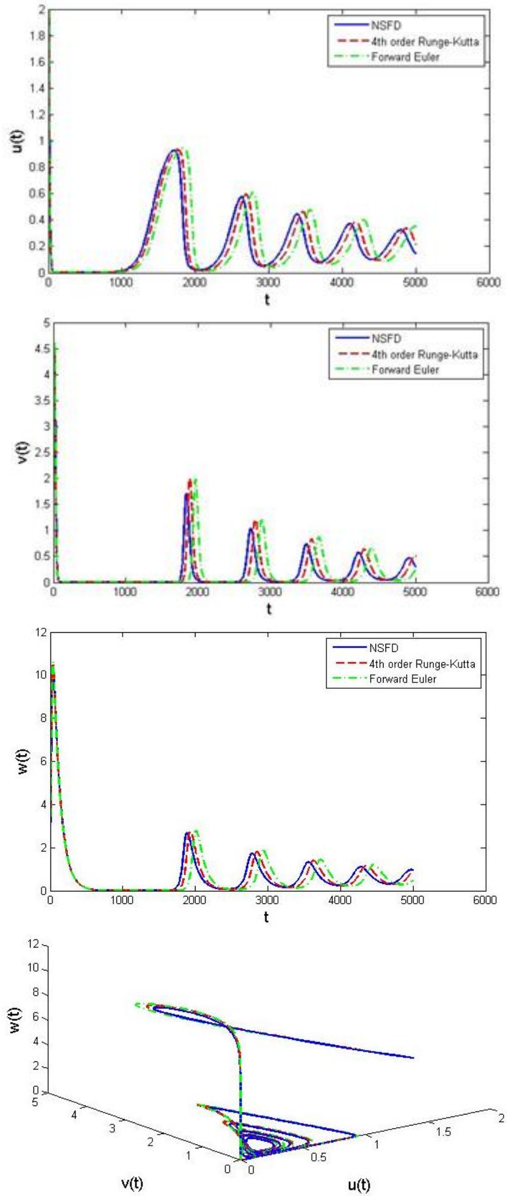


Fig. 11. Comparison with NSFD scheme and 4th order Runge–Kutta and forward Euler solutions of u, v and w converges to fixed point X_5^* with $h = 0.01$

5.1. A Hopf Bifurcation

We claim that for an appropriate choice of constants A and $C-F$, a value of B exists across which a periodic orbit arises through achange in the stability properties; specifically, there is a Hopf bifurcation arising atthis B-value. To demonstrate this, we fix $A=1, C=1, D=1, E=4, F=1$ and considersolutions of (4) as the value of B is varied. The steady state of interest inthis case is $X_B^* = (\frac{1}{B-3}, \frac{B-7}{B-3}, \frac{3}{B-3})$ from which we extract the initial requirementthat $B \geq 7$ so that this is a first-octant steady-state equilibrium. The matrix $J(X_B^*)$ of the linearization about X_B^* has eigenvalues that are roots of the characteristic polynomial (Armstrong and Han, 2012)

$$P_B(\lambda) = \lambda^3 + \frac{1}{B-3}\lambda^2 + \frac{B^2-4B-9}{(B-3)^2}\lambda + \frac{3B-21}{(B-3)^2}.$$

The Schur–Cohn stability criterion (lemma 1) ensures that the roots of P_B lie in the negativecomplex half-plane as long as each coefficient is positive and the product of thecoefficients of λ and λ^2 exceeds the product of the coefficient of λ^3 and the constantterm. Solving these simple inequalities shows that $2+\sqrt{13} < B < 9$ which, togetherwith the initial requirement that $B \geq 7$ means that P_B has three roots two complexconjugates and one real with negative real part as long as $7 \leq B < 9$.
Figure 12: Choosing

$$A=1, B=8, C=1, D=1, E=4, F=1$$

and $h=2.1$ givesrise to (0, 0, 0), (1, 0, 0), $(1/8, 7/8, 0)$, $(1/4, 0, 3/4)$ and $(1/5, 1/5, 3/5)$ as steadystates of (4). With initial conditions $u(0)=2$, $v(0)=2$ and $w(0)=2$ and given that the only stable steady state is $(1/5, 1/5, 3/5)$, the figure confirms that (u_n, v_n, w_n) approaches the steady state solution $X_5^* = (h_1, h_2, h_3) = (1/5, 1/5, 3/5)$ as $n \to \infty$.
Figure 13: With

