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PUSHOUT CROSSED MODULES OF ALGEBROIDS*

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Abstract – In this paper, we presented algebroids and crossed modules of algebroids. We also presented pullback crossed module of algebroids and we then define pushout crossed modules of algebroids.

Keywords – Crossed module, algebroids, pullback, action, pushout

1. INTRODUCTION

The term crossed module was introduced by J. H. C. Whitehead in his work on combinatorial homotopy theory [1]. So many mathematicians and many areas of mathematics have used crossed modules such as homotopy theory, homology and cohomology of groups, Algebra, K-theory etc. The category of crossed module was shown by XMod in this paper. Brown and Mosa replaced algebras by algebroids and defined crossed module of algebroids in [2]. Actor crossed module of algebroid was defined by Alp in [3]. Pullback crossed module was defined by Brown and Wensley in [4]. Pullback crossed module of algebroids was defined by Alp in [5]. Pushout crossed module of profinite groups was presented by Korkes and Porter in [6]. Pushout cat1-profinite groups, Pushout cat1-Lie algebra and Pushout cat1-commutative algebra were presented by Alp and Gürmen in [7-9]. Using this idea we defined pushout crossed module of algebroids in this paper.

In section 1 basic facts on algebroids were recalled as in [2]. In section 2, we presented the algebroid and its examples. In section 3, we presented the action of algebroids and the definition of crossed module of algebroids due to Brown and Mosa in [2]. In section 4, we presented pullback crossed module of algebroids using the definition of pullback crossed module due to Brown and Wensley in [4]. In section 5, we defined pushout crossed module of algebroids due to Korkes and Porter in [6].

2. ALGEBROIDS

Let *R* be a commutative ring. An *R*-category A is a category equipped with an R-module structure, each homomorphism set such that the composition is *R*-bilinear. An *R*-algebroid A is a small *R*-category. More precisely, an *R*-algebroid A on a set of objects A_0 is a directed graph over A_0 such that for each $x, y \in A_0$, $A(x,y)$ has an *R*-module structure and there is an *R*-bilinear function $\circ : A_0(x,y) \times A(y,z) \to A(x,z),\ (a,b) \mapsto a \circ b$ called composition and satisfying the associativity condition, and the existence of identities. A pre-*R*-algebroids has the same structure as an algebroid and the same axioms except that the existence of identities $I_n \in A(x,x)$ is not assumed [2]. We can give algebroid examples as follows [2]:

1. If A_0 has exactly one object, then an *R*-algebroid A over A_0 is just an *R*-algebra. An ideal in A is

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then an example of pre-algebroid.

2. If A is an R-algebroid over A_0 and $x \in A_0$, then $A(x,x)$ inherits the structure of an R-algebra

A morphism $f: B \to C$ of R-algebroids B, C is a functor of the underlying categories which is also R-linear on each $B(x,y) \to C(fx, fy)$. The set of morphisms $B \to C$ of algebroids is written $A\lg(B,C)$.

3. ACTIONS AND CROSSED MODULES

Let A be an R-algebroid over A_0 and let M be a pre-algebroid over A_0. It is convenient to write the compositions in A and in M as juxtaposition [2].

A left action of A on M assigns to each $m \in M(x,y)$ and $a \in A(w,x)$ an element ${}^{a}m \in M(w,y)$ satisfying the axioms:

LAct1: ${}^{c}\left({}^{a}m\right) = {}^{(ca)}m$, ${}^{1}m = m$

LAct2: ${}^{c}\left({}^{a}m\right) = {}^{(ca)}m$, ${}^{1}m = m$

LAct3: ${}^{a}\left(m + m_1\right) = {}^{a}m + {}^{a}m_1$

LAct4: ${}^{a+b}m = {}^{a}m + {}^{b}m$

LAct5: ${}^{a}\left(rm\right) = r\left({}^{a}m\right)$

for all $m, m_1 \in M(x,y)$, $n \in M(y,z)$, $a, b \in A(w,x)$, $c \in A(u,w)$ and $r \in R$.

