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Boundary layer problem for system of first order of ordinary differential equations with linear non-local boundary conditions

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Abstract

In this paper we study the boundary layer problems in which boundary conditions are non-local. Here we try to find the necessary conditions by the help of fundamental solution to the given adjoint equation. By getting help from these conditions, at first the boundary condition is changed from non-local to local. The main aim of this paper is to identify the location of the boundary layer. In other words, at which point the boundary layer is formed.

***Keywords**:* Singular perturbation problems; boundary layer; fundamental solution; necessary conditions

1. Introduction

An important subject in applied mathematics is the theory of singular perturbation problems. The mathematical model for this kind of problem is usually in the form of either ordinary differential equations (O.D.E) or partial differential equations (P.D.E) in which the highest derivative is multiplied by some powers of ε as a positive small parameter [1-3].

The object theory of singular perturbation is to solve differential equation with some initial or boundary conditions with a small parameter. These problems are essentially at the heart of boundary value and initial value problems [3-10]. Through these studies we can find out whether the boundary conditions become a local type (Dirichlet) and the solution of the boundary layer problem is satisfied in boundary conditions then there is no boundary layer. If the limit solution **(**when $\varepsilon \downarrow 0$**)** is not satisfied in the boundary condition**,** then there will be a boundary layer. In book [4] after the first and second chapters, the unsolved boundary layer problem is seen, which shows that boundary layer problems with non-local boundary conditions have not been studied carefully. So, in this paper and some other works: M. Jahanshahi & A. R. Sarakhsi [10-14] and N. Aliev & S. Ashrafi [15], [16], we study the boundary layer problems in which boundary conditions are non-local. Here, an attempt is made to find the necessary conditions with the help of the fundamental solution of the given adjoint equation. By taking advantage of these conditions, first the boundary conditions are changed for non-local to local and finally as before there will be local case and the reason for the boundary layer lock will be studied [12].

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2. Mathematical statement of problem

We consider the following boundary layer problem:

$$l_\varepsilon X_\varepsilon \equiv \varepsilon \dot{X}_\varepsilon(t) + [p(t) + \varepsilon q(t)] X_\varepsilon(t) = f(t), \ t \in (a,b), \quad (1)$$

$$\alpha X_\varepsilon(a) + \beta X_\varepsilon(b) = \gamma \quad (2)$$

where $\varepsilon > 0$ is a small parameter, $p(t)$, $q(t)$ are the square matrices of n order in which the elements are real continuous functions and $f(t)$ and $X_\varepsilon(t)$ are column vectors whose functions are real continuous. $p(t)$, $q(t)$ and $f(t)$ coefficients of equation (1) are known functions while $X_\varepsilon(t)$ is unknown vector function. Data of boundary condition in the problem, that is α and β, is square matrices of n order with real constant elements, and of column vector has n components with real constant elements. Equation (1) easily shows that when $\varepsilon \downarrow 0$, it changes to an algebraic system. Based on this fact, it can be verified whether solutions of linear algebraic system exist in boundary condition (2) or not. As far as we know, if the limit function $X_0(t)$ is satisfied in boundary condition (2), there is no boundary layer in any of the $t=a$ and $t=b$ points. If the limit function is not satisfied in boundary condition (2), then boundary layer exists.

3. The adjoint equation of main equation

To obtain the adjoint equation, we first establish the following lemma.

3.1. Lemma If $p(t)$, $q(t)$ are continuous functions, then the associated adjoint equation of equation (1) will be:

$$l_\varepsilon^* y_\varepsilon = -\varepsilon \dot{y}_\varepsilon(t) + [p^T(t) + \varepsilon q^T(t)]\, y_\varepsilon(t) \qquad (3)$$

where $p^T(t)$, $q^T(t)$ are the transposes of $p(t)$ and $q(t)$ matrices.

