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# On generalized $I$ −statistical convergenceof order $\alpha$

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**Abstract**

The goal of this paper is to generalize the recently introduced summability method and introduce $I_\lambda$ −double statistical convergence of order $\alpha$ by using ideal. We also investigate certain properties of this convergence.

***Keywords***: Ideal; filter; $I-$ double statistical convergence of order $\alpha$; $I_\lambda -$ double statistical convergence of order $\alpha$, closed subspace

## 1. Introduction

The idea of statistical convergence was given by Zygmund [1] in the first edition of his monograph published in Warsaw in 1935. The concept of statistical convergence was introduced by Steinhause [2] and Fast [3] and later reintroduced by Schoenberg [4] independently. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory, number theory, measure theory, trigonometric theory, turnpike theory and Banach spaces. In recent years generalization of statistical convergence has appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on local compact spaces.

In [5], Kostyrko et al. introduced the concept of $I$ -convergence of sequences in a metric space and studied some properties of such convergence. Note that $I$-convergence is an interesting generalization of statistical convergence. More investigations in this direction and more applications of ideals can be found in [6-11] where many important references can be found.

Quite recently in [6, 10] we used ideals to introduce the concepts of $\boldsymbol{I}$ −statistical convergence, $\boldsymbol{I}$ −lacunary statistical convergence and $\boldsymbol{I_\lambda} -$ statistical convergence and investigated their properties. On the other hand, in [12-14] a different direction was taken to the study of these important summability methods where the notions of statistical convergence of order $\boldsymbol{\alpha}$ and $\boldsymbol{\lambda}$ −statistical convergence of order $\boldsymbol{\alpha}$ were introduced and studied.

If $\mathbb{N}$ denotes the set of natural numbers and $\boldsymbol{K} \subset \mathbb{N}$, then $\boldsymbol{K(m,n)}$ denotes the cardinality of the set $\boldsymbol{K} \cap [\boldsymbol{m},\boldsymbol{n}]$. The upper and lower natural density of the subset $\boldsymbol{K}$ is defined by

$$\bar{\boldsymbol{d}}(\boldsymbol{K}) = \lim_{n\to\infty} \sup \frac{K(1,n)}{n} \; and \; \underline{d}(K) = \lim_{n\to\infty} \inf \frac{K(1,n)}{n}.$$

If $\bar{\boldsymbol{d}}(\boldsymbol{K}) = \underline{\boldsymbol{d}}(\boldsymbol{K})$ then we say that the natural density of K exists and it is denoted simply by $\boldsymbol{d}(\boldsymbol{K})$. Clearly $\boldsymbol{d}(\boldsymbol{K}) = \lim_{n\to\infty} \frac{K(1,n)}{n}$.

A sequence $(\boldsymbol{x_k})$ of real numbers is said to be statistically convergent to $\boldsymbol{L}$ if for arbitrary $\boldsymbol{\epsilon} > \boldsymbol{0}$ the set $\boldsymbol{K(\epsilon)} = \{\boldsymbol{k} \in \mathbb{N}: |\boldsymbol{x_k}| \geq \boldsymbol{\epsilon}\}$ has natural density zero. Statistical convergence turned out to be one of the most active areas of research in summability theory after the work of Fridy [15] and Salat [16]. The more general idea of $\boldsymbol{\lambda}$ −statistical convergence was introduced by Mursaleen in [17].

Mursaleen and Edely [18] defined double statistical convergence and examined some properties. Quite recently E. Savas [19] introduced the concepts of $\boldsymbol{I} -$ double statistical convergence, and $\boldsymbol{I_\lambda} -$ double statistical convergence and investigated their properties. Subsequently many interesting investigations have been done by various authors on this convergence (see, for example [20-26]).

In this paper we introduce a new and further general notion namely, $\boldsymbol{I_{\lambda}}$- double statistical convergence of order $\boldsymbol{\alpha}$. We mainly investigate some basic properties of this new summability method.

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## 2. Preliminaries

The following definitions and notions will be needed in the sequel.

**Definition 1.** A family $I \subset 2^{\mathbb{N}}$ is said to be an ideal of $\mathbb{N}$ if the following conditions hold:
(a) $A, B \in I$ implies $A \cup B \in I$,
(b) $A \in I, B \subset I$ implies $B \in I$.

**Definition 2.** A non-empty family $F \subset 2^{\mathbb{N}}$ is said to be a filter of $\mathbb{N}$ if the following conditions hold:
(a) $\emptyset \notin F$,
(b) $A, B \in F$ implies $A \cap B \in F$,
(c) $A \in F, B \subset \mathrm{F}$ implies $B \in F$.

If $I$ is a proper ideal of $\mathbb{N}$ (i.e. $\mathbb{N} \notin I$), then the family of sets $F(I) = \{M \subset \mathbb{N}: \exists A \in I: M = N \backslash A\}$ is a filter of $\mathbb{N}$. It is called the filter associated with the ideal.

**Definition 3.** A proper ideal $I$ is said to be admissible if $\{n\} \in I$ for each $n \in \mathbb{N}$.

Throughout $I$ will stand for a proper admissible ideal of $N$ and by sequence we always mean sequences of real numbers.

