On BC-generalized Landsberg Finsler metrics

M. Zamanzadeh[1]* and B. Najafi[2]

[1]Bijar Branch, Islamic Azad University Bijar, Iran
[2]Department of Mathematics, Faculty of Science, Shahed University of Tehran, Tehran, Iran
E-mails: zamanzadeh.mohammad@gmail.com & najafi@shahed.ac.ir

Abstract

Equality of hh -curvatures of the Berwald and Cartan connections leads to a new class of Finsler metrics, so-called BC-generalized Landsberg metrics. Here, we prove that every BC-generalized Landsberg metric of scalar flag curvature with dimension greater than two is of constant flag curvature.

Keywords: Finsler structure; Landsberg metric; generalized Landsberg metric; scalar flag curvature

1. Introduction

Unlike Riemannian geometry, there are four well-known connections in Finsler geometry: Berwald, Cartan, Rund (Chern) and Hashiguchi connection, which are extensions of the Levi-Civita connection in Reimannian geometry. A connection in Finsler geometry is considered as a triple $(F_{ij}^k, N_i^j, V_{ij}^k)$ which consists of horizontal Christoffel symbols, non-linear connection coefficients and vertical Christoffel symbols.

One of the important and rich classes of Fisnler metrics is the class of Landsberg metrics. There are various ways to define this kind of Finsler metrics [1]. A Landsberg metric can be defined by the requirement that the horizontal Christoffel symbols of Berwald and Rund connections be equal. The counterpart of sectional curvature of Riemannian manifolds for Finslerian manifolds is called flag curvature. Finsler metrics of constant flag curvature are of main interest in both mathematics and physics. In [2], Bejancu and Farran introduced a generalization of Landsberg metric and called it Generalized Landsberg metric (GL-metric). A GL-metric is a Finsler metric for which the hh - curvatures of Berwald and Rund connections are the same. Bejancu and Farran proved that every GL- metric of scalar flag curvature with dimension greater than two is of constant flag curvature [2].

It is well known that Landsberg metrics can also be defined by the requirement that the horizontal Christoffel symbols of Berwald and Cartan connections be equal. Here, we introduce a new

generalization of Landsberg metrics and call it BC-generalized Landsberg metric. A Finsler metric is called a BC-generalized Landsberg metric if the hh - curvatures of Berwald and Cartan connections coincide. Every Landsberg metric is a BC-generalized Landsberg metric. However, the converse is an open problem. Landsberg metrics and their generalizations which are of scalar flag curvature has been studied extensively [3-5]. Numata's well-known theorem states that every Landsberg metric $F^n (n \geq 3)$ of non-zero scalar flag curvature is a Riemannain metric of constant sectional curvature [6]. Then, in [2] Numata's theorem is extended to GL-metrics. Here, we prove a modified version of Numata's theorem for BC-generalized Landsberg metrics. More precisely, we have the following theorem.

Theorem 1. Suppose that $F^n (n \geq 3)$ is a BC-generalized Landsberg metric of scalar flag curvature. Then F is of constant flag curvature.
As Matsumoto pointed out in [7], every GL-metric is a stretch metric and consequently the main theorem of [2] follows from Shibata's theorem which states that every stretch metric of non-zero scalar flag curvature with dimension greater than two is a Riemannain metric of constant sectional curvature. However, a BC-generalized Landsberg metric need not be a stretch metric.

2. Preliminaries

In this section, mainly background material is presented about the basic tools and notations. Let M be an n -dimensional smooth manifold. The

tangent space at $x \in M$ is denoted by $T_x M$ and the tangent bundle of M by TM. Each element of TM has the form (x, y), where $x \in M$ and $y \in T_x M$. Let $TM_0 = TM \setminus \{0\}$. The natural projection $\pi : TM \to M$ is given by $\pi(x, y) = x$. The pull-back tangent bundle $\pi^*(TM)$ is a vector bundle over TM_0 whose fiber $\pi^*_v(TM)$ at $v \in TM_0$ is just $T_x M$ where $\pi(v) = x$. Thus $\pi^*(TM) = \{(x, y, v) \mid y \in T_x M \setminus \{0\}, v \in TM_0\}$. A *Finsler metric* on a manifold M is a function $F : TM \to [0, +\infty)$ with the following properties:

(i) F is C^∞ on TM_0,

(ii) $F(x, \lambda y) = \lambda F(x, y) \quad \forall \lambda > 0$,

(iii) For any tangent vector, the vertical Hessian given by

$$(g_{ij}(x, y)) = \left(\left[\frac{1}{2} F^2 \right]_{y^i y^j} \right)$$

is positive definite.