Proof: To do this we attempt to get the Lagrange formula of system (1) [5].Consider the following scalar product of real functions:

$$(y_\varepsilon, l_\varepsilon X_\varepsilon) = \int_a^b y_\varepsilon^T(t) l_\varepsilon X_\varepsilon(t)dt$$

$$= \varepsilon \int_a^b y_\varepsilon^T(t)\, \dot{X}_\varepsilon(t)dt$$

$$+ \int_a^b y_\varepsilon^T(t)[p(t) + \varepsilon q(t)] X_\varepsilon(t)dt$$

$$= \varepsilon y_\varepsilon^T(t)\, X_\varepsilon(t) \Big|_{t=a}^{b} - \varepsilon \int_a^b \dot{y}_\varepsilon^T(t)\, X_\varepsilon(t)\, dt$$

$$+ \int_a^b y_\varepsilon^T(t)[p\ (t) + \varepsilon q\ (t)] X_\varepsilon(t)$$

$$= \varepsilon y_\varepsilon^T(t)\, X_\varepsilon(t) \Big|_{t=a}^{b}$$

$$+ \int_a^b \{-\varepsilon \dot{y}_\varepsilon(t) + [p^T(t) + \varepsilon q^T(t)]\, y_\varepsilon(t)\}^T X_\varepsilon(t)dt.$$

According to Lagrange formula, the inside of the above integral term, gives the associated adjoint equation (3). If we use part of the integral method for the first sentence on the right hand side, the Lagrange formula leads to the present integrals of the lemma in formula (3).

4. Formal solution

Now we refer to constructing the formal solution of the system,

$$-\varepsilon \dot{y}_\varepsilon(t) + [p^T(t) + \varepsilon q^T(t)]\, y_\varepsilon(t) = 0, \qquad (4)$$

with parameter ε. At first we verify proof of the next lemma:

4.1. Lemma If in the system (4), matrix elements of $p(t)$ and $q(t)$ are infinitely differentiable functions and the roots of the characteristic equation of this system,

$$\det[p^T(t) - \theta(t)E] = 0.$$
(5)

(E is the identity matrix of order n) are distinct and the real parts of them are not zero, that is:

$$\theta_k(t) \neq \theta_s(t), \qquad k \neq s, \quad t \in [a,b],$$
$$\operatorname{Re}\theta_k(t) \neq 0, \qquad k = 1,2,\dots,n.$$

Then the formal solution of system (1) will be

$$y_{\varepsilon k}(t) = e^{\varepsilon^{-1}\int_{\tilde{t}}^{t} \theta_k(\tau)d\tau} \sum_{n=0}^{\infty} \varepsilon^m C_m(t); t \in [a,b], \qquad (6)$$

where $k = 1,2,\dots,n$. Here $\theta_k(t)$ is scaler functions. Moreover, $C_m(t)$ are the column vectors of n components in which the elements are functions that are obtained from these systems:

$$p^T(t)C_0(t) = \theta(t)C_0(t),$$
$$p^T(t)C_{m+1}(t) = \theta\ (t)C_{m+1}(t)$$
$$+C_m^{'}(t) - q^T(t)C_m(t) \qquad m \geq 0, \qquad (7)$$

or

$$\begin{pmatrix} p_{11}(t)-\theta_k(t) & p_{21}(t) & \cdots & p_{n1}(t) \\ p_{12}(t) & p_{22}(t)-\theta_k(t) & \cdots & p_{n2}(t) \\ \vdots & \vdots & \ddots & \vdots \\ p_{1n}(t) & p_{2n}(t) & \cdots & p_{nn}(t)-\theta_k(t) \end{pmatrix}$$

$$\times \begin{pmatrix} C^k_{m+1,1}(t) \\ C^k_{m+1,2}(t) \\ \vdots \\ C^k_{m+1,n}(t) \end{pmatrix} =$$

$$\begin{pmatrix} \dot{C}^k_{m1}(t) & -q_{11}(t)C^k_{m1}(t)\cdots & -q_{n1}(t)C^k_{mn}(t) \\ \dot{C}^k_{m2}(t) & -q_{12}(t)C^k_{m1}(t)\cdots & -q_{n2}(t)C^k_{mn}(t) \\ \vdots & \vdots & \ddots & \vdots \\ \dot{C}^k_{mn}(t) & -q_{1n}(t)C^k_{m1}(t)\cdots & -q_{nm}(t)C^k_{mn}(t) \end{pmatrix}$$