A right action of A on M assigns to each $m \in M(x,y)$ and $a \in A(y,z)$ an element $m^a \in M(x,z)$ satisfying the axioms:

RAct1: $\left(m^a\right)^c = m^{ca}, m^1 = m$

RAct2: $\left(mn\right)^a = mn^a$

RAct3: $\left(m + m_1\right)^a = m^a + m_1^a$

RAct4: $m^{a+b} = m^a + m^b$

RAct5: $\left(rm\right)^a = r\left(m^a\right)$

for all $m, m_1 \in M(x,y)$, $n \in M(y,z)$, $a, b \in A(y,u)$, $c \in A(u,v)$ and $r \in R$. Left and right actions of A on M commute if $\left({}^{a}m\right)^b = {}^{a}\left(m^b\right)$, $m \in M(x,y)$, $a \in A(w,x)$ and $b \in A(y,u)$.

Definition 3.1. A crossed module of algebroids consists of an R-algebroid A, a pre-R-algebroid M, both over the same set of objects, and commuting left and right actions of A on M, together with a pre-algebroid morphism $\mu: M \to A$ over the identity on A_0. These must satisfy the axioms [2]:

CM1: $\mu\left(m^b\right) = (\mu m)b,\ \mu\left({}^{a}m\right) = a(\mu m)$,

CM2: $mn = m^{(\mu n)} = {}^{(\mu m)}n$

for all $m \in M(x,y), n \in M(y,z), a \in A(w,x)$ and $b \in A(y,u)$.

A morphism of crossed modules $(\alpha, \beta): (A, M, \mu) \to (A', M', \mu')$ is an algebroid morphism $\alpha: A \to A'$ and a pre-algebroid morphism $\beta \in M \to M'$ over the same map on objects such that $\alpha\mu = \mu'\beta$ and $\beta\left({}^{a}m\right) = {}^{\alpha a}\left(\beta m\right), \beta\left(m^b\right) = \left(\beta m\right)^{\alpha a}$ for all $a, b \in A, m \in M$.

$$
\begin{tikzcd}
M \arrow[r, "\beta"] \arrow[d, "\mu"'] & M' \arrow[d, "\mu'"] \\
A \arrow[r, "\alpha"] & A'
\end{tikzcd}
$$

Thus we get a category of crossed modules of algebroids. The examples of crossed modules of algebroids are as follows [2]:

1. Let A be an R-algebroid over A_0 and suppose I is a two-sided ideal in A. Let $i: I \to A$ be the

inclusion morphism and let A operate on I by $a^c = ac,\ {}^b a = ba$ for all $a \in I$ and $b, c \in A$ such that these products ac, ba are defined. Then $i : I \to A$ is a crossed module.

2. A two-sided module over the algebroid A is defined as being a crossed module $\mu : M \to A$, in which $\mu m = 0_{xy}$ for all $m \in M(x, y),\ x, y \in A_0$.
3. Let $K \to M \to P$ be morphisms of pre-algebroids over the identity on objects and such that: A is an algebroid; p is surjective. $K = \ker p$; and i is the inclusion. Assume that K is central in M, i. e. that km and mk are zeros for all k in K, m in M. Then $p : M \to A$ can be given the structure of a crossed module with actions ${}^a m = a'm,\ m^b = mb'$ where $a \mapsto a'$ is a set-theoretic section of p.

4. PULLBACK CROSSED MODULES OF ALGEBROIDS

Let (A, M, μ) be a crossed module of algebroids and $\iota : Q \to A$ be a pre-algebroids. Then $(Q, \iota^{**}M, \mu^{**})$ is the pullback of (A, M, μ) by ι where

$$\iota^{**}M = \{(q, m) \in Q \times M \mid \iota q = \mu m\}$$

and $\mu^{**}(q, m) = q.$

$$\begin{tikzcd} \iota^{**}M \arrow[r] \arrow[d, "\mu^{**}"'] & M \arrow[d, "\mu"] \\ Q \arrow[r, "\iota"] & A \end{tikzcd}$$

The right action of Q on $\iota^{**}M$ is given by

$$(q_1, m)^q = (q_1 q, m\iota q)$$

and the left action of Q on $\iota^{**}M$ is given by

$${}^q(q_1, m) = (qq_1, \iota q m)$$

where multiplication in $\iota^{**}M$ is componentwise.

