IJST (2013) 37A3: 265-275
Iranian Journal of Science & Technology

http://ijsts.shirazu.ac.ir

Hemirings characterized by the properties of their $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideals

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Abstract

In this paper we define $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-sub hemirings, $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-ideals, $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-bi-ideals and $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-quasi-ideals. We characterize h-hemiregular and h-intra-hemiregular hemirings by the properties of their $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-ideals, $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-bi-ideals and $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-quasi-ideals.

Keywords: $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-sub hemirings; $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-ideals; $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-bi-ideals; $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-quasi-ideals; h-hemiregular hemirings; h-intra-hemiregular hemirings

1. Introduction

A non-empty set R together with two associative binary operations, addition "$+$" and multiplication "$.$", such that "$.$" distributes over "$+$", is called a semiring and was first introduced by Vandiver [1] in 1934. Semirings have been found useful for solving problems in different areas of applied mathematics and information sciences, since the structure of a semiring provides an algebraic framework for modelling and studying the key factors of these applied areas. Hemirings (semirings with zero element and commutative addition) is a common generalization of rings and distributive lattices. In more recent times semirings have been deeply studied, especially in relation to applications [2, 3].

In the structure theory of semirings, ideals play an important role and are very useful for many purposes. However, in general, they do not coincide with the usual ring ideals. Many results in rings have no apparent analogues in semirings using only ideals. In order to overcome this difficulty, in [4] Henriksen defined a more restricted class of ideals in semirings, called k-ideals, with the property that if the semiring R is a ring, then a complex in R is a k- ideal if and only if it is a ring ideal. Another more restricted, but very important, class of ideals in hemirings, called now h-ideals, has been given and investigated by Iizuka [5]. La Torre [6] thoroughly studied h-ideals and k-ideals.

The concept of fuzzy sets was proposed by Zadeh [7] in 1965. Which provided a useful mathematical tool for describing the systems that are too complex or ill-defined. Since then, fuzzy sets has been applied to many branches of Mathematics. The fuzzification of algebraic structures was initiated by Rosenfeld [8], he introduced the notion of fuzzy subgroups. Since then, the concept of fuzziness has been extensively used in different fields of algebras, for example [9-15]. In [16] J. Ahsan et. al. initiated the study of fuzzy semirings. Fuzzy k-ideals in semirings are studied in [17, 18], and fuzzy h-ideals are studied in [19] among others. The fuzzy algebraic structures play an important role in Mathematics with wide applications in theoretical physics, computer sciences, control engineering, information sciences, coding theory and topological spaces [20, 21].

The notions of "belongingness" and "quasi-coincidence" of fuzzy points and fuzzy sets proposed and discussed by Pu and Liu [22]. After this, many authors used these concepts to generalize some concepts of algebra, for example [23, 24]. In [25, 26], (α, β)-fuzzy ideals of hemirings are

Received: 26 March 2012 / Accepted: 8 July 2012

defined. Generalizing the concept of quasi-coincident of a fuzzy point with a fuzzy set, in [27], Jun defined $(\in, \in \vee q_k)$-fuzzy subgroups and $(\in, \in \vee q_k)$-fuzzy subalgebras in BCK/BCI-algebras, respectively. In [28] $(\in, \in \vee q_k)$-fuzzy h-ideals and $(\in, \in \vee q_k)$-fuzzy k-ideals of hemiring are defined and discussed. Recently Shabir et. al. [29] characterized different classes of semigroups by the properties of their $(\in, \in \vee q_k)$-fuzzy ideals and $(\in, \in \vee q_k)$-fuzzy bi-ideals.

In this paper we define $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-sub hemirings, $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-ideals, $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-bi-ideals and $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-quasi-ideals. We characterize h-hemiregular and h-intra-hemiregular hemirings by the properties of their $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-ideals, $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-bi-ideals and $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-quasi-ideals. Some of these characterizations are generalizations of the characterizations given in [30].

2. Preliminaries

A set $R \neq \phi$ together with two binary operations addition "$+$" and multiplication "$\cdot$" is called a semiring if

(i) $(R,+)$ is semigroup

(ii) $(R,.)$ is semigroup

(iii) Multiplication is distributive from both sides over addition.