Proof: In order to verify lemma , we put the formal solution (6) in system (4) and omit exponential terms from both sides. It can be easily seen that relations (7) are the coefficients of successive powers of small parameter ε. According to the general form of (7), the first system of equations is:

$$[p^T(t)-\theta(t)E]C_0(t)=0 \tag{8}$$

or

$$\begin{pmatrix} p_{11}(t)-\theta\ (t) & p_{21}(t) & \cdots & p_{n1}(t) \\ p_{12}(t) & p_{22}(t)-\theta\ (t) & \cdots & p_{n2}(t) \\ \vdots & \vdots & \ddots & \vdots \\ p_{1n}(t) & p_{2n}(t) & \cdots & p_{nn}(t)-\theta\ (t) \end{pmatrix}$$

$$\times \begin{pmatrix} C_{0,1}(t) \\ C_{0,2}(t) \\ \vdots \\ C_{0,n}(t) \end{pmatrix} = 0$$

In order to have non trivial solutions, $C_0(t)$ the determinant of system (8) should be zero. It means that $\theta_k(t)$ for $k=1,2,...,n$ are the roots of characteristic equation (5). In this case, for fixed k , the elements of $C_0^k(t)$ will be:

$$C_{0j}^{(k)}(t)=\tilde{C}_0^{(k)}(t)P(p_{jr}(t)-e_{jr}\theta_k(t)), \tag{9}$$
$$j=1,2,...,n,$$

where $\tilde{C}_{0j}^{(k)}(t)$ are arbitrary functions and P includes the whole of associated co-factors of r row of matrix $P^T(t)-\theta_k(t)E$, so it is shown as $P(p_{jr}(t)-e_{jr}\theta_k(t))$. Here we should choose r in a such a way that co-factors of the row do not become zero simultaneously (It should be that at most one row of matrix $P^T(t)-\theta_k(t)E$ can be zero as a result of having different roots). Also e_{jr} is delta of Kroncer, that is:

$$e_{jr}=\begin{cases} 0, & j\neq r, \\ 1, & j=r. \end{cases}$$

The column vector $C_0^k(t)$ which consists of the arbitrary functions in (9) is resulted from writing $\theta_k(t)$ instead of $\theta\ (t)$, that is solution of system (8). Because the system (8) is homogeneous, so functions of $\tilde{C}_{0j}^{(k)}(t)$ are arbitrary functions. By factoring these arbitrary functions, finally we multiply the elements of one row of matrix $P^T(t)-\theta_k(t)E$ to co-factors of row r in that matrix. This one again is zero based on linear algebra theorems [6]. So the unknown scalars functions of $\theta\ (t)$ and also $C_0(t)$ vectors of first system in the system set (7) can be obtained. Now, in order to verify the solution of the second system in (7), in which $m=0$, we should find the resulted system from unknown column vector $C_1(t)=\tilde{C}_1^{(k)}(t)$. To make the system solvable, it is necessary that the right hand side of the resulted system from $m=0$ becomes orthogonal to the eigenvector of adjoint equations. Because the right hand includes the arbitrary function $\tilde{C}_0^{(k)}(t)$, so we can apply the orthogonality condition because the determinant of coefficients of the resulted system from $m=0$ is zero. By following this process, the whole solutions of the first and second equations of ystems (7) will be calculated as the determined functions of $C_m(t)$.

Now, Let us confirm this formal solution as a real and classic solution. For this by identifying the

$C_m(t)$ and $\theta_k(t)$ of unknown functions in system (7) and according to formal solution (6), the next matter that should be considerd is exponential terms in (6). Based on choosing $\tilde{t}$ (which was discussed in the above lemma), when $\varepsilon \downarrow 0$, exponential term is will lead to zero. Finally, according to infinity series $(\sum_{m=0}^{\infty} \varepsilon^m C_m(t))$ in the form of formal solution of (6), the asymptotic expansions of linear independent solutions of (4) can be obtained. [14-16]. We can conclude this lemma:

4.2. Lemma On the condition of lemma 4.1, the asymptotic expansions of $y_{\varepsilon k}(t)$ for $k = 1,2,...,n$, leads to $\tilde{y}_{\varepsilon k}(t)$:

$$\tilde{y}_{\varepsilon k}(t) = e^{\varepsilon^{-1}\int_{\tilde{t}}^{t} \theta_k(\tau)d\tau} \left[\sum_{n=0}^{M} \varepsilon^m C_m(t) + O(\varepsilon^{M+1}) \right]; \quad t \in [a,b] \tag{10}$$

where $O(\varepsilon^{M+1})$ is error term.