**Definition 4.** (see [6]). Let $I \subset 2^{\mathbb{N}}$ be a proper admissible ideal in $\mathbb{N}$.
(i) The sequence $(x_k)$ of elements of $\mathbb{R}$ is said to be $I-$convergent to $L \in \mathbb{R}$ if for each $\varepsilon > 0$, the set $A(\varepsilon) = \{k \in \mathbb{N}: |x_k - L| \geq \epsilon\} \in I$.
(ii) The sequence $(x_k)$ of elements of $\mathbb{R}$ is said to be $I^*-$convergent to $L \in \mathbb{R}$ if there exists $M \in F(I)$ such that $(x_k)_{k \in M}$ converges to $L$.

A double sequence $x = (x_{kl})$ of real numbers, $k, l \in \mathbb{N}$ the set of all positive integers, is said to be convergent in the Pringsheim's sense or $P-$convergent if for each $\varepsilon > 0$ there exists $\mathbf{N} \in \mathbb{N}$ such that $|x_{kl} - L| < \varepsilon$ whenever $k, l \geq N$ and $L$ is called Pringsheim limit (denoted by $P - \lim x = L$), [27].

A double sequence $x$ is bounded if there exists a positive number $M$ such that $|x_{kl}| < M$ for all $j$ and $k$, i.e., if

$$\|x\|_{(\infty,2)} = \sup_{j,k}|x_{jk}| < \infty.$$

Note that in contrast to the case for single sequences, a convergent double sequence need not be bounded. Let us denote by $l_\infty^2$, the space of all bounded double sequences.

Let $K \subseteq \mathbb{N} \times \mathbb{N}$ be a two dimensional set of positive integers and let $K_{m,n}$ be the numbers of $(i,j)$ in $K$ such that $i \leq n$ and $j \leq m$. Then the lower asymptotic density of $K$ is defined as

$$P - \lim \inf_{m,n} \frac{K_{m,n}}{mn} = \delta_2(K).$$

In the case when the sequence $\left(\frac{K_{m,n}}{mn}\right)_{m,n=1,1}^{\infty,\infty}$ has a limit, we say that $K$ has a natural density and is defined as

$$P - \lim_{m,n} \frac{K_{m,n}}{mn} = \delta_2(K).$$

For example, let $K = \{(i^2, j^2): i, j \in \mathbb{N}\}$, where $\mathbb{N}$ is the set of natural numbers. Then

$$\delta_2(K) = P - \lim_{m,n} \frac{K_{m,n}}{mn} \leq P - \lim_{m,n} \frac{\sqrt{m}\sqrt{n}}{mn} = 0$$

(i.e. the set $K$ has double natural density zero).

Recently Mursaleen and Edely [18] presented the notion of statistical convergence for double sequence $x = (x_{kl})$ as follows: A real double sequence $x = (x_{kl})$ is said to be statistically convergent to $L$, provided that for each $\varepsilon > 0$

$$\lim_{m,n} \frac{1}{mn} |\{(k,l): k \leq m \text{ and } l \leq n, |x_{kl} - L| \geq \varepsilon\}| = 0.$$

A nontrivial ideal of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $I$ for each $i \in \mathbb{N}$. It is evident that a strongly admissible ideal is also admissible. Let

$$I_0 = \{A \subset \mathbb{N} \times \mathbb{N}: (\exists m(A) \in \mathbb{N}(k, l \geq m(A) \Rightarrow (k,l) \notin A))\}.$$

Then $I_0$ is a nontrivial strongly admissible ideal and clearly an ideal $I$ is if and only if $I_0 \subset I$. Now let $I$ be a nontrivial admissible ideal in $\mathbb{N} \times \mathbb{N}$. A double sequence $x = (x_{kl})$ of real number is said to be convergent to the number $\ell$ with respect to the ideal $I$, if for each $\varepsilon > 0$,

$$A(\varepsilon) = \{(k,l) \in \mathbb{N} \times \mathbb{N}: |x_{kl} - L| \geq \epsilon\} \in I.$$

In this case we write $I - \lim_{kl} x_{kl} = L$.

Note that if $I$ is the ideal $I_0$, then $I-$ convergence coincides with the convergence in Pringsheim's sense and if we take $I_d = \{A \subset \mathbb{N} \times \mathbb{N}: \delta_2(A) = 0\}$ then $I_d-$convergence becomes statistical convergence for double sequences.

## 3. Main results

We now introduce our main definitions.
Definition 5. A sequence $x = (x_{kl})$ is said to be $I-$statistically convergent of order $\alpha$ to $L$ or $s_2(I)^\alpha -$ convergent to $L$, where $0 < \alpha \leq 1$, if for each $\varepsilon > 0$ and $\delta > 0$

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N}: \frac{1}{(mn)^{\alpha}}|\{k \le m \text{ and } l \le m: |x_{kl}-L| \ge \varepsilon\}| \ge \delta\right\} \in I.$$

In this case, we write $x_{kl} \to L(s_2(I)^{\alpha})$. The class of all $I-$ double statistically convergent of order $\alpha$ sequences will be denoted by simply $s_2(I)^{\alpha}$.

**Remark 1.** For $I = I_{fin}, s_2(I)^{\alpha}$ $-$convergence coincides with double statistical convergence of order $\alpha$. For an arbitrary ideal $I$ and for $\alpha = 1$ it coincides with $I-$ double statistical convergence [1]. When $I = I_{fin}$ and $\alpha = 1$ it becomes only double statistical convergence, which is defined by Mursaleen and Edely.