An element $0 \in R$ satisfying the conditions $0x = x0 = 0$ and $0 + x = x + 0 = x,$ for all $x \in R,$ is called a zero of the semiring $(R,+,\cdot)$. An element $1 \in R$ satisfying the condition $1x = x1 = x$ for all $x \in R,$ is called identity of the semiring $(R,+,\cdot)$. A semiring with commutative multiplication is called a commutative semiring. A semiring with commutative addition and zero element is called a hemiring. A non-empty subset A of R is called a sub hemiring of R if it contains zero and is closed with respect to the addition and multiplication of R. A non-empty subset I of R is called a left (right) ideal of R if I is closed under addition and $RI \subseteq I$ $(IR \subseteq I)$. Furthermore, I is called an ideal of R if it is both a left ideal and right ideal of R. A non-empty subset Q of R is called a quasi-ideal of R if Q is closed under addition and $RQ \cap QR \subseteq Q$. A sub hemiring B of a hemiring R is called a bi-ideal of R if $BRB \subseteq B$. Every one sided ideal of a hemiring R is a quasi-ideal and every quasi-ideal is a bi-ideal, but the converse is not true.

A left (right) ideal I of a hemiring R is called a left (right) h-ideal if for all $x, z \in R$ and for any $a, b \in I,$ from $x + a + z = b + z$ it follows $x \in I$. A bi-ideal B of a hemiring R is called an h-bi-ideal of R if for all $x, z \in R$ and $a, b \in B,$ from $x + a + z = b + z$ it follows $x \in B$ [31]. The h-closure $\overline{A}$ of a non-empty subset A of a hemiring R is defined as

$$\overline{A} = \{x \in R \mid x + a + z = b + z \text{ for some } a, b \in A, z \in R\}.$$

A quasi-ideal Q of a hemiring R is called an h-quasi-ideal of R if $\overline{RQ} \cap \overline{QR} \subseteq Q$ and $x + a + z = b + z$ implies $x \in Q$, for all $x, z \in R$ and $a, b \in Q$ [31]. Every left (right) h-ideal of a hemiring R is an h-quasi-ideal of R and every h-quasi-ideal is an h-bi-ideal of R. However, the converse is not true in general.

A fuzzy subset f of a universe X is a function $f : X \to [0,1]$. A fuzzy subset f of X of the form

$$f(z) = \begin{cases} t \in (0,1] & \text{if } z = x \\ 0 & \text{if } z \neq x \end{cases}$$

is called the fuzzy point with support x and value t and is denoted by x_t. Furthermore for $t \in [0,1]$, level subset of a fuzzy subset f of X is denoted and defined by $U(f,t) = \{x \in X : f(x) \geq t\}.$ In [22] Pu and Liu defined and discussed $x_t \alpha f$, where $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$. A fuzzy point x_t is said to belong to $(\text{resp. quasi-coincident with})$ a fuzzy set f written $x_t \in f$ $(\text{resp. } x_t q f)$ if $f(x) \geq t$ $(\text{resp. } f(x) + t > 1)$, and in this case, $x_t \in \vee q f$

$\left(\text{resp. } x_t \in \wedge qf\right)$ means that $x_t \in f$ or $x_t qf$ $\left(\text{resp. } x_t \in f \text{ and } x_t qf\right)$. For $k \in [0,1)$, $x_t q_k f$ means $f(x)+t+k>1$. To say that $x_t \overline{\alpha} f$ means that $x_t \alpha f$ does not hold. For any two fuzzy subsets f and g of X, $f \le g$ means that, for all $x \in X$, $f(x) \le g(x)$. The symbols $f \wedge g$ and $f \vee g$ will mean the following fuzzy subsets of X

$(f \wedge g)(x) = \min\{f(x), g(x)\}$

and $(f \vee g)(x) = \max\{f(x), g(x)\}$

for all $x \in X$.

Definition 1. [31]

The h-intrinsic product of two fuzzy subsets f and g of a hemiring R is defined by

$$\left(f \otimes g\right)\left(x\right) = \begin{cases} \bigvee\limits_{x+\Sigma_{i=1}^{m} a_i b_i + z = \Sigma_{j=1}^{n} a_j' b_j' + z} \left\{ \begin{array}{l} \left(\bigwedge\limits_{i=1}^{m} f\left(a_i\right) \right) \wedge \\ \left(\bigwedge\limits_{i=1}^{m} g\left(b_i\right) \right) \wedge \\ \left(\bigwedge\limits_{j=1}^{n} f\left(a_j'\right) \right) \wedge \\ \left(\bigwedge\limits_{j=1}^{n} g\left(b_j'\right) \right) \end{array} \right\} \\ 0 \quad \text{if } x \text{ cannot be expressed as} \\ x + \Sigma_{i=1}^{m} a_i b_i + z = \Sigma_{j=1}^{n} a_j' b_j' + z \end{cases}$$

3. $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-Fuzzy ideals

Throughout in this paper R will denote hemiring and $\mathbf{R}$ will denote the fuzzy subset of R mapping every element of R on 1.