5. Fundamental solution

Now for calculating the fundamental solution, consider the non-homogeneous system related to system (4):

$$-\varepsilon \dot{Y}_\varepsilon(t) + [p^T(t) + \varepsilon q^T(t)]Y_\varepsilon(t) = g(t) \tag{11}$$

where $p^T(t)$, $q^T(t)$ square matrices of n order in that consist of real functions and $Y_\varepsilon(t)$ is unknown vector. Here $g(t)$ is the known column vector of n order. At first, we establish the following lemma.

5.1. Lemma Under the conditions of lemma 4.1 fundamental solution of system (11) will be:

$$Y_\varepsilon(t,\xi) = -\varepsilon^{-1}\theta(t-\xi)\tilde{Y}_\varepsilon(t)\tilde{Y}_\varepsilon^{-1}(\xi). \tag{12}$$

Here $\theta(t-\xi)$ is the function of Heaviside,

$$\theta(t-\xi) = \begin{cases} 1, & t > \xi \\ \frac{1}{2}, & t = \xi \\ 0, & t < \xi \end{cases}$$

and $\tilde{Y}_\varepsilon(t)$ is the matrix solution of system (4).

Proof: At first we obtain the general solution of non homogeneous system (11) through variation of variable method:

$$Y_\varepsilon(t) = \tilde{Y}_\varepsilon(t)A(t). \tag{13}$$

Here $A(t)$ is the unknown column vector. By using (13) in (11) we will have:

$$-\varepsilon \dot{\tilde{Y}}_\varepsilon(t)A(t) - \varepsilon \tilde{Y}_\varepsilon(t)\dot{A}(t)$$
$$+ [p^T(t) + \varepsilon q^T(t)]\tilde{Y}_\varepsilon(t)A(t) = g(t)$$

where $\tilde{Y}_\varepsilon(t)$ is the matrix solution of inhomogeneous system (4):

$$-\varepsilon \tilde{Y}_\varepsilon(t)\dot{A}(t) = g(t),$$

or

$$\dot{A}(t) = -\varepsilon^{-1}\tilde{Y}_\varepsilon^{-1}(t)g(t),$$

Here we can find this relation for $A(t)$,

$$A(t) = A(a) - \varepsilon^{-1}\int_a^t \tilde{Y}_\varepsilon^{-1}(\xi)g(\xi)d\xi. \tag{14}$$

If in (14) we suppose $A(a) = 0$ and put this one in (12), the solution of the non homogeneous system will be:

$$\begin{aligned} y_\varepsilon(t) &= -\varepsilon^{-1}\int_a^t \tilde{Y}_\varepsilon(t)\tilde{Y}_\varepsilon^{-1}(\xi)g(\xi)d\xi \\ &= -\varepsilon^{-1}\int_a^t \theta(t-\xi)\tilde{Y}_\varepsilon(t)\tilde{Y}_\varepsilon^{-1}(\xi)g(\xi)d\xi \\ &\quad -\varepsilon^{-1}\int_t^b \theta(t-\xi)\tilde{Y}_\varepsilon(t)\tilde{Y}_\varepsilon^{-1}(\xi)g(\xi)d\xi \\ &= -\varepsilon^{-1}\int_a^b \theta(t-\xi)\tilde{Y}_\varepsilon(t)\tilde{Y}_\varepsilon^{-1}(\xi)g(\xi)d\xi. \end{aligned}$$