A symmetric tensor $\mathbf{C}$ is defined by

$$\mathbf{C}(U, V, W) := C_{ijk}(y) U^i V^j W^k,$$

where $U = U^i \frac{\partial}{\partial x^i}$, $V = V^i \frac{\partial}{\partial x^i}$ and $W = W^i \frac{\partial}{\partial x^i}$ is called *Cartan* tensor. Further, let $I_k = g^{ij} C_{ijk}$. Then $\mathbf{I}$ is called *mean Cartan tensor*.

Theorem 2. ([1]) For a Finsler metric F, the following are equivalent
a) $\mathbf{C} = 0$,
b) $\mathbf{I} = 0$,
c) F is Riemannian.

Asymmetric tensors $\mathbf{L}$ on $\pi^*(TM)$ is defined as follows

$$\mathbf{L}(U, V, W) := L_{ijk}(y) U^i V^j W^k,$$

where $L_{ijk} = C_{ijk\|s} y^s$ in which $'\|$' denotes the horizontal covariant derivative with respect to Cartan connection. $\mathbf{L}$ is called *Landsberg tensor*.

Definition 3. A Finsler metric is called a *Landsberg metric* if $\mathbf{L} = 0$.

A global vector field G is induced by F on TM_0. This vector field in a standard coordinate

(x^i, y^i) for TM_0 is given by

$$G = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i},$$

where $G^i(x, y)$ are local functions on TM_0 satisfying $G^i(x, \lambda y) = \lambda^2 G^i(x, y) \quad \lambda > 0$. G is called the associated *spray* to (M, F).

The notion of Riemann curvature for Riemann metrics can be extended to Finsler metrics. For a non-zero vector $y \in T_x M_0$, the Riemann curvature $R_y : T_x M \to T_x M$ is defined by

$$R_y(u) = R^i_k(y) u^k \frac{\partial}{\partial x^i}$$

where

$$R^i_k(y) = 2 \frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^j}{\partial y^k},$$

$$G^i_j = \frac{\partial G^i}{\partial y^j} \text{ and } G^i_{jk} = \frac{\partial G^i_j}{\partial y^k}.$$

Suppose that $P \subset T_x M$ (flag) is an arbitrary plane and $y \in P$ (flag pole). The flag curvature $K(P, y)$ is defined by

$$K(P, y) = \frac{g_y(R_y(v), v)}{g_y(y, y) g_y(v, v) - g_y(v, y) g_y(v, y)}$$

where v is an arbitrary vector in P such that $P = span(y, v)$.

F is said to be of scalar flag curvature, if for a non-zero vector $y \in P$, $K(P, y) = \lambda(y)$ is independent of P, or equivalently,
$R_y = \lambda(y) F^2(y) \{I - g_y(y, .) y\}, \quad y \in T_x M, \quad x \in M$
where $I : T_x M \to T_x M$ denotes the identity map and $g_y(y, .) = \frac{1}{2} [F^2]_{y^i} dx^i$. F is also said to be of constant curvature λ if the above identity holds for a constant λ [3, 4].

Definition 4. A Finsler metric F is said to be *BC-generalized Landsberg metric*, if the Riemannian curvatures of Berwald and Cartan connections coincide.

Remark 2. Every Landsberg metric is a BC-generalized Landsberg metric, but the converse is an open problem.

