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A system of generalized resolvent equations involving generalized pseudocontractive mapping

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Abstract

In this paper, we consider a system of generalized resolvent equations involving generalized pseudocontractive mapping with corresponding system of variational inclusions in real Banach spaces. We establish an equivalence between the system of generalized resolvent equations and the system of variational inclusions using the concept of H(.,.)-co-accretive mapping. Furthermore, we prove the existence of solution of system of generalized resolvent equations and discuss the convergence of iterative sequences generated by the proposed algorithm.

Keywords*:* Generalized resolvent equations; variational inclusions; algorithm; convergence; generalized pseudocontractive mapping

1. Introduction

A useful and important generalization of variational inequalities is a mixed variational inequality containing nonlinear term [1]. Due to the presence of nonlinear term, the project method cannot be used to study the existence of solution for the mixed variational inequalities. In 1994, Hassouni and Moudafi [2] introduced variational inclusions which contain mixed variational inequalities as special cases and they studied perturbed method for solving variational inclusions.

Using the concept of resolvent operator technique, Noor and Noor [3] introduced and studied resolvent equations and established the equivalence between the mixed variational inequalities and the resolvent equations. The resolvent operator technique is being used to develop powerful and efficient numerical technique for solving mixed (quasi) variational inequalities and related optimization problems.

Ahmad and Yao [4] introduced and studied a system of generalized resolvent equations in uniformly smooth Banach spaces by showing its equivalence with a system of variational inclusions. They developed an iterative algorithm for finding the approximate solutions of system of resolvent equations.

In 2008, Zou and Huang [5] introduced and studied $H(.,.)$-accretive mapping and its resolvent operator in Banach spaces. After that Ahmad et al. [6, 7] introduced and studied $H(.,.)$-cocoercive mapping and $H(.,.)$-co-accretive mapping for solving variational inclusion problems.

In this paper, by using the concept of $H(.,.)$-co-accretive mapping, we solve a system of generalized resolvent equations and the convergence criterion is also discussed. Our results can be viewed as a refinement and improvement of some known results of this field.

2. Preliminaries

Throughout the paper, unless otherwise specified, we assume that X be a real Banach space with its norm,$\|.\|$ X^* is the topological dual of X, d is the metric induced by the norm $\|.\|$, $CB(X)$ (respectively, 2^X) is the family of all nonempty closed and bounded subsets (respectively, all nonempty subsets) of X, $\mathcal{D}$ (.,.) is the Hausdorff metric on $CB(X)$ defined by

$$\mathcal{D}(\mathrm{A},\mathrm{B}) = \max\left\{\sup_{x\in A} d(x,B),\ \sup_{y\in B} d(A,y)\right\},$$

where $\mathrm{d}(\mathrm{x},\mathrm{B}) = \inf_{y\in B} d(x,y)$ and $\mathrm{d}(\mathrm{A},\mathrm{y}) = \inf_{x\in A} d(x,y)$.

We also assume that $\langle .,. \rangle$ is the duality pairing between X and X^* and $\mathcal{F}: \mathrm{X} \to 2^{X^*}$ is the normalized duality mapping defined by

$$\mathcal{F}(\mathrm{x}) = \{f \in X^*: \langle x, f\rangle = \|x\|\|f\| \text{ and } \|x\| = \|f\|\}, \forall x \in X.$$

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We need the following concepts and results for the presentation of the main result of this paper.

Proposition 2.1. [8]. Let X be a real Banach space and $\mathcal{F}: X \to 2^{X^*}$ be a normalized duality mapping. Then for any $x, y \in X$,

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \forall\, j(x+y) \in \mathcal{F}(x+y).$$

Definition 2.1. Let $T: X \to X$ be a single-valued mapping and $\mathcal{F}: X \to 2^{X^*}$ be a normalized duality mapping, then T is said to be
(i) cocoercive, if there exists a constant $\mu' > 0$ such that

$\langle Tx - Ty, j(x-y)\rangle \geq \mu'\|Tx - Ty\|^2, \forall\, x, y \in X,\ j(x-y) \in \mathcal{F}(x-y)$;

(ii) relaxed-cocoercive, if there exists a constant $\gamma' > 0$ such that

$\langle Tx - Ty, j(x-y)\rangle \geq (-\gamma')\|Tx - Ty\|^2, \forall\, x, y \in X,\ j(x-y) \in \mathcal{F}(x-y)$;

(iii) η- expansive, if there exists a constant $\eta > 0$ such that

$$\|Tx - Ty\| \geq \eta\|x-y\|, \forall\, x, y \in X;$$

if $\eta = 1$, then it is expansive.

Definition 2.2. Let $H: X \times X \to X$, $A, B: X \to X$ be the single-valued mappings and $\mathcal{F}: X \to 2^{X^*}$ be a normalized duality mapping, then
(i) $H(A,.)$ is said to be cocoercive with respect to A, if there exists a constant $\mu > 0$ such that

$\langle H(Ax,u) - H(Ay,u), j(x-y)\rangle \geq (\mu)\|Ax - Ay\|^2, \forall\, x, y, u \in X, j(x-y) \in \mathcal{F}(x-y)$;

(ii) $H(.,B)$ is said to be relaxed-cocoercive with respect to B, if there exists a constant $\gamma > 0$ such that

$\langle H(u,Bx) - H(u,By), j(x-y)\rangle \geq (-\gamma)\|Bx - By\|^2, \forall\, x, y, u \in X, j(x-y) \in \mathcal{F}(x-y)$;

(iii) $H(A,.)$ is said to be symmetric cocoercive with respect to A and B, if $H(A,.)$ is cocoercive with respect to A and $H(.,B)$ is relaxed-cocoercive with respect to B;
(iv) $H(A,B)$ is said to be mixed Lipschitz continuous with respect to A and B, if there exists a constant $r > 0$ such that

$$\|H(Ax,Bx) - H(Ay,By)\| \leq r\|x-y\|, \forall\, x, y \in X.$$

Definition 2.3. A single-valued mapping $g: X \to X$ is said to be
(i) k-strongly accretive, $k \in (0,1)$ if for any $x, y, \in X$ there exists, $j(x-y) \in \mathcal{F}(x-y)$ such that

$$\langle gx - gy, j(x-y)\rangle \geq k\|x-y\|^2;$$

(ii) Lipschitz continuous, if for any $x, y, \in X$ there exists a constant $\lambda_g > 0$, such that

$$\|gx - gy\| \leq \lambda_g\|x-y\|.$$

Definition 2.4. Let $M: X \times X \to 2^X$ be a multi-valued mapping, $f, g: X \to X$ be the single-valued mappings and $\mathcal{F}: X \to 2^{X^*}$ be a normalized duality mapping, then
(i) $M(f,.)$ is said to be α-strongly accretive with respect to f, if there exists a constant $\alpha > 0$ such that

$\langle u - v, j(x-y)\rangle \geq \alpha\|x-y\|^2, \forall\, x, y, w \in X, u \in M(fx,w), v \in M(fy,w), j(x-y) \in \mathcal{F}(x-y)$;

(ii) $M(.,g)$ is said to be β-relaxed accretive with respect to g, if there exists a constant $\beta > 0$ such that

$\langle u - v, j(x-y)\rangle \geq (-\beta)\|x-y\|^2, \forall x, y, w \in X, u \in M(w,gx), v \in M(w,gy), j(x-y) \in \mathcal{F}(x-y)$;

(iii) $M(.,.)$ is said to be symmetric accretive with respect to f and g, if $M(f,.)$ is strongly accretive with respect to f and $M(.,g)$ is relaxed accretive with respect to g.