Theorem 4.1. There is an algebroid right action of Q on $\iota^{**}M$ given by

$$(q_1, m)^q = (q_1 q, m\iota q)$$

Proof: Proof can be found in [5].

Theorem 4.2. There is an algebroid left action of Q on $\iota^{**}M$ given by

$${}^q(q_1, m) = (qq_1, \iota q m)$$

Proof: Similarly, it can be easily shown that defining left action satisfies the left action conditions of algebroids.

Theorem 4.3. $(Q, \iota^{**}M, \mu^{**})$ has the structure of a crossed module.

Proof: Proof can be found in [5].

5. PUSHOUT CROSSED MODULES OF ALGEBROIDS

Let (M, A, μ) be a crossed module of algebroids over A and let $\phi : A \to H$ be a continuous homomorphism of algebroids. Consider algebroid $\iota_{**}(M)$ topologically generated by the algebroid space $M \times H$ with relations:

$$
\begin{aligned}
& i)(m_1, h_1) + (m_2, h_2) = (m_1 + m_2, h_1 + h_2) \\
& ii)(m, h_1) + (m, h_2) = (m, h_1 + h_2) \\
& iii)r(m, h) = (rm, hr) \\
& iv)(m, h)r = (rm, rh) \\
& v)(m_1, h_1)(m_2, h_2) = (m_1, h_1(\phi\mu m_2)h_2) \\
& vi)(m_1, h_1)(m_2, h_2) = (m_2, (\phi\mu m_1)h_1h_2)
\end{aligned}
$$

$\forall m, m_1, m_2 \in M, h, h_1, h_2 \in H$ and $r \in A$.
Where condition iii) is left scalar multiplication and condition iv) is a right scalar multiplication. Conditions v) and vi) are left and right multiplication respectively.
Define a continuous homomorphism $\mu_{**} : \iota_{**}(M) \to H$ by extending left side $\mu_{**}(m, h) = h(\phi\mu m)$ and right side $\mu_{**}(m, h) = (\phi\mu m)h$ to the whole of $\iota_{**}(M)$ and define $H-$action on the left of $\iota_{**}(M)$ by

$$^{h}(m_1, h_1) = (m_1, hh_1)$$

and on the right of $\iota_{**}(M)$ by

$$(m_1, h_1)^{h} = (m_1, h_1h)$$

For $h, h_1 \in H, m, m_1 \in M$ and a homomorphism $\Psi : M \to \iota_{**}(M)$ by $\Psi(m) = (m, 1)$, then $(H, \iota_{**}(M), \mu_{**})$ is a crossed module of algebroids over H. We can geometrically present all homomorphisms as follows:

$$
\begin{tikzcd}
M \arrow[r, "\Psi"] \arrow[d, "\mu"'] & \iota_{**}(M) \arrow[d, "\mu_{**}"] \\
A \arrow[r, "\phi"'] & H
\end{tikzcd}
$$

Theorem 5.1. Defining $H-$action on the left of $\iota_{**}(M)$ by $^{h}(m_1, h_1) = (m_1, hh_1)$ and on the right of $\iota_{**}(M)$ by $(m_1, h_1)^{h} = (m_1, h_1h)$ are algebroid actions.

Proof: We must show that the right and left action of H on $\iota_{**}(M)$ assigns to each $m \in \iota_{**}M(x, y)$, $h \in H(y, z)$ an element $(m, h)^{h_1} \in \iota_{**}(M)(x, z)$, satisfying the right and left action axioms RAct1-RAct5 and LAct1-LAct4 respectively.