Let F^h_{ij} denote the horizontal Christoffel symbols

of the Cartan connection of Finsler metric F and

$$\delta_k = \frac{\partial}{\partial x^k} - G_k^j \frac{\partial}{\partial y^j}.$$

We know that the hh-curvature of Berwald, Chern and Cartan connections are given by the following, respectively

$$H_{ijk}^h = \delta_k G_{ij}^h + G_{ij}^r G_{rk}^h - (j \mid k) \qquad (1)$$

$$K_{ijk}^h = \delta_k F_{ij}^h + F_{ij}^r F_{rk}^h - (j \mid k) \qquad (2)$$

where $R_{jk}^h = \delta_k G_j^h - \delta_j G_k^h$ and $(j \mid k)$ means interchange then subtract. It is easy to see that

$$H_{ijk}^h = K_{ijk}^h + W_{ijk}^h - C_{ij}^r R_{rk}^h \qquad (3)$$

$$R_{ijk}^h = H_{ijk}^h + W_{ijk}^h \qquad (4)$$

where $W_{ijk}^h = \{ L_{ij\|k}^h + L_{ij}^r L_{rk}^h - (j \mid k) \} + C_{ij}^r R_{rk}^h$ (for more details see [2]).

By definition, a Finsler metric is a BC-generalized Landsberg metric if and only if $W_{ijk}^h = 0$. It is well known that a Finsler metric F is a Landsberg metric if and only if the horizontal Christoffel symbols of the Cartan and Berwald connections coincide [8]. Therefore, if F is a Landsberg metric, then $R_{ijk}^h = H_{ijk}^h$ and consequently $W_{ijk}^h = 0$. Thus the class of Finsler metrics with vanishing W_{ijk}^h is a rich class.

In Finsler geometry, in general, geometric objects depend both on position and direction. In [9] and [10], Basco, Matsumoto and Szilasi studied Finsler metrics with some important tensors such as hh-curvatures of Berwald, Rund and Cartan connections depending only on position. Here, we consider a Finsler metric whose hh-curvature of the Cartan connection depends only on position. More precisely, we have the following.

Proposition 5. Suppose that the hh-curvature of the Cartan connection of a Finsler metric F depends only on position. Then F is a BC-generalized Landsberg metric.

Proof: It is well known that the hh-curvature of Cartan connection satisfies the following

$$y^l \frac{\partial}{\partial y^i}(R_{ljh}^t) = W_{ljh}^t. \qquad (5)$$

Now, suppose that the hh-curvature of the Cartan connection of a Finsler metric F depends only on position. Then, by (5), one can get $R_{ijk}^h = H_{ijk}^h$. This completes the proof.

3. Proof of Theorem 1

First, we recall the following proposition.

Proposition 6. ([8]) Suppose that Finsler metric F is of scalar flag curvature $K(x, y)$. Then

$$R_{ijk} = h_{ik} K_j - h_{ij} K_k, \qquad (6)$$

where $K_i = FK\ell_i + \frac{1}{3} F^2 \frac{\partial K}{\partial y^i}$ and $R_{ijk} = g_{jh} R_{ik}^h$.

We assume that $W_{ijk}^h = 0$. Taking into account the symmetric and anti-symmetric parts of $W_{hijk} = g_{hr} W_{ijk}^r = 0$ in h and k, we see that the vanishing of the tensor W_{hijk} is equivalent to the following equations

$$L_{kjr} L_{ih}^r - L_{hjr} L_{ik}^r = 0, \qquad (7)$$

$$L_{ijh\|k} - L_{ijk\|h} + C_{ijr} R_{kh}^r = 0. \qquad (8)$$

On the other hand, the hh-curvature of Cartan connection satisfies the following

$$R_{ikh}^t = R_{kh/i}^t + R_{kr}^t C_{ih}^r - R_{hr}^t C_{ik}^r \\ + L_{hr}^t L_{ik}^r - L_{kr}^t L_{ih}^r + L_{ih\|k}^t - L_{ik\|h}^t, \qquad (9)$$

where $'\!/$ denotes the vertical covariant derivative with respect to Cartan connection [11].