Definition 2.5. A multi-valued mapping $T: X \to CB(X)$ is said to be $\mathcal{D}$-Lipschitz continuous if for any $x, y \in X$ there exists a constant $\lambda_{\mathcal{D}_T} > 0$ such that

$$\mathcal{D}(T(x), T(y)) \leq \lambda_{\mathcal{D}_T}\|x-y\|.$$

Definition 2.6. Let $A, B: X \to X$ and $H: X \times X \to X$ be the single-valued mappings, then
(i) $H(A,.)$ is said to be generalized pseudocontractive with respect to A, if there exists a constant $s > 0$ such that

$\langle H(Ax,u) - H(Ay,u), j(x-y)\rangle \leq s\|x - y\|^2, \forall x, y, u \in X,\ j(x-y) \in \mathcal{F}(x-y)$;

(ii) $H(.,B)$ is said to be generalized pseudocontractive with respect to B, if there exists a constant $t > 0$ such that

$\langle H(u,Bx) - H(u,By), j(x-y)\rangle \leq t\|x - y\|^2, \forall\, x, y, u \in X,\ j(x-y) \in \mathcal{F}(x-y)$.

Note: If $X = H$, a real Hilbert space, then Definition 2.6 reduces to Definition 2.4(4) of [9].

Definition 2.7. Let $A, B, f, g: X \to X$ and $H: X \times X \to X$ be the single-valued mappings. Let $M: X \times X \to 2^X$ be a multi-valued mapping. The mapping M is said to be $H(.,.)$-co-accretive with respect to A, B, f and g, if $H(A,B)$ is symmetric cocoercive with respect to A and B, $M(f,g)$ is symmetric accretive with respect to f and g and $\big(H(A,B) + \lambda M(f,g)\big)(X) = X$, for every $\lambda > 0$.

Theorem 2.1 [7]. Let X be a real Banach space, let $A, B, f, g: X \to X$ and $H: X \times X \to X$ be the single-valued mappings. Let $M: X \times X \to 2^X$ be an $H(.,.)$-co-accretive mapping with respect to A, B, f and g. Let A be η-expansive and B be σ-Lipschitz continuous. Then the mapping $\big(H(A,B) + \lambda M(f,g)\big)^{-1}$ is single-valued, for every $\lambda > 0$.

Definition 2.8 [7]. Let $A, B, f, g: X \to X$ and $H: X \times X \to X$ be the single-valued mappings. Let $M: X \times X \to 2^X$ be an $H(.,.)$-co-accretive mapping with respect to A, B, f and g. The resolvent operator $R^{H(.,.)}_{\lambda, M(.,.)}: X \to X$ is defined by

$$R^{H(.,.)}_{\lambda, M(.,.)}(x) = \big(H(A,B) + \lambda M(f,g)\big)^{-1}(x), \forall\, x \in X, \lambda > 0. \quad (1)$$

Theorem 2.2 [7]. Let $A, B, f, g: X \to X$ and $H: X \times X \to X$ be the single-valued mappings. Suppose $M: X \times X \to 2^X$ be an $H(.,.)$-co-accretive mapping with respect to A, B, f and g with constants μ, γ, α and β, respectively. Let A be η-expansive and B be σ-Lipschitz continuous such that $\alpha > \beta, \mu > \gamma$ and $\eta > \sigma$. Then the resolvent operator $R^{H(.,.)}_{\lambda, M(.,.)}: X \to X$ is Lipschitz continuous with constant L, that is,

$$\left\| R^{H(.,.)}_{\lambda, M(.,.)}(x) - R^{H(.,.)}_{\lambda, M(.,.)}(y) \right\| \le L \|x - y\|, \forall\, x, y \in X, \quad (2)$$

where $L =: \frac{1}{\lambda(\alpha - \beta) + (\mu\eta^2 - \gamma\sigma^2)}$.

3. Iterative algorithms and convergence result

Let X_1 and X_2 be two real Banach spaces, $A_1, B_1, f_1, g_1: X_1 \to X_1$, $A_2, B_2, f_2, g_2: X_2 \to X_2$, $H_1: X_1 \times X_1 \to X_1$, $H_2: X_2 \times X_2 \to X_2$, $S: X_1 \times X_2 \to X_1$ and $T: X_1 \times X_2 \to X_2$ be the single-valued mappings, $E: X_1 \to 2^{X_1}$, $F: X_2 \to 2^{X_2}$ be the multi-valued mappings. Let $M_1: X_1 \times X_1 \to 2^{X_1}$ and $M_2: X_2 \times X_2 \to 2^{X_2}$ be the $H_1(A_1, B_1)$-co-accretive and $H_2(A_2, B_2)$-co-accretive mappings, respectively. We consider the problem of finding

$$(x,y) \in X_1 \times X_2, u \in E(x), v \in F(y),\ z' \in X_1,\ z'' \in X_2$$

such that

$$\begin{cases} S(x,v) + {\lambda_1}^{-1} J^{H_1(.,.)}_{\lambda_1, M_1(.,.)}(z') = 0, & \lambda_1 > 0 \\ T(u,y) + {\lambda_2}^{-1} J^{H_2(.,.)}_{\lambda_2, M_1(.,.)}(z'') = 0, & \lambda_2 > 0 \end{cases} \quad (3)$$

where

$$J^{H_1(.,.)}_{\lambda_1, M_1(.,.)} = I - H_1\left[A_1\left(R^{H_1(.,.)}_{\lambda_1, M_1(.,.)}(.)\right), B_1\left(R^{H_1(.,.)}_{\lambda_1, M_1(.,.)}(.)\right)\right],$$

$$J^{H_2(.,.)}_{\lambda_2, M_2(.,.)} = I - H_2\left[A_2\left(R^{H_2(.,.)}_{\lambda_2, M_2(.,.)}(.)\right), B_2\left(R^{H_2(.,.)}_{\lambda_2, M_2(.,.)}(.)\right)\right],$$

$R^{H_1(.,.)}_{\lambda_1, M_1(.,.)}$ and $R^{H_2(.,.)}_{\lambda_2, M_2(.,.)}$ are the resolvent operators associated with M_1 and M_2 respectively,

$$H_1(A_1(R^{H_1(.,.)}_{\lambda_1, M_1(.,.)}(z'))) = H_1(A_1(R^{H_1(.,.)}_{\lambda_1, M_1(.,.)}))(z')$$

and

$$H_2(A_2(R^{H_2(.,.)}_{\lambda_2, M_2(.,.)}(z''))) = H_2(A_2(R^{H_2(.,.)}_{\lambda_2, M_2(.,.)}))(z'').$$

The system (3) is called system of generalized resolvent equations.