Definition 2.

(i) A fuzzy subset f of R is called an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-sub hemiring of R if for any $x, y \in R$ and $t, r \in (0,1]$, it satisfies

$(1b')$ $\quad \left(x+y\right)_{\min\{t,r\}} \overline{\in} f \Rightarrow x_t \overline{\in} \vee \overline{q_k} f$ or $y_r \overline{\in} \vee \overline{q_k} f$.

$(2b')$ $\quad \left(xy\right)_{\min\{t,r\}} \overline{\in} f \Rightarrow x_t \overline{\in} \vee \overline{q_k} f$ or $y_r \overline{\in} \vee \overline{q_k} f$.

$(5b')$ $\quad \left(x\right)_{\min\{t,r\}} \overline{\in} f \Rightarrow a_t \overline{\in} \vee \overline{q_k} f$ or $b_r \overline{\in} \vee \overline{q_k} f$, for all $a, b, x, z \in R$ with $x+a+z = b+z$. (ii) A fuzzy subset f of R is called an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left (resp. right) ideal of R if for any $x, y \in R$ and $t, r \in (0,1]$, if it satisfies $(1b')$ and

$(4b')$ $\left(yx\right)_t \overline{\in} f \Rightarrow x_t \overline{\in} \vee \overline{q_k} f$

(resp. $(3b')$ $\left(xy\right)_t \overline{\in} f \Rightarrow x_t \overline{\in} \vee \overline{q_k} f$)

An $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left and right ideal of R is called $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideal of R.

(iii) An $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideal of R is said to be an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-ideal of R if it satisfies $(5b')$ $x+a+z = b+z$, $\left(x\right)_{\min\{t,r\}} \overline{\in} f \Rightarrow a_t \overline{\in} \vee \overline{q_k} f$ or $b_r \overline{\in} \vee \overline{q_k} f$, for all $x, z, a, b \in R$ and $t, r \in (0,1]$.

Theorem 1.

Let I be an h-sub hemiring of R. Define a fuzzy subset f of R as follows:

$$f\left(x\right) = \begin{cases} < \frac{1-k}{2} & \text{for } x \notin I \\ 1 & \text{otherwise} \end{cases}$$

Then f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-sub hemiring of R.

Proof: Let $x, y \in R$ and $t, r \in (0,1]$, such that $(x+y)_{\min\{t,r\}} \overline{\in} f$,

Then

$f(x+y) < \min\{t,r\} \Rightarrow f(x+y) \neq 1$

$\Rightarrow f(x+y) < \frac{1-k}{2}$

$\Rightarrow x+y \notin I \Rightarrow x \notin I$ or $y \notin I \Rightarrow f(x) < \frac{1-k}{2}$ or $f(y) < \frac{1-k}{2}$.

If $\min\{t,r\} > \frac{1-k}{2}$, then $f(x) < \frac{1-k}{2} < \min\{t,r\}$ or $f(y) < \frac{1-k}{2} < \min\{t,r\}$

$\Rightarrow \quad f(x) < t$ or $f(y) < r \Rightarrow x_t \overline{\in} f$ or $y_r \overline{\in} f$

$\Rightarrow x_t \overline{\in} \vee \overline{q_k} f$ or $y_r \overline{\in} \vee \overline{q_k} f$.

If $\min\{t,r\} \leq \frac{1-k}{2}$, then $t \leq \frac{1-k}{2}$ or $r \leq \frac{1-k}{2}$

$\Rightarrow f(x)+t+k < \frac{1-k}{2}+\frac{1-k}{2}+k=1$ or $f(y)+r+k < \frac{1-k}{2}+\frac{1-k}{2}+k=1$

$\Rightarrow x_t \overline{q_k} f$ or $y_r \overline{q_k} f \Rightarrow x_t \overline{\in} \vee \overline{q_k} f$ or $y_r \overline{\in} \vee \overline{q_k} f$. This proves $(1b')$.

Proofs of $(2b')$ and $(5b')$ are similar. Hence f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k} f)$-fuzzy h-sub hemiring of R.

Definition 3.
A fuzzy subset f of a hemiring R is said to be an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-bi-ideal of R, if it satisfies $(1b'), (2b'), (5b')$ and

$(6b') (xyz)_{\min\{t,r\}} \overline{\in} f \Rightarrow x_t \overline{\in} \vee \overline{q} f$ or $z_r \overline{\in} \vee \overline{q_k} f$, for all $x, y, z \in R$.

Theorem 2.
Let I be an h-ideal (h-bi-ideal) of R. Define a fuzzy subset f of R as follows:

$$f(x) = \begin{cases} < \frac{1-k}{2} & \text{for } x \notin I \\ 1 & \text{otherwise} \end{cases}$$

Then f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-ideal (h-bi-ideal) of R.