Using the relations $g_{tj\|i} = g_{tj/i} = 0$, we get

$$g_{tj} R_{kh/i}^t = R_{jkh/i}. \qquad (10)$$

Contracting (9) with g_{tj} implies that

$$R_{ijkh} = R_{jkh/i} + R_{jkr} C_{ih}^r - R_{jhr} C_{ik}^r + L_{jhr} L_{ik}^r \\ - L_{jkr} L_{ih}^r + L_{jih\|k} - L_{jik\|h}, \qquad (11)$$

Since we have $R_{ijkh} + R_{jikh} = 0$, then (11) yields

$$R_{jkh/i} + R_{ikh/j} = -R_{jkr} C_{ih}^r - R_{ikr} C_{jh}^r \\ + R_{jhr} C_{ik}^r + R_{ihr} C_{jk}^r + 2R_{kh}^r C_{rij}, \qquad (12)$$

and by definition of the $'/\hat{}$

$$\frac{\partial}{\partial y^i}(R_{jkh}) + \frac{\partial}{\partial y^j}(R_{ikh}) = 4R_{kh}^r C_{ijr} \tag{13}$$

$$+ (R_{jhr} - R_{jrh}).C_{ik}^r + (R_{ihr} - R_{irh}).C_{kj}^r$$

Contracting (13) with y^k and using the relations $C_{ik}^r y^k = C_{kj}^r y^k = 0$, we get

$$\frac{\partial}{\partial y^i}(R_{jkh}y^k) + \frac{\partial}{\partial y^j}(R_{ikh}y^k) \tag{14}$$

$$- R_{jih} - R_{ijh} = 4R_{kh}^r C_{ijr} y^k.$$

Suppose that F is of scalar flag curvature $K(x,y)$. Then, by using Proposition 6, the equation (14) can be rewritten as follows

$$\frac{\partial}{\partial y^i}(F^2 K h_{jh}) = 2F\ell_i K h_{jh} + F^2 \frac{\partial K}{\partial y^i} h_{jh} + F^2 K\left(2C_{ijh} - F^{-1}(h_{ji}\ell_h + h_{hi}\ell_j)\right)$$

$$= FK(2\ell_i h_{jh} - h_{ji}\ell_h - h_{hi}\ell_j) + 2F^2 K C_{ijh} + F^2 \frac{\partial K}{\partial y^i} h_{jh}$$

$$= FK(\ell_i h_{jh} - h_{ji}\ell_h - h_{hi}\ell_j) + 2F^2 K C_{ijh} + 3K_i h_{jh} - 3FK\ell_i h_{jh}$$

$$= -FK(\ell_j h_{ih} + \ell_i h_{hj} + \ell_h h_{ji}) + 2F^2 K C_{jih} + 3K_i h_{jh}. \tag{17}$$

Similarly

$$\frac{\partial}{\partial y^j}(F^2 K h_{ih}) = -FK(\ell_j h_{ih} + \ell_i h_{hj} + \ell_h h_{ji}) \tag{18}$$

$$+ 2F^2 K C_{jih} + 3K_j h_{ih}.$$

Plugging (16), (17) and (18) into (15) implies that

$$(K_i - FK\ell_i)h_{jh} + (K_h - FK\ell_h)h_{ji} \tag{19}$$

$$+ (K_j - FK\ell_j)h_{ih} = 0.$$

Contracting (19) with g^{ji}, we get

$$(n+1)(K_h - FK\ell_h) = 0, \tag{20}$$

and consequently

$$K_h - FK\ell_h = \frac{1}{3}F^2 \frac{\partial K}{\partial y^h} = 0, \tag{21}$$

which implies that $K = K(x)$ is isotropic. Now, the proof follows from Schur's lemma.

By Proposition 5, if hh-curvature of Cartan connection depends only on position, then

$$\frac{\partial}{\partial y^i}(F^2 K h_{jh}) + \frac{\partial}{\partial y^j}(F^2 K h_{ih}) = h_{jh}K_i - h_{ji}K_h \tag{15}$$

$$+ h_{ih}K_j - h_{ij}K_h + 4C_{ijr}F^2 K h_h^r.$$

Note that, we have

$$C_{ijr}h_h^r = C_{ijr}\delta_h^r - C_{ijr}\ell^r \ell_h = C_{ijh}. \tag{16}$$

Now, we simplify the term $\frac{\partial}{\partial y^i}(F^2 K h_{jh})$ as follows

$W_{ijk}^h = 0$. Thus, we have the following.

Corollary 7. If hh-curvature of the Cartan connection depends only on position and $F^n (n \geq 3)$ is of scalar flag curvature, then F is of constant flag curvature.

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