In [17], Mursaleen introduced the idea of $\lambda-$ statistical convergence for single sequence as follows:

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to $\infty$ such that

$$\lambda_{n+1} \le \lambda_n + 1, \lambda_1 = 1.$$

The generalized de la Vallee-Pousin mean is defined by

$$t_n(x) := \frac{1}{\lambda_n}\sum_{k\in I_n} x_k$$

where $I_n = [n-\lambda_n+1, n]$.

A sequence $x = (x_k)$ is said to be $(V,\lambda)$ $-$summable to a number $L$ if

$$t_n(x) \to \ell \text{ as } n \to \infty.$$

The number sequence $x = (x_k)$ is said to be $\lambda-$ statistically convergent to the number $\mathbf{L}$ if for each $\varepsilon > 0$,

$$\lim_n \frac{1}{\lambda_n}|\{n-\lambda_n+1 \le k \le n, |x_k - L| \ge \varepsilon\}| = 0$$

In this case we write $s_\lambda - lim_k x_k = L$ and we denote the set of all $\lambda-$ statistically convergent sequences by $\mathbf{s}_\lambda$.

Let $\lambda = (\lambda_m)$ and $\mu = (\mu_n)$ be two non-decreasing sequences of positive numbers, both of which tend to $\infty$ as $m$ and $n$ approach $\infty$, respectively. Also, let $\lambda_{m+1} \le \lambda_m + 1, \lambda_1 = 1$ and $\mu_{n+1} \le \mu_n + 1$, $\mu_1 = 1$. The collection of such sequence $\bar{\lambda}_{mn}$ will be denoted by $\boldsymbol{\Delta}$.

We write the generalized double de la Valee-Poussin mean by

$$t_{m,n}(x) := \frac{1}{\lambda_m^{\alpha}\mu_n^{\alpha}}\sum_{k\in I_m, l\in J_n} x_{k,l}$$

where $I_m = [m-\lambda_m+1, m]$ and $J_n = [n-\mu_n+1, n]$.

A sequence $x = (x_{kl})$ is said to be $[V,\bar{\lambda}]^{\alpha}(I)$ $-$summable to a number $L \in X$, if

$$I - \lim_{m,n} t_{m,n}(x) \to L$$

i.e. for any $\delta > 0$,

$$\{(m,n) \in \mathbb{N}\times\mathbb{N}: |t_{m,n}(x) - L| \ge \delta\} \in I.$$

Throughout this paper we shall denote $\lambda_m\mu_n$ by $\bar{\lambda}_{mn}$ and $(k \in I_m, l \in I)$ by $(k,l) \in I_{m,n}$.

If $I = I_{fin}$, $[V,\bar{\lambda}]^{\alpha}(I)$ $-$summability becomes $[V,\bar{\lambda}]$ summability. If we take $\alpha = 1$, $[V,\bar{\lambda}]^{\alpha}(I)$ $-$summability reduce to $[V,\bar{\lambda}]$ summability.

**Definition 6.** A sequence $x = (x_{kl})$ is said to be $I_\lambda$-double statistically convergent of order $\alpha$ or $s_{\bar{\lambda}}(I)^{\alpha}$ convergent to L, if for each $\varepsilon > 0$ and $\delta > 0$

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N}: \frac{1}{\bar{\lambda}_{mn}^{\alpha}}|\{(k,l) \in I_{nm}: |x_{kl}-L| \ge \varepsilon\}| \ge \delta\right\} \in I.$$

In this case we write $s_{\bar{\lambda}}(I)^{\alpha} - limx = L$ or $x_{kl} \to L(s_{\bar{\lambda}}(I)^{\alpha})$.

**Remark 2.** If we take $I = I_{fin} = \{A \subseteq N: A \textbf{ is a finite subset}\}$, then $I_f$ is a non-trivial admissible ideal of N, and $s_{\bar{\lambda}}(I)^{\alpha}$convergence again coincides with $\lambda-$ double statistically convergent of order $\alpha$. For an arbitrary ideal $I$ and for $\alpha = 1$ it coincides with $I_\lambda$-double statistically convergence [1]. Finally, for $I = I_{fin}$ and $\alpha = 1$ it becomes $\lambda$ $-$double statistically convergence. Also, note that taking $\bar{\lambda}_{mn} = mn$ we get definition 5 from definition 6.

We shall denote by $s_{\bar{\lambda}}(I)^{\alpha}$ and $[V,\bar{\lambda}]^{\alpha}(I)$ the collection of all $I_\lambda-$ double statistically convergent of order $\alpha$ and $[V,\bar{\lambda}](I)$ $-$convergent of order $\alpha$ sequences respectively.

**Theorem 1.** Let $0 < \alpha \le \beta \le 1$. Then $s_{\bar{\lambda}}(I)^{\alpha} \subset s_{\bar{\lambda}}(I)^{\beta}$.