Proof: Proof is straightforward by using Theorem 1.

Theorem 3.
For a fuzzy subset f of a hemiring R and for all $x, y, z, a, b \in R$, $(1b')$ to $(6b')$ are equivalent to $(1c')$ to $(6c')$, where

$(1c')$ $\max\{f(x+y), \frac{1-k}{2}\} \geq \min\{f(x), f(y)\}$

$(2c')$ $\max\{f(xy), \frac{1-k}{2}\} \geq \min\{f(x), f(y)\}$

$(3c')$ $\max\{f(xy), \frac{1-k}{2}\} \geq f(x)$

$(4c')$ $\max\{f(yx), \frac{1-k}{2}\} \geq f(x)$

$(5c')$ $\max\{f(x), \frac{1-k}{2}\} \geq \min\{f(a), f(b)\}$, when $x+a+z=b+z$.

$(6c')$ $\max\{f(xyz), \frac{1-k}{2}\} \geq \min\{f(x), f(z)\}$

Proof: We prove $(1b')$ is equivalent to $(1c')$. Others follow in an analogous way.

$(1b') \Rightarrow (1c')$ Suppose $(1c')$ does not hold. Then there exists $x, y \in R$, such that $\max\{f(x+y), \frac{1-k}{2}\} < \min\{f(x), f(y)\}$. Then we can choose $t \in (\frac{1-k}{2}, 1]$, such that $\max\{f(x+y), \frac{1-k}{2}\} < t < \min\{f(x), f(y)\}$. Then $(x+y)_t \overline{\in} f$ but $x_t \in \wedge q_k$ and $y_t \in \wedge q_k$, which is a contradiction. So $(1c')$ holds.

$(1c') \Rightarrow (1b')$ Let $x, y \in R$, and $t, r \in (0,1]$, such that $(x+y)_{\min\{t,r\}} \overline{\in} f$. Then $f(x+y) < \min\{t,r\}$. If $\max\{f(x+y), \frac{1-k}{2}\} = f(x+y)$, then $\min\{f(x), f(y)\} \leq f(x+y) < \min\{t,r\}$

$\Rightarrow f(x) < t$ or $f(y) < r \Rightarrow x_t \overline{\in} f$ or $y_r \overline{\in} f \Rightarrow x_t \overline{\in} \vee \overline{q_k} f$ or $y_r \overline{\in} \vee \overline{q_k} f$. If $\max\{f(x+y), \frac{1-k}{2}\} = \frac{1-k}{2}$, then $\min\{f(x), f(y)\} \leq \frac{1-k}{2}$. Suppose $x_t \in f$ and $y_r \in f$, then $f(x) \geq t$ and $f(y) \geq r$, then $t \leq f(x) < \frac{1-k}{2}$ or $r \leq f(y) < \frac{1-k}{2} \Rightarrow x_t \overline{q_k} f$ or $y_r \overline{q_k} f \Rightarrow x_t \overline{\in} \vee \overline{q_k} f$ or $y_r \overline{\in} \vee \overline{q_k} f$.

Theorem 4.
(i) A fuzzy subset f of a hemiring R is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-sub hemiring of R if and only if it satisfies $(1c'), (2c')$ and $(5c')$.

(ii) A fuzzy subset f of a hemiring R is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-ideal of R if and only if it satisfies $(1c'), (3c'), (4c')$ and $(5c')$.

(iii) A fuzzy subset f of a hemiring R is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-bi-ideal of R if and only if it satisfies $(1c'), (2c'), (5c')$ and $(6c')$.

Proof: Proof is straightforward by using Theorem 3.

Definition 4.
A fuzzy subset f of a hemiring R is said to be an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-quasi-ideal of R if it satisfies $(1c'), (5c')$ and

$(7c') \max\{f(x), \frac{1-k}{2}\}$

$\geq \min\{(f\otimes \mathbf{R})(x), (\mathbf{R}\otimes f))(x)\}$

for all $x \in R$.

Proofs of the following Theorems are straightforward and hence omitted.

Theorem 5.

A fuzzy subset f of R is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-sub hemiring (h-ideal, h-bi-ideal, h-quasi-ideal) of R if and only if for all $t \in (\frac{1-k}{2}, 1]$, its level subset $U(f,t) \neq \phi$ is h-sub hemiring (h-ideal, h-bi-ideal, h-quasi-ideal) of R.

Theorem 6.