We mention the following system of variational inclusions and we will show its equivalence with system of generalized resolvent equations (3).

Find $(x,y) \in X_1 \times X_2, u \in E(x), v \in F(y)$, such that

$$\begin{cases} 0 \in S(x,v) + M_1(f_1(x), g_1(x)), \\ 0 \in T(u,y) + M_2(f_2(y), g_2(y)), \end{cases} \quad (4)$$

Lemma 3.1. $(x,y) \in X_1 \times X_2, u \in E(x), v \in F(y)$ is a solution of system of variational inclusions (4) if and only if (x,y,u,v) satisfies

$$x = R^{H_1(.,.)}_{\lambda_1, M_1(.,.)}\big[H_1\big(A_1(x), B_1(x)\big) - \lambda_1 S(x,v)\big],\ \lambda_1 > 0,$$

$$y = R^{H_2(.,.)}_{\lambda_2, M_2(.,.)}\big[H_2\big(A_2(y), B_2(y)\big) - \lambda_2 T(u,y)\big],\ \lambda_2 > 0.$$

Proof: The proof of Lemma 3.1 follows directly from the definition of resolvent operators $R^{H_1(.,.)}_{\lambda_1, M_1(.,.)}$ and $R^{H_2(.,.)}_{\lambda_2, M_2(.,.)}$.

Proposition 3.1. The system of variational inclusions (4) has a solution (x,y,u,v) with $(x,y) \in X_1 \times X_2, u \in E(x)$ and $v \in F(y)$, if and only if system of generalized resolvent equations (3) has a solution (z', z'', x, y, u, v) with $(x,y) \in X_1 \times X_2, u \in E(x), v \in F(y),\ z' \in X_1, z'' \in X_2$, where

$$x = R^{H_1(.,.)}_{\lambda_1, M_1(.,.)}(z'), \quad (5)$$

$$y = R^{H_2(.,.)}_{\lambda_2, M_2(.,.)}(z''), \quad (6)$$

and

$$z' = \left[H_1\big(A_1(x), B_1(x)\big) - \lambda_1 S(x,v)\right],$$

$$z'' = \left[H_2\big(A_2(y), B_2(y)\big) - \lambda_2 T(u,y)\right].$$

Proof: Let (x,y,u,v) be a solution of system of variational inclusions (4), then by Lemma (3.1) it satisfies the following equations:

$$x = R^{H_1(.,.)}_{\lambda_1, M_1(.,.)}\left[H_1\big(A_1(x), B_1(x)\big) - \lambda_1 S(x,v)\right], \ \lambda_1 > 0,$$

$$y = R^{H_2(.,.)}_{\lambda_2, M_2(.,.)}\left[H_2\big(A_2(y), B_2(y)\big) - \lambda_2 T(u,y)\right], \ \lambda_2 > 0.$$

Let $z' = \left[H_1\big(A_1(x), B_1(x)\big) - \lambda_1 S(x,v)\right]$ and $z'' = \left[H_2\big(A_2(y), B_2(y)\big) - \lambda_2 T(u,y)\right]$, then we have

$$x = R^{H_1(.,.)}_{\lambda_1, M_1(.,.)}(z'),$$

$$y = R^{H_2(.,.)}_{\lambda_2, M_2(.,.)}(z''),$$

and

$$z' = H_1\left[A_1\left(R^{H_1(.,.)}_{\lambda_1, M_1(.,.)}(z')\right), B_1\left(R^{H_1(.,.)}_{\lambda_1, M_1(.,.)}(z')\right)\right] - \lambda_1 S(x,v),$$

$$z'' = H_2\left[A_2\left(R^{H_2(.,.)}_{\lambda_2, M_2(.,.)}(z'')\right), B_2\left(R^{H_2(.,.)}_{\lambda_2, M_2(.,.)}(z'')\right)\right] - \lambda_2 T(u,y),$$

it follows that

$$\left[I - H_1\left(A_1\left(R^{H_1(.,.)}_{\lambda_1, M_1(.,.)}(.)\right), B_1\left(R^{H_1(.,.)}_{\lambda_1, M_1(.,.)}(.)\right)\right)\right](z') = -\lambda_1 S(x,v),$$

and

$$\left[I - H_2\left(A_2\left(R^{H_2(.,.)}_{\lambda_2, M_2(.,.)}(.)\right), B_2\left(R^{H_2(.,.)}_{\lambda_2, M_2(.,.)}(.)\right)\right)\right](z'') = -\lambda_2 T(u,y),$$

that is,

$$S(x,v) + {\lambda_1}^{-1} J^{H_1(.,.)}_{\lambda_1, M_1(.,.)}(z') = 0,$$

$$T(u,y) + {\lambda_2}^{-1} J^{H_2(.,.)}_{\lambda_2, M_1(.,.)}(z'') = 0.$$

Thus (z', z'', x, y, u, v) is a solution of system of generalized resolvent equations (3).