Here we can see that the fundamental solution of (4) is in the form of (12). In fact if we put solution (12) in the left side of (4) it will be:

$$-\varepsilon \dot{Y}_\varepsilon(t,\xi)+[p^T(t)+\varepsilon q^T(t)]Y_\varepsilon(t,\xi)$$
$$=-\varepsilon[-\varepsilon^{-1}\delta(t-\xi)\dot{\tilde{Y}}_\varepsilon(t)\tilde{Y}_\varepsilon^{-1}(\xi)$$
$$-\varepsilon^{-1}\theta(t-\xi)\tilde{Y}_\varepsilon(t)\tilde{Y}_\varepsilon^{-1}(\xi)]$$
$$+[p^T(t)+\varepsilon q^T(t)](-\varepsilon^{-1})$$
$$\times\theta(t-\xi)\tilde{Y}_\varepsilon(t)\tilde{Y}_\varepsilon^{-1}(\xi)$$
$$=\delta(t-\xi)-\varepsilon^{-1}\theta(t-\xi)\{-\varepsilon\dot{\tilde{Y}}_\varepsilon(t)$$
$$+[p^T(t)+\varepsilon q^T(t)]\}\tilde{Y}_\varepsilon(t)\}\tilde{Y}_\varepsilon^{-1}(\xi).$$

Note that the second term of the last relation will be zero, because $\tilde{Y}_\varepsilon(t)$ is solution of related homogenous system (4). So, this solution is the fundamental solution of the system (4), that is:

$$-\varepsilon \dot{Y}_\varepsilon(t,\xi)+[p^T(t)+\varepsilon q^T(t)]Y_\varepsilon(t,\xi)=\delta(t-\xi)E. \qquad (15)$$

Where $\delta(t-\xi)$ is the Dirac-delta function and E is the identity matrix of n order.

6. Fundamental relations and necessary conditions

At first we verify the following lemma:

6.1. Lemma Under the conditions of lemma 4.1, the arbitrary solution of system (1) is satisfied in the following relations:

$$\int_a^b Y_\varepsilon^T(t,\xi)f(t)dt+\varepsilon Y_\varepsilon^T(a,\xi)X_\varepsilon(a)$$
$$-\varepsilon Y_\varepsilon^T(b,\xi)X_\varepsilon(b) \qquad (16)$$
$$=\begin{cases} X_\varepsilon(\xi), & \xi\in(a,b) \\ \frac{1}{2}X_\varepsilon(\xi), & \xi=a, \quad \xi=b\end{cases}$$

In which the second case gives us the necessary conditions.

Proof: Next,, we multiply the two sides of system (1) to transpose fundamental solution (12) in the left hand and then obtain the integral in $[a,b]$,

$$\varepsilon\int_a^b Y_\varepsilon^T(t,\xi)\dot{X}_\varepsilon(t)dt$$
$$+\int_a^b Y_\varepsilon^T(t,\xi)[p(t)+\varepsilon q(t)]X_\varepsilon(t)dt$$
$$=\int_a^b Y_\varepsilon^T(t,\xi)f(t)dt$$

or

$$\varepsilon Y_\varepsilon^T(t,\xi)X_\varepsilon(t)\Big|_{t=a}^{b}-\varepsilon\int_a^b \dot{Y}_\varepsilon^T(t,\xi)X_\varepsilon(t)dt$$
$$+\int_a^b Y_\varepsilon^T(t,\xi)[p(t)+\varepsilon q(t)]X_\varepsilon(t)dt$$
$$=\int_a^b Y_\varepsilon^T(t,\xi)f(t)dt.$$

The relation (15), and from the properties of the Dirac-delta function [8], [9] give us relations (16).

$$\int_a^b Y_\varepsilon^T(t,\xi)f(t)dt+\varepsilon Y_\varepsilon^T(a,\xi)X_\varepsilon(a)-\varepsilon Y_\varepsilon^T(b,\xi)X_\varepsilon(b)$$
$$=\int_a^b\{-\varepsilon\dot{Y}_\varepsilon(t,\xi)+Y_\varepsilon^T(t,\xi)[p^T(t)+\varepsilon q^T(t)]\}X_\varepsilon(t)dt$$
$$=\int_a^b\delta(t-\xi)X_\varepsilon(t)dt=\begin{cases} X_\varepsilon(\xi), & \xi\in(a,b) \\ \frac{1}{2}X_\varepsilon(\xi), & \xi=a, \quad \xi=b\end{cases}$$