RAct1:

$$\left((m_1,h_1)^{h_2}\right)^{h_3} = (m_1,h_1h_2)^{h_3} = (m_1,h_1h_2h_3) = (m_1,h_1)^{h_2h_3}$$

$$(m_1,h_1)^1 = (m_1,h_1)$$

LAct1:

$${}^{h_3}\left({}^{h_2}(m_1,h_1)\right) = {}^{h_3}(m_1,h_2h_1) = (m_1,h_3h_2h_1) = {}^{h_3h_2}(m_1,h_1)$$

$${}^{1}(m_1,h_1) = (m_1,h_1)$$

RAct2:

$$\left((m_1,h_1)(m_2,h_2)\right)^{h_3} = (m_2,(\phi\mu m_1)h_1h_2)^{h_3} = (m_2,\mu_{**}(m_1,h_1)h_2)^{h_3} = (m_2,\mu_{**}(m_1,h_1)h_2h_3)$$

$$(m_1,h_1)((m_2,h_2))^{h_3} = (m_1,h_1)(m_2,h_2h_3) = (m_2,\mu_{**}(m_1,h_1)h_2h_3)$$

LAct2:

$${}^{h_3}\left((m_1,h_1)(m_2,h_2)\right) = {}^{h_3}(m_1,h_1(\phi\mu m_2)h_2) = {}^{h_3}(m_1,h_1\mu_{**}(m_2,h_2)) = (m_1,h_3h_1\mu_{**}(m_2,h_2))$$

$${}^{h_3}((m_1,h_1))(m_2,h_2) = (m_1,h_3h_1)(m_2,h_2) = (m_1,h_3h_1\mu_{**}(m_2,h_2))$$

RAct3:

$$\left((m_1,h_1)+(m_2,h_2)\right)^{h_3} = \left((m_1+m_2,h_1+h_2)\right)^{h_3} = \left((m_1+m_2),(h_1+h_2)h_3\right) = (m_1+m_2,h_1h_3+h_2h_3)$$

$$(m_1,h_1)^{h_3}+(m_2,h_2)^{h_3} = (m_1,h_1h_3)+(m_2,h_2h_3) = (m_1+m_2,h_1h_3+h_2h_3)$$

LAct3:

$${}^{h_3}\left((m_1,h_1)+(m_2,h_2)\right) = {}^{h_3}\left((m_1+m_2,h_1+h_2)\right) = \left((m_1+m_2),h_3(h_1+h_2)\right) = (m_1+m_2,h_3h_1+h_3h_2)$$

$${}^{h_3}(m_1,h_1)+{}^{h_3}(m_2,h_2) = (m_1,h_3h_1)+(m_2,h_3h_2) = (m_1+m_2,h_3h_1+h_3h_2)$$

RAct4:

$$(m_1,h_1)^{h_2+h_3} = (m_1,h_1(h_2+h_3)) = (m_1,h_1h_2+h_1h_3)$$

$$(m_1,h_1)^{h_2}+(m_1,h_1)^{h_3} = (m_1,h_1h_2)+(m_1,h_1h_3) = (m_1,h_1h_2+h_1h_3)$$

LAct4:

$${}^{h_2+h_3}(m_1,h_1) = (m_1,(h_2+h_3)h_1) = (m_1,h_2h_1+h_3h_1)$$

$${}^{h_2}(m_1,h_1)+{}^{h_3}(m_1,h_1) = (m_1,h_2h_1)+(m_1,h_3h_1) = (m_1,h_2h_1+h_3h_1)$$

RAct5:

$$(rm_1,rh_1)^{h_2} = (rm_1,rh_1h_2) = (m_1,h_1h_2)r = (m_1,h_1)^{h_2}r$$

LAct5:

$${}^{h_2}(rm_1,h_1r) = (rm_1,h_2h_1r) = r(m_1,h_2h_1) = r\,{}^{h_2}(m_1,h_1)$$

$\forall (m_1,h_1),(m_2,h_2)\in \iota_{**}(M)(x,y)$, $h_2,h_3\in Q(y,u)$ and $r\in R$.