A non-empty subset A of R is an h-sub hemiring (h-ideal, h-bi-ideal, h-quasi-ideal) of R if and only if C_A is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-sub hemiring (h-ideal, h-bi-ideal, h-quasi-ideal) of R.

Lemma 1.

Every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left *(right)* h-ideal *of a* hemiring R *is an* $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-quasi-ideal *of* R.

Lemma 2.

Every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-quasi-ideal *of* R *is an* $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-bi-ideal *of* R.

Remark 1.

Converses of the Lemma 1 and Lemma 2 are not true in general.

Example 1.

Let $Z_0 = Z^+ \cup \{0\}$,

$$R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in Z_0 \right\}$$

and $Q = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in Z_0 \right\}$.

Then R is a hemiring under the usual addition and multiplication of matrices, and Q is h-quasi-ideal of R but Q is not left (right) h-ideal of R. Then by Theorem 6, C_Q is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-quasi-ideal of R and C_Q is not an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left (right) h-ideal of R.

Example 2.

Let Z^+ and R^+ be the sets of all positive integers and positive real numbers, respectively. And

$$R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} : a, b \in R^+, c \in Z^+ \right\}$$

$$I = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

$$\cup \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} : a, b \in R^+, c \in Z^+, a < b \right\}$$

$$L = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

$$\cup \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} : a, b \in R^+, c \in Z^+, b > 3 \right\}$$

Then R is a hemiring under the usual addition and multiplication of matrices, and I is right h-ideal and L is left h-ideal of R. Now, product IL is an h-bi-ideal of R and it is not an h-quasi-ideal of R. Then by Theorem 6, C_{IJ} is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-bi-ideal of R and it is not an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-quasi-ideal of R.

Definition 5.

Let f, g be fuzzy subsets of a hemiring R. Then the fuzzy subset $f+g$ of R is defined by

$$(f+g)(x) = \bigvee_{x+(a_1+b_1)+z=(a_2+b_2)+z} \left\{ \begin{array}{l} f(a_1) \wedge f(a_2) \\ \wedge g(b_1) \wedge g(b_2) \end{array} \right\}$$

for all $a_1, a_2, b_1, b_2, x, z \in R$.

Definition 6.

Let f and g be fuzzy subsets of a hemiring R then the fuzzy subsets $f \vee \frac{1-k}{2}$, $f \wedge^{\frac{1-k}{2}} g$, $f \otimes^{\frac{1-k}{2}} g$ and $f +^{\frac{1-k}{2}} g$ of R are defined as

$$\left(f \vee \tfrac{1-k}{2}\right)(x) = f(x) \vee \tfrac{1-k}{2}$$

$$\left(f \wedge^{\frac{1-k}{2}} g\right)(x) = (f \wedge g)(x) \vee \tfrac{1-k}{2}$$

$$\left(f \otimes^{\frac{1-k}{2}} g\right)(x) = (f \otimes g)(x) \vee \tfrac{1-k}{2}$$

$$\left(f+^{\frac{1-k}{2}}g\right)(x)=\left(f+g\right)(x)\vee\tfrac{1-k}{2}$$

for all $x\in R$.

Lemma 3.
If f and g are $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy right and left h-ideals of R respectively, then $f\otimes^{\frac{1-k}{2}}g\le f\wedge^{\frac{1-k}{2}}g$.

Proof: Proof is straightforward.

4. h-hemiregular hemirings

In this section we characterize h-hemiregular hemirings by the properties of their $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy h-ideals, $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy h-bi-ideals and $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy h-quasi-ideals.

Definition 6. [31]
A hemiring R is said to be h-hemiregular if for each $x\in R$, there exist $a,b,z\in R$ such that $x+xax+z=xbx+z$.

Lemma 4. [31]
A hemiring R is h-hemiregular if and only if for any right h-ideal I and any left h-ideal L of R we have $\overline{IL}=I\cap L$.

Lemma 5. [31]
Let R be a hemiring. Then the following conditions are equivalent.
(i) R is h-hemiregular.
(ii) $B=\overline{BRB}$ for every h-bi-ideal B of R.
(iii) $Q=\overline{QRQ}$ for every h-quasi-ideal Q of R.

Theorem 7.
For a hemiring R the following conditions are equivalent.
(i) R is h-hemiregular.
(ii) $(f\wedge^{\frac{1-k}{2}}g)=(f\otimes^{\frac{1-k}{2}}g)$ for every $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy right h-ideal f and every $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy left h-ideal g of R.