We use the second case of last relation and get the necessary condition as follows:

$$\frac{1}{2}X_\varepsilon(a)=\int_a^b Y_\varepsilon^T(t,a)f(t)dt+\varepsilon(-\varepsilon^{-1})\theta(0)X_\varepsilon(a)$$
$$-\varepsilon(-\varepsilon^{-1})\theta(b-a)\tilde{Y}_\varepsilon^{T^{-1}}(a)\tilde{Y}_\varepsilon^{T}(b)X_\varepsilon(b)$$

$$\frac{1}{2}X_\varepsilon(b)=\int_a^b Y_\varepsilon^T(t,b)f(t)dt+\varepsilon(-\varepsilon^{-1})\theta(a-b)$$
$$\times\tilde{Y}_\varepsilon^{T^{-1}}(b)\tilde{Y}_\varepsilon^{T}(a)-\varepsilon(-\varepsilon^{-1})\theta(0)X_\varepsilon(b).$$

According to first relation we will have:

$$X_\varepsilon(a)=\int_a^b Y_\varepsilon^T(t,a)f(t)dt-\varepsilon Y_\varepsilon^T(b,a)X_\varepsilon(b), \qquad (17)$$

Note that we show

$$-\varepsilon^{-1}\tilde{Y}_{\varepsilon}^{T^{-1}}(a)\tilde{Y}_{\varepsilon}^{T}(b)=Y_{\varepsilon}^{T}(b,a)$$

We should consider that in the second relation the above will be as identified as follows.

$$\frac{1}{2}X_{\varepsilon}(b)=\int_{a}^{b}Y_{\varepsilon}^{T}(t,b)\,f(t)dt+\frac{1}{2}X_{\varepsilon}(b),$$

the integral term will be zero from (12).

7. Localization of boundary conditions

In this part, we convert non-local conditions (2) by using the above necessary conditions for local boundary conditions. At first we prove the following lemma.

7.1. Lemma Under the conditions of lemma 4.1 we assume:

$$\det[\beta-\varepsilon\alpha]Y_{\varepsilon}^{T}(b,a)\neq 0, \tag{18}$$

so for boundary values of unknown function we have the following relations:

$$X_{\varepsilon}(a)=\int_{a}^{b}Y_{\varepsilon}^{T}(t,a)\,f(t)dt-\varepsilon Y_{\varepsilon}^{T}(b,a)$$
$$\times[\beta-\varepsilon\alpha Y_{\varepsilon}^{T}(b,a)]^{-1}[\gamma-\alpha\int_{a}^{b}Y_{\varepsilon}^{T}(t,a)\,f(t)dt],$$
$$X_{\varepsilon}(b)=[\beta-\varepsilon\alpha Y_{\varepsilon}^{T}(b,a)]^{-1}[\gamma-\alpha\int_{a}^{b}Y_{\varepsilon}^{T}(t,a)\,f(t)dt] \tag{19}$$

Proof: If we consider the relation (17) and boundary condition (2) as an algebraic system for the unknowns $X_{\varepsilon}(a)$ and $X_{\varepsilon}(b)$, and solve this system by Cramer rule, we will have the relations (19) for boundary values of $X_{\varepsilon}(t)$. In this case, the nonlocal boundary conditions were converted to local boundary conditions.

8. Solution of main boundary value problem

In order to give analytical statement to solution of main problem (1)-(2), we prove this theorem:

8.1. Theorem Assume in the system of main problem (1), $\varepsilon>0$ is a small parameter and $p(t),\ q(t)$ are the square matrices of n order in which the elements are functions of infinitely differentiable. $f(t)$ is function of column vectors that in continuous. $X_{\varepsilon}(t)$ is unknown vector of n order, and α,β are square matrices of n order which has constant elements, and γ is constant column vector of n order and non-local boundary conditions (2) are linear independent. Moreover the roots of the characteristic equation (5) are distinct and their real parts will not be zero. Also, we assume condition (18) will hold. Then the solution of the boundary layer problem (1)-(2) will be:

$$X_{\varepsilon}(\xi)=\int_{a}^{b}Y_{\varepsilon}^{T}(t,\xi)\,f(t)dt+\varepsilon Y_{\varepsilon}^{T}(a,\xi)$$
$$\times[\int_{a}^{b}Y_{\varepsilon}^{T}(t,a)\,f(t)dt-\varepsilon Y_{\varepsilon}^{T}(b,a)]$$
$$\times[\beta-\varepsilon\alpha Y_{\varepsilon}^{T}(b,a)]^{-1}[\gamma-\alpha\int_{a}^{b}Y_{\varepsilon}^{T}(t,a)\,f(t)dt]$$
$$-\varepsilon Y_{\varepsilon}^{T}(b,\xi)[\beta-\varepsilon\alpha Y_{\varepsilon}^{T}(b,a)]^{-1}$$
$$\times[\gamma-\alpha\int_{a}^{b}Y_{\varepsilon}^{T}(t,a)\,f(t)dt], \tag{20}$$

where $Y^{T}{}_{\varepsilon}(t,\xi)$ is the transpose of fundamental solution (12).