Conversely, let (z', z'', x, y, u, v) be a solution of system of generalized resolvent equations (3), then

$$\lambda_1 S(x,v) = -J^{H_1(.,.)}_{\lambda_1, M_1(.,.)}(z'), \tag{7}$$

$$\lambda_2 T(u,y) = -J^{H_2(.,.)}_{\lambda_2, M_1(.,.)}(z''). \tag{8}$$

Now

$$\begin{aligned}\lambda_1 S(x,v) &= -J^{H_1(.,.)}_{\lambda_1, M_1(.,.)}(z') \\ &= -\left[I - H_1\left(A_1\left(R^{H_1(.,.)}_{\lambda_1, M_1(.,.)}(.)\right), B_1\left(R^{H_1(.,.)}_{\lambda_1, M_1(.,.)}(.)\right)\right)\right](z') \\ &= H_1\left[A_1\left(R^{H_1(.,.)}_{\lambda_1, M_1(.,.)}(z')\right), B_1\left(R^{H_1(.,.)}_{\lambda_1, M_1(.,.)}(z')\right)\right] - z'.\end{aligned}$$

It follows that

$$z' = H_1\left[A_1\left(R^{H_1(.,.)}_{\lambda_1, M_1(.,.)}(z')\right), B_1\left(R^{H_1(.,.)}_{\lambda_1, M_1(.,.)}(z')\right)\right] - \lambda_1 S(x,v),$$

$$\begin{aligned}&R^{H_1(.,.)}_{\lambda_1, M_1(.,.)}(z') \\ &= R^{H_1(.,.)}_{\lambda_1, M_1(.,.)}\left[H_1\left(A_1\left(R^{H_1(.,.)}_{\lambda_1, M_1(.,.)}(z')\right), B_1\left(R^{H_1(.,.)}_{\lambda_1, M_1(.,.)}(z')\right)\right) - \lambda_1 S(x,v)\right]\end{aligned}$$

i.e., $x = R^{H_1(.,.)}_{\lambda_1, M_1(.,.)}\left[H_1\big(A_1(x), B_1(x)\big) - \lambda_1 S(x,v)\right]$.

Further,

$$\begin{aligned}\lambda_2 T(u,y) &= -J^{H_2(.,.)}_{\lambda_2, M_1(.,.)}(z'') \\ &= -\left[I - H_2\left(A_2\left(R^{H_2(.,.)}_{\lambda_2, M_1(.,.)}(.)\right), B_2\left(R^{H_2(.,.)}_{\lambda_2, M_2(.,.)}(.)\right)\right)\right](z'') \\ &= H_2\left[A_2\left(R^{H_2(.,.)}_{\lambda_2, M_1(.,.)}(z'')\right), B_2\left(R^{H_2(.,.)}_{\lambda_2, M_2(.,.)}(z'')\right)\right] - z''.\end{aligned}$$

It follows that

$$z'' = H_2\left[A_2\left(R^{H_2(.,.)}_{\lambda_2, M_2(.,.)}(z'')\right), B_2\left(R^{H_2(.,.)}_{\lambda_2, M_2(.,.)}(z'')\right)\right] - \lambda_2 T(u,y),$$

$$\begin{aligned}&R^{H_2(.,.)}_{\lambda_2, M_2(.,.)}(z'') \\ &= R^{H_2(.,.)}_{\lambda_2, M_2(.,.)}\left[H_2\left(A_2\left(R^{H_2(.,.)}_{\lambda_2, M_2(.,.)}(z'')\right), B_2\left(R^{H_2(.,.)}_{\lambda_2, M_2(.,.)}(z'')\right)\right) - \lambda_2 T(u,y)\right]\end{aligned}$$

i.e., $y = R^{H_2(.,.)}_{\lambda_2, M_2(.,.)}\left[H_2\big(A_2(y), B_2(y)\big) - \lambda_2 T(u,y)\right]$.

Thus, we have

$$x = R^{H_1(.,.)}_{\lambda_1, M_1(.,.)}\left[H_1\big(A_1(x), B_1(x)\big) - \lambda_1 S(x,v)\right],$$

$$y = R^{H_2(.,.)}_{\lambda_2, M_2(.,.)}\left[H_2\big(A_2(y), B_2(y)\big) - \lambda_2 T(u,y)\right].$$

Then by Lemma 3.1, (x,y,u,v) is a solution of system of variational inclusions (4).

Algorithm 3.1. For the given $(x_0, y_0) \in X_1 \times X_2, u_0 \in E(x_0), v \in F(y_0), {z'}_0 \in X_1$ and ${z''}_0 \in X_2$, compute the sequences $\{z'_n\}$, $\{z''_n\}$, $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ by the following iterative schemes:

$$x_n = R^{H_1(.,.)}_{\lambda_1, M_1(.,.)}({z'}_n), \tag{9}$$

$$y_n = R^{H_2(.,.)}_{\lambda_2, M_2(.,.)}({z''}_n), \tag{10}$$

$$u_n \in E(x_n): \|u_{n+1} - u_n\| \le \mathcal{D}(E(x_{n+1}), E(x_n)), \tag{11}$$

$$v_n \in F(y_n): \|v_{n+1} - v_n\| \le \mathcal{D}(F(y_{n+1}), F(y_n)), \tag{12}$$

$$z'_{n+1} = H_1(A_1(x_n), B_1(x_n)) - \lambda_1 S(x_n, v_n), \tag{13}$$

$$z''_{n+1} = H_2(A_2(y_n), B_2(y_n)) - \lambda_2 T(u_n, y_n), \tag{14}$$

where $n = 0,1,2,\ldots$ and $\lambda_1 > 0, \lambda_2 > 0$ are two constants.

The system of generalized resolvent equations (3) can also be written as

$$z' = H_1\big(A_1(x), B_1(x)\big) - S(x,v) + (I - \lambda_1{}^{-1})J^{H_1(.,.)}_{\lambda_1, M_1(.,.)}(z'),$$

$$z'' = H_2\big(A_2(y), B_2(y)\big) - T(u,y) + (I - \lambda_2{}^{-1})J^{H_2(.,.)}_{\lambda_2, M_2(.,.)}(z'').$$

We use the above fixed point formulation to suggest the following iterative algorithm.

Algorithm 3.2. For the given $(x_0, y_0) \in X_1 \times X_2, u_0 \in E(x_0), v \in F(y_0), z'_0 \in X_1$ and $z''_0 \in X_2$, compute the sequences $\{z'_n\}$, $\{z''_n\}$, $\{x_n\}, \{y_n\}$, $\{u_n\}$ and $\{v_n\}$ by the following iterative schemes:

$$x_n = R^{H_1(.,.)}_{\lambda_1, M_1(.,.)}(z'_n),$$

$$y_n = R^{H_2(.,.)}_{\lambda_2, M_2(.,.)}(z''_n),$$

$$u_n \in E(x_n): \|u_{n+1} - u_n\| \le \mathcal{D}(E(x_{n+1}), E(x_n)),$$

$$v_n \in F(y_n): \|v_{n+1} - v_n\| \le \mathcal{D}(F(y_{n+1}), F(y_n)),$$

$$z'_{n+1} = H_1\big(A_1(x_n), B_1(x_n)\big) - S(x_n, v_n) + (I - \lambda_1{}^{-1})J^{H_1(.,.)}_{\lambda_1, M_1(.,.)}(z'_n),$$

$$z''_{n+1} = H_2\big(A_2(y_n), B_2(y_n)\big) - T(u_n, y_n) + (I - \lambda_2{}^{-1})J^{H_2(.,.)}_{\lambda_2, M_2(.,.)}(z''_n).$$

We now prove the following existence and convergence result for the system of generalized resolvent equations (3).