**Proof:** Let $0 < \alpha \le \beta \le 1$. Then

$$\frac{|\{(k,l) \in I_{mn}: |x_{kl}-L| \ge \epsilon\}|}{\bar{\lambda}_{mn}^{\beta}} \le \frac{|\{(k,l) \in I_{mn}: |x_{kl}-L| \ge \epsilon\}|}{\bar{\lambda}_{mn}^{\alpha}}$$

and so for any $\delta > 0$,

$$\left\{(\boldsymbol{m},\boldsymbol{n}) \in \mathbb{N} \times \mathbb{N}: \frac{|\{(\boldsymbol{k},\boldsymbol{l}) \in \boldsymbol{I_{mn}}: |\boldsymbol{x_{kl}} - \boldsymbol{L}| \geq \boldsymbol{\epsilon}\}|}{\bar{\boldsymbol{\lambda}}_{\boldsymbol{mn}}^{\boldsymbol{\beta}}} \geq \boldsymbol{\delta}\right\} \subset \left\{(\boldsymbol{m},\boldsymbol{n}) \in \mathbb{N} \times \mathbb{N}: \frac{|\{(\boldsymbol{k},\boldsymbol{l}) \in \boldsymbol{I_{mn}}: |\boldsymbol{x_{kl}} - \boldsymbol{L}| \geq \boldsymbol{\epsilon}\}|}{\bar{\boldsymbol{\lambda}}_{\boldsymbol{mn}}^{\boldsymbol{\alpha}}} \geq \boldsymbol{\delta}\right\}$$

Hence if the set on the right hand side belongs to the ideal $\boldsymbol{I}$ then obviously the set on the left hand side also belongs to $\boldsymbol{I}$. This shows that $\boldsymbol{s_{\bar{\lambda}}(I)^{\alpha}} \subset \boldsymbol{s_{\bar{\lambda}}(I)^{\beta}}$.

**Remark 3.** We can see that the inclusion is sometimes strict. Suppose that $\boldsymbol{\alpha},\boldsymbol{\beta}$ are such that there is a $\boldsymbol{k},\boldsymbol{l} \in \boldsymbol{N} \times \boldsymbol{N}$ with $\boldsymbol{\alpha} < \frac{\boldsymbol{1}}{\boldsymbol{k},\boldsymbol{l}} < \boldsymbol{\beta}$, and $\boldsymbol{I} = \boldsymbol{I_{fin}}$. To prove that the inclusion is strict consider the sequence $\boldsymbol{x} = (\boldsymbol{x_{kl}})$ defined by
$\boldsymbol{x_{kl}} = \boldsymbol{1}$, if $\boldsymbol{k} = \boldsymbol{i^p}$ and $\boldsymbol{l} = \boldsymbol{j^q}$
$\boldsymbol{x_{kl}} = \boldsymbol{0}$, if $\boldsymbol{k} \neq \boldsymbol{i^p}$ and $\boldsymbol{l} \neq \boldsymbol{j^q}, \boldsymbol{i},\boldsymbol{j} \in \mathbb{N} \times \mathbb{N}$.

Then $\boldsymbol{s_2(I)^{\beta}} - \mathbf{lim}\boldsymbol{x_{k,l}} = \boldsymbol{0}$ i.e. $\boldsymbol{x} \in \boldsymbol{s_2(I)^{\beta}}$ and so $\boldsymbol{x} \in \boldsymbol{s_{\bar{\lambda}}(I)^{\beta}}$, but $\boldsymbol{x} \notin \boldsymbol{s_2(I)^{\alpha}}$ and so not in $\boldsymbol{s_{\bar{\lambda}}(I)^{\alpha}}$.

**Corollary 1.** If a sequence is $\boldsymbol{I_{\bar{\lambda}}}$ −statistically convergent of order $\boldsymbol{\alpha}$ to $\boldsymbol{L}$ for some $\boldsymbol{0} < \boldsymbol{\alpha} \leq \boldsymbol{1}$ then it is $\boldsymbol{I_{\bar{\lambda}}}$ −statistically convergent to $\boldsymbol{L}$, i.e. $\boldsymbol{S_{\bar{\lambda}}}\ (\boldsymbol{I})^{\boldsymbol{\alpha}} \subset \boldsymbol{S_{\bar{\lambda}}}\ (\boldsymbol{I})$
We also show that

**Theorem 2.** Let $\boldsymbol{0} < \boldsymbol{\alpha} \leq \boldsymbol{\beta} \leq \boldsymbol{1}$. Then
(i) $\boldsymbol{s_2(I)^{\alpha}} \subset \boldsymbol{s_2(I)^{\beta}}$
(ii) $\circledcirc \boldsymbol{V}, \bar{\boldsymbol{\lambda}} \circledcirc^{\boldsymbol{\alpha}}\ (\boldsymbol{I}) \subset \circledcirc \boldsymbol{V}, \bar{\boldsymbol{\lambda}} \circledcirc^{\boldsymbol{\beta}}\ (\boldsymbol{I})$
(iii) In particular, $\boldsymbol{S(I)^{\alpha}} \subset \boldsymbol{s(I)}$ and $[\boldsymbol{V},\boldsymbol{\lambda}]^{\boldsymbol{\alpha}}\ (\boldsymbol{I}) \subset [\boldsymbol{V},\bar{\boldsymbol{\lambda}}]\ (\boldsymbol{I})$.

**Theorem 3.** $\boldsymbol{s_{\bar{\lambda}}}\ (\boldsymbol{I})^{\boldsymbol{\alpha}} \cap \boldsymbol{l_{\infty}^2}$ is closed subset of $\boldsymbol{l_{\infty}^2}$ where as usual, $\boldsymbol{l_{\infty}^2}$ is the banach space of all bounded real squences endowed with the sup norm.