Proof: $(i)\Rightarrow(ii)$: Let f be an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy right h-ideal and g be an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy left h-ideal, then by Lemma 6, $f\otimes^{\frac{1-k}{2}}g\le f\wedge^{\frac{1-k}{2}}g$. Let $a\in R$, then there exist $x_1,x_2,z\in R$ such that $a+ax_1a+z=ax_2a+z$. Thus we have

$$\left(f\otimes^{\frac{1-k}{2}}g\right)(a)$$

$$=\left(\bigvee_{a+\sum_{i=1}^{m}a_ib_i+z=\sum_{j=1}^{n}c_jd_j+z}\left(\begin{array}{l}\left(\bigwedge_{i=1}^{m}f(a_i)\right)\wedge\\ \left(\bigwedge_{i=1}^{m}g(b_i)\right)\wedge\\ \left(\bigwedge_{j=1}^{n}f(c_j)\right)\wedge\\ \left(\bigwedge_{j=1}^{n}g(d_j)\right)\end{array}\right)\right)\vee\frac{1-k}{2}$$

$$\ge\{(f(a)\wedge g(x_1a)\wedge g(x_2a))\vee\frac{1-k}{2}\}$$

$$\ge\{f(a)\wedge g(a)\}\vee\frac{1-k}{2}$$

$$=\left(f\wedge^{\frac{1-k}{2}}g\right)(a).$$

So $\left(f\otimes^{\frac{1-k}{2}}g\right)\ge\left(f\wedge^{\frac{1-k}{2}}g\right)$.

Hence $\left(f\otimes^{\frac{1-k}{2}}g\right)=\left(f\wedge^{\frac{1-k}{2}}g\right)$.

$(ii)\Rightarrow(i)$: Let I and L be right and left h-ideals of R, respectively. Then by Theorem 6, C_I and C_L are $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy right h-ideal and $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy left h-ideal of R. Hence by hypothesis $C_I\otimes^{\frac{1-k}{2}}C_L=C_I\wedge^{\frac{1-k}{2}}C_L$. Thus $(C_I\otimes C_L)\vee\frac{1-k}{2}=(C_I\wedge C_L)\vee\frac{1-k}{2}$. This implies $C_{\overline{IL}}\vee\frac{1-k}{2}=C_{I\cap L}\vee\frac{1-k}{2}$. Hence $\overline{IL}=I\cap L$, so by Lemma 4, R is h-hemiregular.

Theorem 8.
For a hemiring R, the following conditions are equivalent.
(i) R is h-hemiregular.
(ii) $f\vee\frac{1-k}{2}\le(f\otimes^{\frac{1-k}{2}}\mathbf{R}\otimes^{\frac{1-k}{2}}f)$ for every $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy h-bi-ideal f of R.

(iii) $f \vee \frac{1-k}{2} \le (f \otimes^{\frac{1-k}{2}} \mathbf{R} \otimes^{\frac{1-k}{2}} f)$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-quasi-ideal f of R.

Proof: $(i) \Rightarrow (ii)$: Let f be an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-bi-ideal f of R and $a \in R$. Then there exist $x, x', z \in R$ such that $a + axa + z = ax'a + z$. Thus we have

$$(f \otimes^{\frac{1-k}{2}} \mathbf{R} \otimes^{\frac{1-k}{2}} f)(a)$$

$$= \left(\bigvee_{a+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} c_j d_j + z} \left(\left(\bigwedge_{i=1}^{m} \begin{pmatrix} f \otimes^{\frac{1-k}{2}} \mathbf{R})(a_i) \\ \wedge f(b_i) \end{pmatrix} \right) \wedge \left(\bigwedge_{j=1}^{n} \begin{pmatrix} f \otimes^{\frac{1-k}{2}} \mathbf{R})(c_j) \\ \wedge f(d_j) \end{pmatrix} \right) \right) \right) \vee \frac{1-k}{2}$$

$$\ge \{(f \otimes^{\frac{1-k}{2}} \mathbf{R})(ax) \wedge f(a) \wedge (f \otimes^{\frac{1-k}{2}} \mathbf{R})(ax') \vee \tfrac{1-k}{2}\}$$

$$\ge \{(f(axa) \wedge f(a) \wedge f(ax'a) \vee \tfrac{1-k}{2}\}$$

$$\ge f(a) \vee \tfrac{1-k}{2}.$$

$(ii) \Rightarrow (iii)$: This is straightforward.

$(iii) \Rightarrow (i)$: Let Q be an h-quasi-ideal of R. Then by Theorem 6, C_Q is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy h-quasi-ideal of R. Now by hypothesis $C_Q \vee \frac{1-k}{2} \le (C_Q \otimes^{\frac{1-k}{2}} \mathbf{R} \otimes^{\frac{1-k}{2}} C_Q) = C_{\overline{QRQ}} \vee \frac{1-k}{2}$ $\Rightarrow Q \subseteq \overline{QRQ},$ but $\overline{QRQ} \subseteq Q$. Therefore $Q = \overline{QRQ}$. Hence by Lemma 5, R is h-hemiregular.