Theorem 3.1. Let X_1 and X_2 be two real Banach spaces. Let $E: X_1 \to CB(X_1)$, $F: X_2 \to CB(X_2)$ be the $\mathcal{D}$-Lipschitz continuous mappings with constants $\lambda_{\mathcal{D}_E}$ and $\lambda_{\mathcal{D}_F}$, respectively. Let $H_1: X_1 \times X_1 \to X_1$, $H_2: X_2 \times X_2 \to X_2$ be the single-valued mappings such that H_1 is r_1-mixed Lipschitz continuous with respect to A_1 and B_1, s_1-generalized pseudocontractive with respect to A_1 and t_1-generalized pseudocontractive with respect to B_1 and H_2 is r_2-mixed Lipschitz continuous with respect to A_2 and B_2, s_2-generalized pseudocontractive with respect to A_2 and t_2-generalized pseudocontractive with respect to B_2. Let $S: X_1 \times X_2 \to X_1$, $T: X_1 \times X_2 \to X_2$ be Lipschitz continuous in the first and second arguments with constants $\lambda_{S_1}, \lambda_{S_2}, \lambda_{T_1}$ and λ_{T_2}, respectively. Let $M_1: X_1 \times X_1 \to 2^{X_1}$ and $M_2: X_2 \times X_2 \to 2^{X_2}$ be $H_1(A_1, B_1)$-co-accretive and $H_2(A_2, B_2)$-co-accretive mappings such that resolvent operators associated with M_1 and M_2 are Lipschitz continuous with constants L_1 and L_2, respectively,
where

$$L_1 =: \frac{1}{\lambda_1(\alpha_1 - \beta_1) + (\mu_1\eta_1^2 - \gamma_1\sigma_1^2)}, L_2 =: \frac{1}{\lambda_1(\alpha_2 - \beta_2) + (\mu_2\eta_2^2 - \gamma_2\sigma_2^2)}.$$

If there exist constants $\lambda_1 > 0$ and $\lambda_2 > 0$, such that

$$\begin{cases} 0 < L_1\big(K_1 + \sqrt{\theta_1} + \sqrt{\theta_3}\big) < 1 \\ 0 < L_2\big(K_2 + \sqrt{\theta_2} + \sqrt{\theta_4}\big) < 1 \end{cases} \tag{15}$$

where

$$K_1 = \sqrt{\frac{1+2(s_1+t_1)+3r_1}{1-r_1}} \text{ and } K_2 = \sqrt{\frac{1+2(s_2+t_2)+3r_2}{1-r_2}}.$$

Then there exist $(x, y) \in X_1 \times X_2, z' \in X_1, z'' \in X_2, u \in E(x)$ and $v \in F(y)$ satisfy the system of resolvent equations (3) and the iterative sequences $\{z'_n\}$, $\{z''_n\}$, $\{x_n\}, \{y_n\}$, $\{u_n\}$ and $\{v_n\}$ generated by Algorithm 3.1 converge strongly to z', z'', x, y, u and v, respectively.

Proof: From Algorithm 3.1, we have

$$\begin{aligned} \|z'_{n+1} - z'_n\| &= \big\|H_1\big(A_1(x_n), B_1(x_n)\big) \\ &\quad - \lambda_1 S(x_n, v_n) \\ &\quad - [H_1(A_1(x_{n-1}), B_1(x_{n-1})) \\ &\quad - \lambda_1 S(x_{n-1}, v_{n-1})]\big\| \\ &\le \big\|x_n - x_{n-1} + [H_1\big(A_1(x_n), B_1(x_n)\big) \\ &\quad - H_1(A_1(x_{n-1}), B_1(x_{n-1}))]\big\| \\ &+ \|x_n - x_{n-1} + \lambda_1(S(x_n, v_n) - S(x_{n-1}, v_{n-1}))\| \end{aligned} \tag{16}$$

By Proposition 2.1, using the mixed Lipschitz continuity and generalized pseudoconractivity of H_1, we have

$$\begin{aligned} &\big\|x_n - x_{n-1} + [H_1\big(A_1(x_n), B_1(x_n)\big) \\ &\quad - H_1(A_1(x_{n-1}), B_1(x_{n-1}))]\big\|^2 \\ &\le \|x_n - x_{n-1}\|^2 + 2\langle [H_1\big(A_1(x_n), B_1(x_n)\big) - \\ &H_1(A_1(x_{n-1}), B_1(x_{n-1}))], \\ &(j(x_n - x_{n-1} + [H_1\big(A_1(x_n), B_1(x_n)\big) - \\ &H_1(A_1(x_{n-1}), B_1(x_{n-1}))])\rangle \\ &\le \|x_n - x_{n-1}\|^2 + 2\langle [H_1\big(A_1(x_n), B_1(x_n)\big) - \\ &H_1(A_1(x_{n-1}), B_1(x_{n-1}))], j(x_n - x_{n-1})\rangle \\ &\le \|x_n - x_{n-1}\|^2 + 2\langle [H_1\big(A_1(x_n), B_1(x_n)\big) - \\ &H_1(A_1(x_{n-1}), B_1(x_{n-1}))], j(x_n - x_{n-1})\rangle \\ &+2\langle \big[H_1\big(A_1(x_n), B_1(x_n)\big) - \\ &H_1\big(A_1(x_{n-1}), B_1(x_{n-1})\big)\big], j(x_n - x_{n-1} + \\ &[H_1\big(A_1(x_n), B_1(x_n)\big) \\ &-H_1(A_1(x_{n-1}), B_1(x_{n-1}))]) - j(x_n - x_{n-1})\rangle \\ &\le \|x_n - x_{n-1}\|^2 + 2\langle [H_1\big(A_1(x_n), B_1(x_n)\big) - \\ &H_1(A_1(x_{n-1}), B_1(x_{n-1}))], j(x_n - x_{n-1})\rangle \\ &+2[\big\|H_1\big(A_1(x_n), B_1(x_n)\big) - \\ &H_1(A_1(x_{n-1}), B_1(x_{n-1}))\big\| \\ &\quad \times \{\big\|x_n - x_{n-1} + H_1\big(A_1(x_n), B_1(x_n)\big) - \\ &H_1(A_1(x_{n-1}), B_1(x_{n-1}))\big\| + \|x_n - x_{n-1}\|\}] \\ &\le \|x_n - x_{n-1}\|^2 + 2\langle [H_1\big(A_1(x_n), B_1(x_n)\big) - \\ &H_1(A_1(x_{n-1}), B_1(x_{n-1}))], j(x_n - x_{n-1})\rangle \end{aligned}$$