**Proof:** Suppose that $(\boldsymbol{x^{nm}}) \subset \boldsymbol{s_{\bar{\lambda}}}\ (\boldsymbol{I})^{\boldsymbol{\alpha}} \cap \boldsymbol{l_{\infty}^2}, \boldsymbol{0} < \boldsymbol{\alpha} \leq \boldsymbol{1}$ is a convergent sequence and converges to $\boldsymbol{x} \in \boldsymbol{l_{\infty}^2}$. We need to prove that $\boldsymbol{x} \in \boldsymbol{s_{\bar{\lambda}}}\ (\boldsymbol{I})^{\boldsymbol{\alpha}} \cap \boldsymbol{l_{\infty}^2}$ since $\boldsymbol{s_{\bar{\lambda}}}\ (\boldsymbol{I})^{\boldsymbol{\alpha}} \subset \boldsymbol{s_{\bar{\lambda}}}\ (\boldsymbol{I})$ and $\boldsymbol{s_{\bar{\lambda}}}\ (\boldsymbol{I}) \cap \boldsymbol{l_{\infty}^2}$ is closed in $\boldsymbol{l_{\infty}^2}$, (see, theorem 2.4 [1]). So $\boldsymbol{x} \in \boldsymbol{s_{\lambda}}\ (\boldsymbol{I}) \cap \boldsymbol{l_{\infty}^2}$. Again since $\boldsymbol{x^{nm}} \in \boldsymbol{s_{\bar{\lambda}}}\ (\boldsymbol{I})^{\boldsymbol{\alpha}} \subset \boldsymbol{s_{\bar{\lambda}}}\ (\boldsymbol{I})$ for all $\boldsymbol{m} = \boldsymbol{1},\boldsymbol{2},\boldsymbol{3},\ldots$, and $\boldsymbol{n} = \boldsymbol{1},\boldsymbol{2},\boldsymbol{3},\ldots$, $\boldsymbol{x^{nm}}$ is $\boldsymbol{I_{\bar{\lambda}}}$ −statistically convergent to some number $\boldsymbol{L_{mn}}$ for $\boldsymbol{m} = \boldsymbol{1},\boldsymbol{2},\boldsymbol{3},\ldots$, and $\boldsymbol{n} = \boldsymbol{1},\boldsymbol{2},\boldsymbol{3},\ldots$. We first show that the sequence $(\boldsymbol{L_{mn}})$ is convergent to some number $\boldsymbol{L}$ and the sequence $\boldsymbol{x} = (\boldsymbol{x_{kl}})$ is $\boldsymbol{I_{\bar{\lambda}}}$ −statistically convergent of order $\boldsymbol{\alpha}$ to $\boldsymbol{L}$. Take a positive strictly decreasing sequence $(\boldsymbol{\varepsilon_{mn}})$ converging to 0. Choose a positive integer $\boldsymbol{m},\boldsymbol{n}$ such that $\|\boldsymbol{x} - \boldsymbol{x^{mn}}\|_{\infty} < \frac{\boldsymbol{\epsilon_{mn}}}{\boldsymbol{4}}$. Let $\boldsymbol{0} < \boldsymbol{\delta} < \frac{\boldsymbol{1}}{\boldsymbol{3}}$. Write

$$\boldsymbol{A} = \left\{(\boldsymbol{m},\boldsymbol{n}) \in \boldsymbol{N} \times \boldsymbol{N}: \frac{\boldsymbol{1}}{\bar{\boldsymbol{\lambda}}_{\boldsymbol{mn}}} \left|\left\{(\boldsymbol{k},\boldsymbol{l}) \in \boldsymbol{I_{mn}} : |\boldsymbol{x_{kl}^{mn}} - \boldsymbol{L_{mn}}| \geq \frac{\boldsymbol{\epsilon_{mn}}}{\boldsymbol{4}}\right\}\right| < \boldsymbol{\delta}\right\} \in \boldsymbol{F(I)}$$

and

$$\boldsymbol{B} = \left\{(\boldsymbol{m},\boldsymbol{n}) \in \mathbb{N} \times \mathbb{N} : \frac{\boldsymbol{1}}{\bar{\boldsymbol{\lambda}}_{\boldsymbol{mn}}} \left|\left\{(\boldsymbol{k},\boldsymbol{l}) \in \boldsymbol{I_{mn}} : |\boldsymbol{x_{kl}^{mn}} - \boldsymbol{L_{mn}}| \geq \frac{\boldsymbol{\epsilon_{mn}}}{\boldsymbol{4}}\right\}\right| < \boldsymbol{\delta}\right\} \in \boldsymbol{F(I)}$$

Since $\boldsymbol{A} \cap \boldsymbol{B} \in \boldsymbol{F(I)}$ **and** $\boldsymbol{\phi} \notin \boldsymbol{F}\ (\boldsymbol{I})$, we can choose $\boldsymbol{m} \in \boldsymbol{A} \cap \boldsymbol{B}$. Then

$$\frac{\boldsymbol{1}}{\bar{\boldsymbol{\lambda}}_{\boldsymbol{mn}}} \left\{(\boldsymbol{k},\boldsymbol{l}) \in \boldsymbol{I_{mn}} : |\boldsymbol{x_{kl}^{mn}} - \boldsymbol{L_{mn}}| \geq \frac{\boldsymbol{\epsilon_{mn}}}{\boldsymbol{4}} \vee \left|\boldsymbol{x_{kl}^{m+1,n+1}} - \boldsymbol{L_{m+1,n+1}}\right| \geq \frac{\boldsymbol{\epsilon_{mn}}}{\boldsymbol{4}}\right\} \leq \frac{\boldsymbol{2}}{\boldsymbol{\delta}} < \boldsymbol{1}.$$