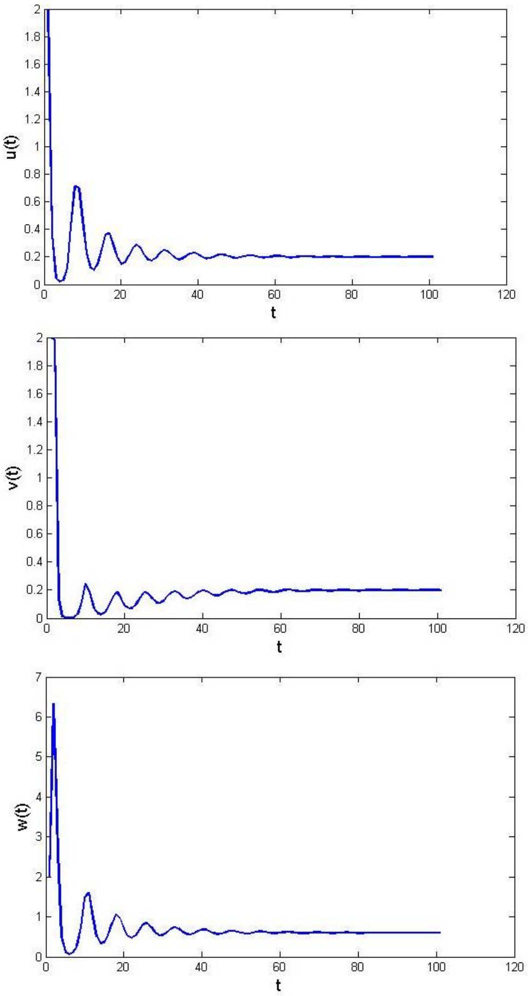
$$A=1, B=11, C=1, D=1, E=4, F=1$$

and $h=2.5$ all equilibria are unstable. Choosing $u(0)=2$, $v(0)=2$ and $w(0)=2$, the figure confirms that (u_n, v_n, w_n) will not approach (for example) the steady state solution $(h_1, h_2, h_3) = (1/7, 3/7, 3/7)$ as $n \to \infty$ although $A/B < D/E$ but rathera periodic solution. This suggests a limit cycle.

Remark 1. The foregoing computations show that the system (4) undergoes a Hopf bifurcation for

$$A=1, C=1, D=1, E=4, F=1$$

across $B=9$. For $7 \leq B < 9$ the system has a stable equilibrium point X_B^* as described above where as X_9^* is a stable center. For $B > 9$ solutions of the system approach a limit cycle asdemonstrated in Fig. 13.