$$+2[r_1\|x_n - x_{n-1}\| \times \{\|x_n - x_{n-1} + H_1(A_1(x_n), B_1(x_n)) - H_1(A_1(x_{n-1}), B_1(x_{n-1}))\| + \|x_n - x_{n-1}\|\}]$$
$$\leq \|x_n - x_{n-1}\|^2 + 2\langle[H_1(A_1(x_n), B_1(x_n)) - H_1(A_1(x_{n-1}), B_1(x_{n-1}))], j(x_n - x_{n-1})\rangle + r_1(\|x_n - x_{n-1}\|^2 + \|x_n - x_{n-1} + H_1(A_1(x_n), B_1(x_n)) - H_1(A_1(x_{n-1}), B_1(x_{n-1}))\|^2) + 2r_1\|x_n - x_{n-1}\|^2$$
$$\leq \|x_n - x_{n-1}\|^2 + 2(s_1 + t_1)\|x_n - x_{n-1}\|^2 + 3r_1\|x_n - x_{n-1}\|^2 + r_1\|x_n - x_{n-1} + H_1(A_1(x_n), B_1(x_n)) - H_1(A_1(x_{n-1}), B_1(x_{n-1}))\|^2$$

which implies that

$$(1 - r_1)\|x_n - x_{n-1} + H_1(A_1(x_n), B_1(x_n)) - H_1(A_1(x_{n-1}), B_1(x_{n-1}))\|^2 \leq [1 + 2(s_1 + t_1) + 3r_1]\|x_n - x_{n-1}\|^2.$$

Thus, we have

$$\|x_n - x_{n-1} + H_1(A_1(x_n), B_1(x_n)) - H_1(A_1(x_{n-1}), B_1(x_{n-1}))\| \leq K_1\|x_n - x_{n-1}\|, \quad (17)$$

where $K_1 = \sqrt{\frac{1+2(s_1+t_1)+3r_1}{1-r_1}}$.

Since S is Lipschitz continuous in both the arguments and F is $\mathcal{D}$-Lipschitz continuous, we have

$$\|S(x_n, v_n) - S(x_{n-1}, v_{n-1})\| \leq \|S(x_n, v_n) - S(x_{n-1}, v_n)\| + \|S(x_{n-1}, v_n) - S(x_{n-1}, v_{n-1})\| \leq \lambda_{S_1}\|x_n - x_{n-1}\| + \lambda_{S_2}\|v_n - v_{n-1}\| \leq \lambda_{S_1}\|x_n - x_{n-1}\| + \lambda_{S_2}\mathcal{D}(F(x_n), F(y_{n-1})) \leq \lambda_{S_1}\|x_n - x_{n-1}\| + \lambda_{S_2}\lambda_{\mathcal{D}_F}\|y_n - y_{n-1}\|. \quad (18)$$

Using (18) and Proposition 2.1, it follows that

$$\|x_n - x_{n-1} + \lambda_1(S(x_n, v_n) - S(x_{n-1}, v_{n-1}))\|^2 \leq \|x_n - x_{n-1}\|^2 + 2\lambda_1\langle S(x_n, v_n) - S(x_{n-1}, v_{n-1}), j(x_n - x_{n-1} + \lambda_1(S(x_n, v_n) - S(x_{n-1}, v_{n-1})))\rangle$$
$$\leq \|x_n - x_{n-1}\|^2 + 2\lambda_1[\|S(x_n, v_n) - S(x_{n-1}, v_{n-1})\| \times \|x_n - x_{n-1} + \lambda_1(S(x_n, v_n) - S(x_{n-1}, v_{n-1}))\|]$$
$$\leq \|x_n - x_{n-1}\|^2 + 2\lambda_1(\lambda_{S_1}\|x_n - x_{n-1}\| + \lambda_{S_2}\lambda_{\mathcal{D}_F}\|y_n - y_{n-1}\|) \times \|x_n - x_{n-1} + \lambda_1(S(x_n, v_n) - S(x_{n-1}, v_{n-1}))\|$$
$$\leq \|x_n - x_{n-1}\|^2 + \lambda_1\lambda_{S_1}(\|x_n - x_{n-1}\|^2 + \|x_n - x_{n-1} + \lambda_1(S(x_n, v_n) - S(x_{n-1}, v_{n-1}))\|^2) + \lambda_1\lambda_{S_2}\lambda_{\mathcal{D}_F}(\|y_n - y_{n-1}\|^2 + \|x_n - x_{n-1} + \lambda_1(S(x_n, v_n) - S(x_{n-1}, v_{n-1}))\|^2)$$
$$= \|x_n - x_{n-1}\|^2 + \lambda_1\lambda_{S_1}\|x_n - x_{n-1}\|^2 + \lambda_1\lambda_{S_2}\lambda_{\mathcal{D}_F}\|y_n - y_{n-1}\|^2 + (\lambda_1\lambda_{S_1} + \lambda_1\lambda_{S_2}\lambda_{\mathcal{D}_F})\|x_n - x_{n-1} + \lambda_1(S(x_n, v_n) - S(x_{n-1}, v_{n-1}))\|^2,$$

$$[1 - \lambda_1(\lambda_{S_1} + \lambda_{S_2}\lambda_{\mathcal{D}_F})]\|x_n - x_{n-1} + \lambda_1(S(x_n, v_n) - S(x_{n-1}, v_{n-1}))\|^2 \leq (1 + \lambda_1\lambda_{S_1})\|x_n - x_{n-1}\|^2 + \lambda_1\lambda_{S_2}\lambda_{\mathcal{D}_F}\|y_n - y_{n-1}\|^2.$$

It follows that

$$\|x_n - x_{n-1} + \lambda_1(S(x_n, v_n) - S(x_{n-1}, v_{n-1}))\|^2 \leq \frac{(1 + \lambda_1\lambda_{S_1})}{1 - \lambda_1(\lambda_{S_1} + \lambda_{S_2}\lambda_{\mathcal{D}_F})}\|x_n - x_{n-1}\|^2 + \frac{\lambda_1\lambda_{S_2}\lambda_{\mathcal{D}_F}}{1-\lambda_1(\lambda_{S_1}+\lambda_{S_2}\lambda_{\mathcal{D}_F})}\|y_n - y_{n-1}\|^2$$
$$= \theta_1\|x_n - x_{n-1}\|^2 + \theta_2\|y_n - y_{n-1}\|^2 \leq \theta_1\|x_n - x_{n-1}\|^2 + \theta_2\|y_n - y_{n-1}\|^2 + 2\sqrt{\theta_1\theta_2}\|x_n - x_{n-1}\|\|y_n - y_{n-1}\|,$$

$$\|x_n - x_{n-1} + \lambda_1(S(x_n, v_n) - S(x_{n-1}, v_{n-1}))\| \leq \sqrt{\theta_1}\|x_n - x_{n-1}\| + \sqrt{\theta_2}\|y_n - y_{n-1}\|, \quad (19)$$

where

$$\theta_1 = \frac{(1+\lambda_1\lambda_{S_1})}{1-\lambda_1(\lambda_{S_1}+\lambda_{S_2}\lambda_{\mathcal{D}_F})}, \ \theta_2 = \frac{\lambda_1\lambda_{S_2}\lambda_{\mathcal{D}_F}}{1-\lambda_1(\lambda_{S_1}+\lambda_{S_2}\lambda_{\mathcal{D}_F})}.$$