Since $\bar{\boldsymbol{\lambda}}_{\boldsymbol{mn}} \to \infty$ and $\boldsymbol{A} \cap \boldsymbol{B} \in \boldsymbol{F(I)}$ is infinite, we can actually choose the above $\boldsymbol{m},\boldsymbol{n}$ so that $\bar{\boldsymbol{\lambda}}_{\boldsymbol{mn}} > \mathbf{5}$ **(say)**. Therefore there must exist $(\boldsymbol{k},\boldsymbol{l}) \in \boldsymbol{I_{mn}}$ for which we have, $|\boldsymbol{x_{kl}^{mn}} - \boldsymbol{L_{mn}}| < \frac{\boldsymbol{\epsilon_{mn}}}{\boldsymbol{4}}$ *and* $\left|\boldsymbol{x_{kl}^{m+1,n+1}} - \boldsymbol{L_{m+1,n+1}}\right| < \frac{\boldsymbol{\epsilon_{mn}}}{\boldsymbol{4}}$.

Hence it follows that

$$|\boldsymbol{L_{mn}} - \boldsymbol{L_{m+1,n+1}}| \leq |\boldsymbol{L_{mn}} - \boldsymbol{x_{kl}^{mn}}| + |\boldsymbol{x_{kl}^{mn}} - \boldsymbol{x_{kl}^{m+1,n+1}}| + \left|\boldsymbol{x_{kl}^{m+1,n+1}} - \boldsymbol{L_{m+1,n+1}}\right|$$
$$\leq |\boldsymbol{x_{kl}^{mn}} - \boldsymbol{L_{mn}}| + \left|\boldsymbol{x_{kl}^{m+1,n+1}} - \boldsymbol{L_{m+1,n+1}}\right| + \|\boldsymbol{x} - \boldsymbol{x^{mn}}\|_{\infty} + \|\boldsymbol{x} - \boldsymbol{x^{m+1,n+1}}\|_{\infty}$$
$$< \frac{\boldsymbol{\epsilon_{mn}}}{\boldsymbol{4}} + \frac{\boldsymbol{\epsilon_{mn}}}{\boldsymbol{4}} + \frac{\boldsymbol{\epsilon_{mn}}}{\boldsymbol{4}} + \frac{\boldsymbol{\epsilon_{mn}}}{\boldsymbol{4}} = \boldsymbol{\epsilon_{mn}}.$$

This implies that $(\boldsymbol{L_{mn}})$ is a Cauchy sequence of real numbers and let $\boldsymbol{L_{mn}} \to \boldsymbol{L}$ as $\boldsymbol{m},\boldsymbol{n} \to \infty$. We prove that $\boldsymbol{x} \to \boldsymbol{L}(\boldsymbol{s_{\bar{\lambda}}(I)^{\alpha}})$. Choose $\boldsymbol{\varepsilon} > \boldsymbol{0}$ and choose $(\boldsymbol{m},\boldsymbol{n}) \in \mathbb{N} \times \mathbb{N}$ such that $\boldsymbol{\epsilon}_{\mathbf{mn}} < \frac{\boldsymbol{\epsilon}}{\boldsymbol{4}}$, $\|\boldsymbol{x} - \boldsymbol{x^{mn}}\|_{\infty} < \frac{\boldsymbol{\epsilon}}{\boldsymbol{4}}$, $|\boldsymbol{L_{mn}} - \boldsymbol{L}| < \frac{\boldsymbol{\epsilon}}{\boldsymbol{4}}$. Now since

$$\frac{\boldsymbol{1}}{\bar{\boldsymbol{\lambda}}_{\boldsymbol{\gamma\mu}}^{\boldsymbol{\alpha}}} |\{(\boldsymbol{k},\boldsymbol{l}) \in \boldsymbol{I_{\gamma,\mu}}: |\boldsymbol{x_{kl}} - \boldsymbol{L}| \geq \boldsymbol{\epsilon}\}|$$
$$\leq \frac{\boldsymbol{1}}{\bar{\boldsymbol{\lambda}}_{\boldsymbol{\gamma\mu}}^{\boldsymbol{\alpha}}} |\{(\boldsymbol{k},\boldsymbol{l}) \in \boldsymbol{I_{\gamma,\mu}}: |\boldsymbol{x_{kl}} - \boldsymbol{x_{kl}^{nm}}| + |\boldsymbol{x_{kl}^{nm}} - \boldsymbol{L_{mn}}| + |\boldsymbol{L_{mn}} - \boldsymbol{L}| \geq \boldsymbol{\epsilon}\}|$$
$$\leq \frac{\boldsymbol{1}}{\bar{\boldsymbol{\lambda}}_{\boldsymbol{\gamma\mu}}^{\boldsymbol{\alpha}}} |\{(\boldsymbol{k},\boldsymbol{l}) \in \boldsymbol{I_{\gamma,\mu}}: |\boldsymbol{x_{kl}^{nm}} - \boldsymbol{L_{mn}}| \geq \frac{\boldsymbol{\epsilon}}{\boldsymbol{2}}\}|.$$