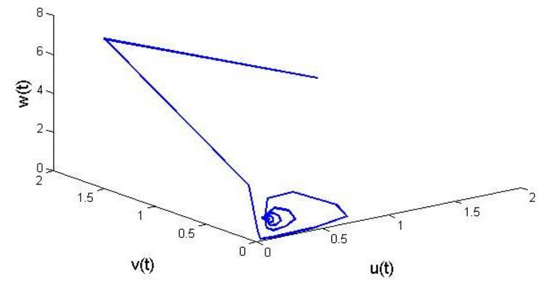


Fig. 12. Solutions for $u(0)=2, v(0)=2$ and $w(0)=2$ with $\mathrm{h}=2.1$

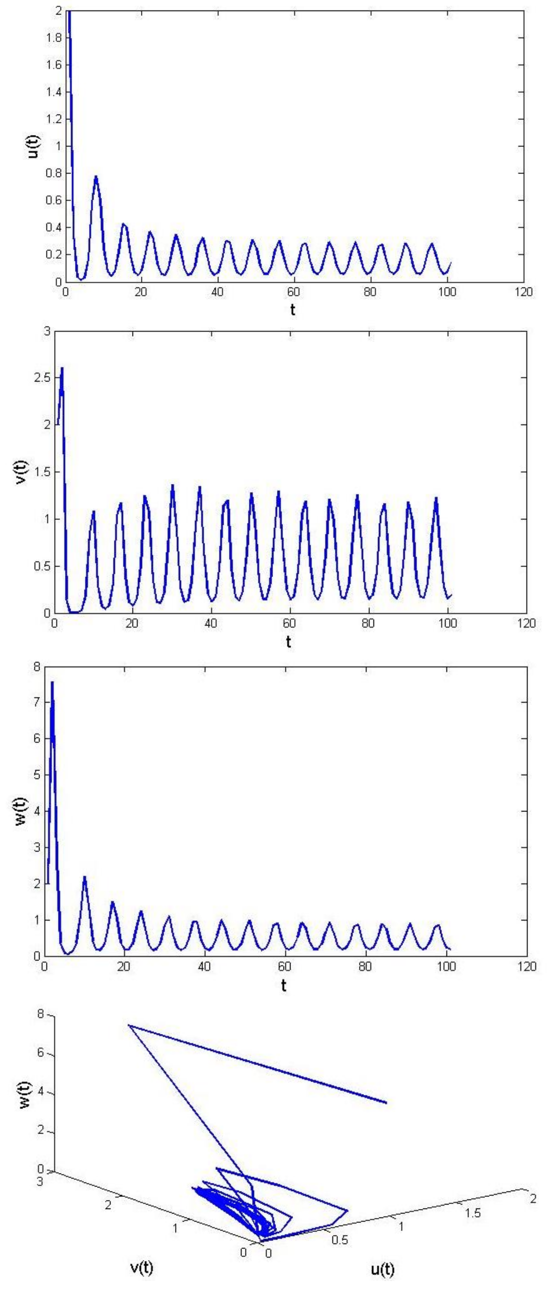


Fig. 13. Solutions for $u(0)=2, v(0)=2$ and $w(0)=2$ with $\mathrm{h}=2.5$

References

Arenas, A. J., Gonzalez–Parra, G., & Chen–Charpentier, B. M. (2010). A nonstandard numerical scheme ofpredictor–corrector type for epidemic models. *Computers Mathematics with Applications,* 59(12), 3740–3749.

Arenas, A. J., Morano, J. A., & Cortes, J. C. (2008). Nonstandard numerical method for a mathematical model of RSV epidemiological transmission.*Computers & Mathematics with Applications,* 56(3),670–678.

Armstrong, S., & Han, J. (2012). A method for numerical analysis of a Lotka–Volterra food web model.*journal of numerical analysis and modeling, series B,* 3(4), 442–459.

Chen, X., & Cohen, J. E. (2001). Transient dynamics and food–web complexity in the Lotka–Volterra cascade model. *Proc. Bio. Sci.,* 268(1469), 869–877.

Elaydi, S. N. (1999).*An Introduction to Difference Equations*.Springer, New York.

Gonzalez–Parra, G., Arenas, A. J., & Chen–Charpentier, B. M. (2010). Combination of nonstandardschemes and Richardsons extrapolation to improve the numerical solution of population models.*Mathematical and Computer Modelling,* 52(7–8), 1030–1036.

Jordan, P. M. (2003). A nonstandard finite difference scheme for nonlinear heat transfer in a thin finite rod.*Journal of Difference Equations and Applications,* 9(11), 1015–1021.

Liao, C., & Ding, X. (2012). Nonstandard finite difference variational integrators for multisymplectic PDEs. *Journal of Applied Mathematics, 2012,* Article ID 705–179, 22 pages.

Malek, A. (2011). Applications of nonstandard finite difference methods to nonlinear heat transfer problems.*In heat transfer–mathematical modeling, Numerical Methods and Information Technology,* doi:10.5772/14439.

Mickens, R. E. (2003). A nonstandard finite difference scheme for the diffusionless Burgers equation withlogistic reaction. *Mathematics and Computers in Simulation,* 62(1–2), 117–124.

Mickens, R. E. (2005). *Advances in the Applications of Nonstandard Finite Difference Schemes*. Wiley-Interscience,Singapore.

Mickens, R. E. (2005). A numerical integration technique for conservative oscillators combining nonstandard finite difference methods with a Hamilton's principle. *Journal of Sound and Vibration,* 285(1–2), 477–482.

Mickens, R. E. (2007). Calculation of denominator functions fornonstandard finite difference schemes for differential equationssatisfying a positivity condition. *Numerical Methods for Partial Differential Equations,* 23(3), 672–691.

Mickens, R. E. (2005). Dynamic consistency: a fundamental principle for constructing nonstandard finite difference schemes for differential equations. *Journal of Difference Equations and Applications,* 11(7), 645–653.

Mickens, R. E. (2002). Nonstandard difference schemes for differential equations. *Journal of Difference Equations and Applications,* 8(9), 823–847.

Mickens, R. E. (1994). *Nonstandard finite difference models of differential equations*. World Scientific, Singapore.

Moghadas, S. M., Alexander, M. E., Corbett, B. D., & Gumel, A. B.(2003). A positivity–preserving Mickens–type discretization of an epidemic model. *Journal of Difference Equations and Applications,* 9(11), 1037–1051.

Moghadas, S. M., Alexander, M. E., & Corbett, B. D. (2004). A nonstandard numerical scheme for a generalized Gause–typepredator–prey model. *D,* 188(1–2),134–151.

Murray, J. D. (2003). *Mathematical Biology, II,* Third edition, Springer.

Roeger, L. W. (2008). Dynamically consistent discrete Lotka–Volterra competition models derived from nonstandard finite differenceschemes, *Discrete and Continuous Dynamical SystemsB,* 9(2), 415–429.

Roeger, L. W. (2004). Local stability of Eulers and Kahans methods.*Journal of Difference Equations and Applications,* 10(6), 601–614.