Using (17) and (19), (16) becomes

$$\|z'_{n+1} - z'_n\| \leq (K_1 + \sqrt{\theta_1})\|x_n - x_{n-1}\| + \sqrt{\theta_2}\|y_n - y_{n-1}\|. \quad (20)$$

Again, from Algorithm 3.1, we have

$$\|z''_{n+1} - z''_n\| = \|H_2(A_2(y_n), B_2(y_n)) - \lambda_2 T(u_n, y_n) - [H_2(A_2(y_{n-1}), B_2(y_{n-1})) - \lambda_2 T(u_{n-1}, y_{n-1})]\|$$
$$\leq \|y_n - y_{n-1} + [H_2(A_2(y_n), B_2(y_n)) - H_2(A_2(y_{n-1}), B_2(y_{n-1}))]\| + \|y_n - y_{n-1} + \lambda_2(T(u_n, y_n) - T(u_{n-1}, y_{n-1}))\|. \quad (21)$$

Using the same argument as for (17), we have

$$\|y_n - y_{n-1} + H_2(A_2(y_n), B_2(y_n)) - H_2(A_2(y_{n-1}), B_2(y_{n-1}))\| \leq K_2\|y_n - y_{n-1}\|^2, \quad (22)$$

where $K_2 = \sqrt{\frac{1+2(s_2+t_2)+3r_2}{1-r_2}}$.

Since T is Lipschitz continuous in both the arguments, E is $\mathcal{D}$-Lipschitz continuous, we have

$$\begin{aligned}\|T(u_n,y_n)-T(u_{n-1},y_{n-1})\| &\le \|T(u_n,y_n)-T(u_{n-1},y_n)\| \\ &\quad + \|T(u_{n-1},y_n) \\ &\quad - T(u_{n-1},y_{n-1})\| \\ &\le \lambda_{T_1}\|u_n-u_{n-1}\| + \lambda_{T_2}\|y_n-y_{n-1}\| \\ &\le \lambda_{T_1}\mathcal{D}(E(x_n),E(x_{n-1}))\|u_n-u_{n-1}\| + \\ &\le \lambda_{T_1}\lambda_{\mathcal{D}_E}\|x_n-x_{n-1}\| + \\ \lambda_{T_2}\|y_n-y_{n-1}\|. \end{aligned} \quad (23)$$

Using (23) and Proposition 2.1, we have

$$\begin{aligned}\|y_n-y_{n-1}+\lambda_2(T(u_n,y_n)-T(u_{n-1},y_{n-1}))\|^2 &\le \|y_n-y_{n-1}\|^2 \\ +2\lambda_2\langle T(u_n,y_n)-T(u_{n-1},y_{n-1}), j(y_n-y_{n-1} &+ \lambda_2(T(u_n,y_n) \\ &- T(u_{n-1},y_{n-1})))\rangle \\ \le \|y_n-y_{n-1}\|^2 + 2\lambda_2[\|T(u_n,y_n) &- T(u_{n-1},y_{n-1})\| \\ \times \|y_n-y_{n-1}+\lambda_2(T(u_n,y_n)-T(u_{n-1},y_{n-1}))\|] &\le \|y_n-y_{n-1}\|^2 + 2\lambda_2(\lambda_{T_1}\lambda_{\mathcal{D}_E}\|x_n- \\ x_{n-1}\| + \lambda_{T_2}\|y_n-y_{n-1}\|) &\times \|y_n-y_{n-1}+\lambda_2(T(u_n,y_n)- \\ T(u_{n-1},y_{n-1}))\| &\le \|y_n-y_{n-1}\|^2 + \lambda_2\lambda_{T_1}\lambda_{\mathcal{D}_E}(\|x_n- \\ x_{n-1}\|^2 + \|y_n-y_{n-1} &\end{aligned}$$

$$\begin{aligned}+\lambda_2(T(u_n,y_n)-T(u_{n-1},y_{n-1}))\|^2) &+\lambda_2\lambda_{T_2}(\|y_n-y_{n-1}\|^2 + \\ \|y_n-y_{n-1}+\lambda_2(T(u_n,y_n)-T(u_{n-1},y_{n-1}))\|^2) &\le \|y_n-y_{n-1}\|^2 + \lambda_2\lambda_{T_1}\lambda_{\mathcal{D}_E}(\|x_n- \\ x_{n-1}\|^2 + \lambda_2\lambda_{T_2}\|y_n-y_{n-1}\|^2 &+\lambda_2(\lambda_{T_1}\lambda_{\mathcal{D}_E}+\lambda_{T_2})\|y_n-y_{n-1}+ \\ \lambda_2(T(u_n,y_n)-T(u_{n-1},y_{n-1}))\|^2, &\end{aligned}$$

$$\begin{aligned}[1-\lambda_2(\lambda_{T_1}\lambda_{\mathcal{D}_E}+\lambda_{T_2})]\|y_n-y_{n-1}+\lambda_2(T(u_n,y_n) &- T(u_{n-1},y_{n-1}))\|^2 \\ \le \lambda_2\lambda_{T_1}\lambda_{\mathcal{D}_E}\|x_n-x_{n-1}\|^2 + 1 + \lambda_2\lambda_{T_2}\|y_n- &\\ y_{n-1}\|^2. &\end{aligned}$$

It follows that

$$\|y_n-y_{n-1}+\lambda_2(T(u_n,y_n)-T(u_{n-1},y_{n-1}))\|^2 \le \frac{\lambda_2\lambda_{T_1}\lambda_{\mathcal{D}_E}}{1-\lambda_2(\lambda_{T_1}\lambda_{\mathcal{D}_E}+\lambda_{T_2})}\|x_n-x_{n-1}\|^2$$

$$+\frac{1+\lambda_2\lambda_{T_2}}{1-\lambda_2(\lambda_{T_1}\lambda_{\mathcal{D}_E}+\lambda_{T_2})}\|y_n-y_{n-1}\|^2$$

$$= \theta_3\|x_n-x_{n-1}\|^2 + \theta_4\|y_n-y_{n-1}\|^2$$

$$\le \theta_3\|x_n-x_{n-1}\|^2 + \theta_4\|y_n-y_{n-1}\|^2$$

$$+2\sqrt{\theta_3\theta_4}\|x_n-x_{n-1}\|\|y_n-y_{n-1}\|$$

$$\|y_n-y_{n-1}+\lambda_2(T(u_n,y_n)-T(u_{n-1},y_{n-1}))\| \le \sqrt{\theta_3}\|x_n-x_{n-1}\| + \sqrt{\theta_4}\|y_n-y_{n-1}\|, \quad (24)$$

where

$\theta_3 = \frac{\lambda_2\lambda_{T_1}\lambda_{\mathcal{D}_E}}{1-\lambda_2(\lambda_{T_1}\lambda_{\mathcal{D}_E}+\lambda_{T_2})}$, $\theta_4 = \frac{1+\lambda_2\lambda_{T_2}}{1-\lambda_2(\lambda_{T_1}\lambda_{\mathcal{D}_E}+\lambda_{T_2})}$.