So it follows that

$$\{(\boldsymbol{\gamma},\boldsymbol{\mu}) \in \mathbb{N} \times \mathbb{N}: \frac{\boldsymbol{1}}{\bar{\boldsymbol{\lambda}}_{\boldsymbol{\gamma\mu}}^{\boldsymbol{\alpha}}} |\{(\boldsymbol{k},\boldsymbol{l}) \in \boldsymbol{I_{\gamma,\mu}}: |\boldsymbol{x_{kl}} - \boldsymbol{L}| \geq \boldsymbol{\epsilon}\}| \geq \boldsymbol{\delta}\}$$
$$\subset \{(\boldsymbol{\gamma},\boldsymbol{\mu}) \in \mathbb{N} \times \mathbb{N}: \frac{\boldsymbol{1}}{\bar{\boldsymbol{\lambda}}_{\boldsymbol{\gamma\mu}}^{\boldsymbol{\alpha}}} |\{(\boldsymbol{k},\boldsymbol{l}) \in \boldsymbol{I_{\gamma,\mu}}: |\boldsymbol{x_{kl}} - \boldsymbol{L}$$

$| \geq \frac{\epsilon}{2}\}| \geq \boldsymbol{\delta}\}$

for any given $\boldsymbol{\delta > 0}$. This shows that $\boldsymbol{x \to L\,(s_{\bar{\lambda}}(I)^{\alpha})}$ and this completes the proof of the theorem.

**Theorem 4.** Let $\bar{\lambda} = (\bar{\lambda}_{mn}) \in \Delta$. Then $\boldsymbol{x_{kl} \to L \circledcirc V, \bar{\lambda}\circledcirc^{\alpha}\,(I)} \implies \boldsymbol{x_{kl} \to L\,(s_{\bar{\lambda}}(I)^{\alpha})}$ and the inclusion $\boldsymbol{\circledcirc V, \bar{\lambda}\circledcirc^{\alpha}\,(I) \subset s_{\bar{\lambda}}(I)^{\alpha})}$ is proper for every ideal *I*.

**Proof:** Let $\boldsymbol{\epsilon > 0}$ and $\boldsymbol{x_{kl} \longrightarrow L \circledcirc V, \bar{\lambda}\circledcirc^{\alpha}\,(I)}$. We write

$\sum_{(kl)\in I_{mn}} |\, x_{kl} \to L| \geq \sum_{(kl)\in I_{mn}\&|\, x_{kl} \to L|>\varepsilon} |\, x_{kl} \to L| \geq \epsilon.\ |\{(kl) \in I_{mn}: |\, x_{kl} \to L| \geq \epsilon\}|.$

So for a given $\boldsymbol{\delta > 0}$,

$\frac{1}{\bar{\lambda}^{\alpha}_{mn}} |\{(k,l) \in I_{mn}: |\, x_{kl} - L\,| \geq \epsilon\}| \geq \delta \implies \frac{1}{\bar{\lambda}^{\alpha}_{mn}} \sum_{(kl)\in I_{mn}} |\, x_{kl} \to L| \geq \epsilon\delta,$

*i.e.* $\{(m,n) \in \mathbb{N}\times\mathbb{N}: \frac{1}{\bar{\lambda}^{\alpha}_{mn}} |\{(k,l) \in I_{mn} : |\, x_{kl} - L\,| \geq \epsilon\}| \geq \delta\} \subset \{(m,n) \in \mathbb{N}\times\mathbb{N}: \frac{1}{\bar{\lambda}^{\alpha}_{mn}} \sum_{(kl)\in I_{mn}} |\, x_{kl} \to L| \geq \epsilon\}.$

Since $\boldsymbol{x_{kl} \to L \circledcirc V, \bar{\lambda}\circledcirc^{\alpha}\,(I)}$, the set on the right hand side belongs to $\boldsymbol{I}$ and so it follows that $\boldsymbol{x_{kl} \longrightarrow L\,(s_{\bar{\lambda}}(I)^{\alpha})}$.

To show that $\boldsymbol{s_{\bar{\lambda}}\,(I)^{\alpha} \subsetneq \circledcirc V, \bar{\lambda}\circledcirc^{\alpha}\,(I)}$, take a fixed $\boldsymbol{A \in I}$. Define $\boldsymbol{x = (x_{kl})}$ by

$$x_{kl} = \begin{cases} kl, & \text{if } m - \circledcirc\sqrt{\lambda^{\alpha}_{m}}\circledcirc + 1 \leq k \leq m,\ n - \circledcirc\sqrt{\mu^{\alpha}_{n}}\circledcirc + 1 \leq l \leq n,\ (m,n) \notin A \\ kl, & \text{if } m - \lambda_m + 1 \leq k \leq m,\ n - \mu_n + 1 \leq l \leq n,\ (m,n) \in A \\ 0, & \text{otherwise.} \end{cases}$$