The following characterizations are similar.

Theorem 9.

For a hemiring R, the following conditions are equivalent.

(i) R is h-hemiregular.

(ii) $\left(f \wedge^{\frac{1-k}{2}} g\right) \le \left(f \otimes^{\frac{1-k}{2}} g \otimes^{\frac{1-k}{2}} f\right)$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-bi-ideal f and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-ideal g of R.

(iii) $\left(f \wedge^{\frac{1-k}{2}} g\right) \le \left(f \otimes^{\frac{1-k}{2}} g \otimes^{\frac{1-k}{2}} f\right)$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-quasi-ideal and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-ideal g of R.

Theorem 10.

For a hemiring R, the following conditions are equivalent.

(i) R is h-hemiregular.

(ii) $\left(f \wedge^{\frac{1-k}{2}} g\right) \le \left(f \otimes^{\frac{1-k}{2}} g\right)$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-bi-ideal f and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left h-ideal g of R.

(iii) $\left(f \wedge^{\frac{1-k}{2}} g\right) \le \left(f \otimes^{\frac{1-k}{2}} g\right)$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-quasi-ideal f and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left h-ideal g of R.

(iv) $\left(f \wedge^{\frac{1-k}{2}} g\right) \le \left(f \otimes^{\frac{1-k}{2}} g\right)$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right h-ideal f and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-bi-ideal g of R.

(v) $\left(f \wedge^{\frac{1-k}{2}} g\right) \le \left(f \otimes^{\frac{1-k}{2}} g\right)$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right h-ideal f and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-quasi-ideal g of R.

(vi) $\left(f \wedge^{\frac{1-k}{2}} g \wedge^{\frac{1-k}{2}} h\right) \le \left(f \otimes^{\frac{1-k}{2}} g \otimes^{\frac{1-k}{2}} h\right)$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right h-ideal f, every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-bi-ideal g and every $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy right h-ideal h of R.

(vii) $\left(f \wedge^{\frac{1-k}{2}} g \wedge^{\frac{1-k}{2}} h\right) \le \left(f \otimes^{\frac{1-k}{2}} g \otimes^{\frac{1-k}{2}} h\right)$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right h-ideal f, every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-quasi-ideal g and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right h-ideal h of R.

5. h-intra-hemiregular hemirings

In this section we characterize h-hemiregular and h-intra-hemiregular hemirings by the properties of their $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-ideals, $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-bi-ideals and $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-quasi-ideals.

Definition 7. [31]

A hemiring R is said to be h-intra-hemiregular if for each $x \in R$, there exist $a_i, a_i', b_j, b_j', z \in R$ such that $x + \sum_{i=1}^{m} a_i x^2 a_i' + z = \sum_{j=i}^{n} b_j x^2 b_j' + z$.

Lemma 6. [31]
A hemiring R is h-intra-hemiregular if and only if for any right h-ideal I and any left h-ideal L of R we have $I \cap L \subseteq \overline{LI}$.

Example 3. [31]
(i) Let $R = \{0, a, b\}$ be a hemiring with addition "$+$" and multiplication "$\cdot$" defined by the following table:

$+$	0	a	b
0	0	a	b
a	a	a	b
b	b	b	b

$\cdot$	0	a	b
0	0	0	0
a	0	a	a
b	0	a	a

Then R is both h-hemiregular and h-intra-hemiregular hemiring.
(ii) Consider the hemiring N_0. It is not h-hemiregular and h-intra-hemiregular hemiring. Indeed $2 \in \mathrm{N}$ cannot be written as $2 + 2a2 + z = 2a^{'}2 + z$ and $2 + \sum_{i=1}^{m} a_i 2^2 b_i + z = \sum_{j=1}^{n} a_j^{'} 2^2 b_j^{'} + z$ for all $a_i, a_i^{'}, b_j, b_j^{'}, z \in \mathrm{N}$.

Lemma 7. [31]
The following conditions are equivalent for a hemiring R.
(i) R is both h-hemiregular and h-intra-hemiregular.
(ii) $B = \overline{B}^2$ for every h-bi-ideal B of R.
(iii) $Q = \overline{Q}^2$ for every h-quasi-ideal Q of R.

Lemma 8.
A hemiring R is an h-intra-hemiregular if and only if $f \wedge^{\frac{1-k}{2}} g \le f \otimes^{\frac{1-k}{2}} g$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left h-ideal f and for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right h-ideal g of R.