Using (22) and (24), (21) becomes

$$\|z''_{n+1}-z''_n\| \le \sqrt{\theta_3}\|x_n-x_{n-1}\| + (K_2+\sqrt{\theta_4})\|y_n-y_{n-1}\|. \quad (25)$$

Adding (20) and (25), we have

$$\begin{aligned}\|z'_{n+1}-z'_n\| + \|z''_{n+1}-z''_n\| &\le \left(K_1+\sqrt{\theta_1}+\sqrt{\theta_3}\right)\|x_n \\ &\quad - x_{n-1}\| \\ &+\left(K_2+\sqrt{\theta_2}+\sqrt{\theta_4}\right)\|y_n-y_{n-1}\|. \end{aligned} \quad (26)$$

Also, from (9) and (10), we have

$$\|x_n-x_{n-1}\| = \left\|R^{H_1(.,.)}_{\lambda_1,M_1(.,.)}(z'_n) - R^{H_1(.,.)}_{\lambda_1,M_1(.,.)}(z'_{n-1})\right\| \le L_1\|z'_n-z'_{n-1}\| \quad (27)$$

and

$$\|y_n-y_{n-1}\| = \left\|R^{H_2(.,.)}_{\lambda_2,M_2(.,.)}(z''_n) - R^{H_2(.,.)}_{\lambda_2,M_2(.,.)}(z''_{n-1})\right\| \le L_2\|z''_n-z''_{n-1}\| \quad (28)$$

Using (27) and (28), (26) becomes

$$\begin{aligned}\|z'_{n+1}-z'_n\| + \|z''_{n+1}-z''_n\| &\le L_1\left(K_1+\sqrt{\theta_1}+\sqrt{\theta_3}\right)\|z'_n \\ &\quad - z'_{n-1}\| \\ &+L_2\left(K_2+\sqrt{\theta_2}+\sqrt{\theta_4}\right)\|z''_n-z''_{n-1}\| \\ \le \zeta(\|z'_n-z'_{n-1}\| + \|z''_n-z''_{n-1}\|) &\end{aligned} \quad (29)$$

where $\zeta = max\{L_1\left(K_1+\sqrt{\theta_1}+\sqrt{\theta_3}\right), L_2\left(K_2+\sqrt{\theta_2}+\sqrt{\theta_4}\right)\}$.

By (15), we know that $0<\zeta<1$ and so (29) implies that $\{z'_n\}$ and $\{z''_n\}$ are both Cauchy sequences. Thus, there exist $z'\epsilon X_1$ and $z''\epsilon X_2$ such that $z'_n \to z'$ and $z''_n \to z''$ as $n \to \infty$. From (27) and (28), it follows that $\{x_n\}$ and $\{y_n\}$ are also Cauchy sequences in X_1 and X_2, respectively, that is, there exist $x\epsilon X_1$, $y\epsilon X_2$ such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$. Also, from (11) and (12) we have

$$\|u_{n+1}-u_n\| \le \mathcal{D}(E(x_{n+1}),E(x_n)) \le \lambda_{\mathcal{D}_E}\|x_{n+1}-x_n\|,$$

$$\|v_{n+1}-v_n\| \le \mathcal{D}(F(y_{n+1}),F(y_n)) \le \lambda_{\mathcal{D}_F}\|y_{n+1}-y_n\|,$$

and hence, $\{u_n\}$ and $\{v_n\}$ are also Cauchy sequences such that $u_n \to u$ and $v_n \to v$, respectively. Now we will show that $u\epsilon E(x)$ and $v\epsilon F(y)$. Since $u_n\epsilon E(x_n)$ and

$$\begin{aligned}&d(u_n,E(x)) \\ &\le max\{d(u_n,E(x)), \sup_{w_1\epsilon E(x)}(E(x_n),w_1)\}\end{aligned}$$

$$\leq \max\{\sup_{w_2 \epsilon E(x_n)} (w_2, E(x)), \sup_{w_1 \epsilon E(x)} (E(x_n), w_1)\}$$
$$= \mathcal{D}(E(x_n), E(x)),$$

we have

$$d\big(u, E(x)\big) \leq \|u - u_n\| + d\big(u_n, E(x)\big)$$
$$\leq \|u - u_n\| + \mathcal{D}(E(x_n), E(x))$$
$$\leq \|u - u_n\| + \lambda_{\mathcal{D}_E}\|x_n - x\| \to 0, as\ n \to \infty,$$

which implies that $d\big(u, E(x)\big) = 0$. Since $E(x) \epsilon CB(X)$, it follows that $u \epsilon E(x)$. Similarly, we can show that $v \epsilon F(y)$. By continuity of $H_1, H_2, A_1, A_2, B_1, B_2, M_1, M_2, E, F, S, T$, $R^{H_1(.,.)}_{\lambda_1, M_1(.,.)}$, $R^{H_2(.,.)}_{\lambda_2, M_2(.,.)}$ and Algorithm 3.1, we have

$$z' = H_1\big(A_1(x), B_1(x)\big) - \lambda_1 S(x, v)$$
$$= H_1\left(A_1\left(R^{H_1(.,.)}_{\lambda_1, M_1(.,.)}(z')\right), B_1\left(R^{H_1(.,.)}_{\lambda_1, M_1(.,.)}(z')\right)\right)$$
$$- \lambda_1 S(x, v),$$

$$z'' = H_2\big(A_2(y), B_2(y)\big) - \lambda_2 T(u, y)$$
$$= H_2\left(A_2\left(R^{H_2(.,.)}_{\lambda_2, M_2(.,.)}(z'')\right), B_1\left(R^{H_2(.,.)}_{\lambda_2, M_2(.,.)}(z'')\right)\right)$$
$$- \lambda_2 T(u, y).$$

By Proposition (3.1) the required result follows.

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