Then for every $\boldsymbol{\epsilon > 0\ (0 < \epsilon < 1)}$ since

$$a_{mn} := \frac{1}{\lambda^{\alpha}_{m}\mu^{\alpha}_{n}} |\{(k,l) \in I_{mn}: x_{kl} - 0| \geq \epsilon\}| = \frac{\circledcirc\sqrt{\lambda^{\alpha}_{m}}\circledcirc\left[\sqrt{\mu^{\alpha}_{n}}\right]}{\lambda^{\alpha}_{m}\mu^{\alpha}_{n}}$$

tends to zero in Pringsheim's sense as $\boldsymbol{m,n \to \infty}$ and $\boldsymbol{(m,n) \notin A}$. Hence for each $\boldsymbol{\delta > 0}$ $\{(m,n) \in \mathbb{N}\times\mathbb{N}: a_{mn} \geq \delta\} \subset A \cup \{(\mathbb{N}\times\mathbb{N} \setminus A) \cap ((\{1,2,\dots,i_o - 1\}\times\mathbb{N}) \cup (\mathbb{N}\times\{1,2,\dots i_o - 1\}))\}$. for some $\boldsymbol{i_o \in \mathbb{N}}$. Since $\boldsymbol{I}$ is strongly admissible ideal, it follows that $\boldsymbol{x_{kl} \to L(s_{\bar{\lambda}})^{\alpha}\,(I)}$ ).

Obviously, $\boldsymbol{x_{kl} \nrightarrow 0 \circledcirc V, \bar{\lambda}\circledcirc^{\alpha}(I)}$. Note that if $\boldsymbol{A \in I}$ is infinite then $\boldsymbol{x_{kl} \nrightarrow \theta(s_{\bar{\lambda}})^{\alpha}\,(I)}$. This example also shows that $\boldsymbol{I_{\lambda} -}$ double statistical convergence of order $\boldsymbol{\alpha}$ is more general than $\boldsymbol{\lambda -}$ double statistical convergence of order $\boldsymbol{\alpha}$.

**Theorem 5.** $\boldsymbol{s_2(I)^{\alpha} \subset s_{\lambda}(I)^{\alpha}}$ if and only if

$$\liminf_{mn} \frac{\bar{\lambda}^{\alpha}_{mn}}{(mn)^{\alpha}} > 0. \qquad (1)$$

**Proof:** For given $\boldsymbol{\epsilon > 0}$,

$\frac{1}{(mn)^{\alpha}} |k \leq m, l \leq n : |\, x_{kl} - L| \geq \epsilon\}| \geq \frac{1}{(mn)^{\alpha}} \{(k,l) \in I_{mn} : |\, x_{kl} - L| \geq \epsilon\}|$

$\geq \frac{\bar{\lambda}^{\alpha}_{mn}}{(mn)^{\alpha}} \frac{1}{\bar{\lambda}^{\alpha}_{mn}} \{(k,l) \in I_{mn} : |\, x_{kl} - L| \geq \epsilon\}|.$

If $\liminf_{nm\to\infty} \frac{\bar{\lambda}^{\alpha}_{mn}}{(mn)^{\alpha}} = \boldsymbol{a}$ then from definition $\{(m,n) \in N \times N : \frac{\bar{\lambda}^{\alpha}_{mn}}{(mn)^{\alpha}} < \frac{a}{2}\}$ is finite.

For $\boldsymbol{\delta > 0}$,

$\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{\bar{\lambda}^{\alpha}_{mn}} \{(k,l) \in I_{mn} : |\, x_{kl} - L| \geq \epsilon\}| \geq \delta\}$
$\subset \{(m,n) \in \mathbb{N}\times\mathbb{N}\ \frac{1}{(mn)^{\alpha}} \{(k,l) \in I_{mn} : |\, x_{kl} - L| \geq \epsilon\}| \geq \frac{a}{2}\delta\} \cup \{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{\bar{\lambda}^{\alpha}_{mn}}{(mn)^{\alpha}} < \frac{a}{2}\}.$

Since $\boldsymbol{I}$ is admissible, the set on the right hand side belongs to $\boldsymbol{I}$.

Conversely, suppose that $\boldsymbol{x \in s_2(I)^{\alpha}}$ and either $\liminf_{n} \frac{\lambda^{\alpha}_{m}}{m^{\alpha}}$ or $\liminf_{m} \frac{\mu^{\alpha}_{n}}{n^{\alpha}}$ or both are zero. Then as in [18], we can choose subsequences $\boldsymbol{(m(p))_{p=1}^{\infty}}$ and $\boldsymbol{(n(q))_{q=1}^{\infty}}$ such that $\frac{\lambda^{\alpha}_{m}(p)}{m^{\alpha}(p)} < \frac{1}{p}$ and $\frac{\mu^{\alpha}_{n}(q)}{n^{\alpha}(q)} < \frac{1}{q}$. Define a sequence $\boldsymbol{x = (x_{kl})}$ by

$$x_{kl} = \begin{cases} 1, & \text{if } k \in k_{m(p)} \text{ and } l \in I_{n(q)}, \\ 0, & \text{otherwise.} \end{cases}$$

Then, $\boldsymbol{x}$ is bounded and statistical convergence of order $\boldsymbol{\alpha}$ and hence $\boldsymbol{x \in s_2\,(I)^{\alpha}}$. But on the other hand, $\boldsymbol{x \notin [V, \bar{\lambda}]}$ and from theorem 4, $\boldsymbol{x \notin s_2\,(I)^{\alpha}}$. This contradiction implies that condition (1) must hold. This completes the proof.

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