Left and right actions of H on $\iota_*(M)$ commute if $\left({}^{h_2}(m_1,h_1)\right)^{h_3} = {}^{h_2}\left((m_1,h_1)^{h_3}\right)$ for all $(m_1,h_1)\in \iota_{**}(M)(x,y)$, $h_2\in H(w,x)$ and $h_3\in A(y,u)$.

$$\left({}^{h_2}(m_1,h_1)\right)^{h_3} = (m_1,h_2h_1)^{h_3} = (m_1,h_2h_1h_3)$$

$${}^{h_2}\left((m_1,h_1)^{h_3}\right) = {}^{h_2}(m_1,h_1h_3) = (m_1,h_2h_1h_3)$$

Theorem 5.2. $(H,\iota_{**}(M),\mu_{**})$ has the structure of a crossed module.

Proof: We must show that axioms of the crossed module of algebroids are satisfied:

CM1:

$$\mu_{**}((m_1,h_1)^{h_2}) = \mu_{**}(m_1,h_1h_2) \qquad\qquad \mu_{**}({}^{h_2}(m_1,h_1)) = \mu_{**}(m_1,h_2h_1)$$
$$= (\phi\mu m_1)h_1h_2 \qquad\qquad = h_2h_1(\phi\mu m_1)$$
$$= \mu_{**}(m_1,h_1)h_2 \qquad\qquad = h_2\mu_{**}(m_1,h_1)$$

CM2:

$$(m_1,h_1)^{\mu_{**}(m_2,h_2)} = (m_1,h_1)^{(\phi\mu m_2)h_2} \qquad\qquad {}^{\mu_{**}(m_1,h_1)}(m_2,h_2) = {}^{h_1(\phi\mu m_1)}(m_2,h_2)$$
$$= (m_1,h_1(\phi\mu m_2)h_2) \qquad\qquad = (m_2,h_1(\phi\mu m_1)h_2)$$
$$= (m_1,h_1\mu_{**}(m_2,h_2)) \qquad\qquad = (m_2,\mu_{**}(m_1,h_1)h_2)$$
$$= (m_1,h_1)(m_2,h_2) \qquad\qquad = (m_1,h_1)(m_2,h_2)$$

A morphism of crossed modules $(\alpha,\beta):(H,\iota_{**}(M),\mu_{**}) \to (H',\iota_{**}(M'),\mu'_{**})$ is an algebroid morphism $\alpha: H \to H'$ and a pre-algebroid morphism $\beta \in \iota_{**}(M) \to \iota_{**}(M')$ over the same map on objects such that $\alpha\mu_{**} = \mu'_{**}\beta$ and $\beta({}^{h_2}(m_1,h_1)) = {}^{\alpha h_2}(\beta(m_1,h_1)), \beta((m_1,h_1)^{h_3}) = (\beta(m_1,h_1))^{\alpha h_3}, \forall h_2,h_3 \in H \quad (m_1,h_1) \in \iota_{**}(M)$.

$$\begin{tikzcd}
\iota_{**}(M) \arrow[r, "\beta"] \arrow[d, "\mu_{**}"'] & \iota_{**}(M') \arrow[d, "\mu'_{**}"] \\
H \arrow[r, "\alpha"'] & H'
\end{tikzcd}$$

Then we get the category of pushout crossed module of algebroids.

$(\Psi,\phi):(A,M,\mu) \to (H,\iota_{**}(M),\mu_{**})$ is a morphism of algeroids having the following universal property. Given any crossed module of algebroids (H,D,μ') over H and continuous morphism $(\theta,\phi):(A,M,\mu) \to (H,D,\mu')$, (θ,ϕ) factorises in a unique way via (Ψ,ϕ) and a morphism $(\bar{\theta},1_H)$ of crossed module of algebroids over H.

$$\begin{tikzcd}
M \arrow[r, "\Psi"] \arrow[d, "\mu"'] & \iota_{**}(M) \arrow[r, "\bar{\theta}"] \arrow[d, "\mu_{**}"] & D \arrow[dl, "\mu'"] \\
A \arrow[r, "\phi"'] & H &
\end{tikzcd}$$

For a unique $(\bar{\theta},1_H)$ in AlgXMod/*H.*

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