Proof: According to the distinct roots in equation (5) and the form of formal solution (6) in the system (4), we can obtain a normal form of it in lemma 5.1. On the other hand, asymptotic expansion of normal linear independent solution (4) has been given in lemma 4.2 and in lemma 5.1 the fundamental solution of system (4) based on the relation (12) and also the basic relation and necessary condition by relation (17) is offered. Finally, in lemma 7.1 we can find boundary condition in local form. In this situation, in order to verify the theorem it is enough to put the boundary conditions of localization (19) in the left hand side of (16). First case of relation (16) gives the solution (20) of problem (1)-(2).

9. Limiting behavior of solution

Now let us study the limited situation of solution where $\varepsilon\downarrow 0$. For this, when $\varepsilon\downarrow 0$ the limit of system (1) is

$$p(t)X_{0}(t)=f(t) \tag{21}$$

$$\alpha X_{0}(a)+\beta X_{0}(b)=\gamma. \tag{22}$$

If the determinant of the above system is not zero, that is:

$$\det[p(t)] \neq 0. \qquad (23)$$

So

$$X_0(t) = p(t)^{-1} f(t) \qquad (24)$$

In the system (15) where the fundamental solution is usable, when $\varepsilon \downarrow 0$, the solution will be:

$$p(t)^T Y_0(t,\xi) = \delta(t-\xi)E \qquad (25)$$

or such as (21) we will have,

$$Y_0(t,\xi) = [p(t)^T]^{-1}\delta(t-\xi) \qquad (26)$$

Now, by considering (26), the limiting relations (19) and (20), when $\varepsilon \downarrow 0$, will be:

$$\begin{cases} X_0(a) = \dfrac{1}{2} p^{-1}(a) f(a), \\ X_0(b) = \beta^{-1}\gamma - \dfrac{1}{2}\beta^{-1}\alpha p^{-1}(a) f(a), \end{cases} \qquad (27)$$

and

$$X_0(\xi) = p^{-1}(\xi) f(\xi) \qquad (28)$$

We consider the relation (28) is the same as relation (24) which is limiting state of problem (1)-(2).

10. The formation of boundary layers

As mentioned in the introduction, the formation or non formation of boundary layers is important for constructing approximate solutions [2], [3]. They easily see the limiting situation of solution of the main problem (20) is adjustable in limiting situation of system (21), hence, we have this theorem:

10.1. Theorem Under the conditions of theorem 8.1 and the condition,

$$\det[p(t)] \neq 0.$$

If we have this relation,

$$\alpha p^{-1}(a) f(a) + \beta p^{-1}(b) f(b) = \gamma \qquad (29)$$

then there will be no boundary layer in any of the boundary point's $t = a, t = b$.

Proof: We can see from (29) that the limiting situation of solution in problem (1)-(2), when $\varepsilon \downarrow 0$ in (20), are satisfied with the limiting situation of boundary condition (22). Hence in this case, according to boundary layer problems, we have no boundary layers.

10.2. Theorem Under the conditions of theorem 10.1 and the condition:

$$\det[p(t)] \neq 0.$$

Also, the condition (29) does not hold, so

I) if $f(a) = 0$, the boundary layer forms just in $t = b$,

II) if $f(a) \neq 0$, boundary layer will be in both $t = a$, and $t = b$.

Proof: In fact, if we don't have condition (29), there will be a formed boundary layer. But if $f(a) = 0$, we can adopt the limiting situation of solution (20) in first relation (19). However, (20) does not satisfy with second relation (19). Because of this, boundary layer in $t = b$ will be formed. Referring to the fact that the limit function (20) is not satisfied with relations (19), finally in both $t = a$, and $t = b$ points we will have boundary layer.

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