Proof: Let f and g be $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left h-ideal and $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right h-ideal of R, respectively. Let $a \in R$, then there exist $x_i, x_i^{'}, y_j, y_j^{'}, z \in R$ such that $a + \sum_{i=1}^{m} x_i a^2 x_i^{'} + z = \sum_{j=i}^{n} y_j a^2 y_j^{'} + z$. Thus we have

$$\left(f \otimes^{\frac{1-k}{2}} g\right)(a)$$

$$= \bigvee_{a + \sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a_j^{'} b_j^{'} + z} \left(\begin{array}{l} \left(\bigwedge_{i=1}^{m} f(a_i) \right) \wedge \\ \left(\bigwedge_{i=1}^{m} g(b_i) \right) \wedge \\ \left(\bigwedge_{j=1}^{n} f(a_j^{'}) \right) \wedge \\ \left(\bigwedge_{j=1}^{n} g(b_j^{'}) \right) \end{array} \right) \vee \frac{1-k}{2}$$

$$\ge \left(\begin{array}{l} \left(\bigwedge_{i=1}^{m} f(x_i a) \right) \wedge \left(\bigwedge_{i=1}^{m} g(a x_i^{'}) \right) \wedge \\ \left(\bigwedge_{j=1}^{n} f(y_j a) \right) \wedge \left(\bigwedge_{j=1}^{n} g(a y_j^{'}) \right) \end{array} \right) \vee \frac{1-k}{2}$$

$$\ge \{ f(a) \wedge g(a) \} \vee \frac{1-k}{2}$$

$$= \{ f(a) \wedge g(a) \vee \frac{1-k}{2} \} = \left(f \wedge^{\frac{1-k}{2}} g \right)(a).$$

So $f \otimes^{\frac{1-k}{2}} g \ge f \wedge^{\frac{1-k}{2}} g$.

Conversely assume that P and Q are left and right h-ideals of R, respectively. Then by Theorem 6, C_P and C_Q are $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left h-ideal and $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right h-ideal of R, respectively. Then by hypothesis $C_P \wedge_{\frac{1-k}{2}} C_Q \le C_P \otimes_{\frac{1-k}{2}} C_Q$

$$\Rightarrow C_{P \cap Q} \wedge \tfrac{1-k}{2} \le C_{\overline{PQ}} \wedge \tfrac{1-k}{2} \Rightarrow P \cap Q \subseteq \overline{PQ} \Rightarrow$$

by Lemma 6, R is h-intra-hemiregular.

Theorem 11.
The following conditions are equivalent for a hemiring R :
(i) R is both h-hemiregular and h-intra-hemiregular.
(ii) $f \vee \frac{1-k}{2} = f \otimes^{\frac{1-k}{2}} f$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-bi-ideal f of R.
(iii) $f \vee \frac{1-k}{2} = f \otimes^{\frac{1-k}{2}} f$ for every

$(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-quasi-ideal f of R.

Proof: $(i) \Rightarrow (ii)$ Let f be an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-bi-ideal of R and $x \in R$.

Then, as R is both h-hemiregular and h-intra-hemiregular, there exist $a, a', a_i, a_i', b_j, b_j', z \in R$ such that $x + xax + z = xa'x + z$ and $x + \sum_{i=1}^m a_i x^2 a_i' + z = \sum_{j=1}^n b_j x^2 b_j' + z$. Then (as given in Lemma 5.6, [31]) there exist $a_1, a_2, p_i, p_i', q_j, q_j', z \in R$ such that

$$x + \sum_{j=1}^n (xa_2q_jx)(xq_j'a_1x) + \sum_{j=1}^n (xa_1q_jx)(xq_j'a_2x)$$
$$+\sum_{i=1}^m (xa_1p_ix)(xp_i'a_1x) + \sum_{i=1}^m (xa_2p_ix)(xp_i'a_2x) + z$$
$$= \sum_{i=1}^m (xa_2p_ix)(xp_i'a_1x) + \sum_{i=1}^m (xa_1p_ix)(xp_i'a_2x)$$
$$+\sum_{j=1}^n (xa_1q_jx)(xq_j'a_1x) + \sum_{j=1}^n (xa_2q_jx)(xq_j'a_2x) + z$$

Now

$$(f \otimes^{\frac{1-k}{2}} f)(x)$$
$$= \bigvee_{x+\sum_{i=1}^m a_ib_i + z = \sum_{j=1}^n a_j'b_j' + z} \begin{pmatrix} \left(\bigwedge_{i=1}^m f(a_i)\right) \wedge \\ \left(\bigwedge_{i=1}^m f(b_i)\right) \wedge \\ \left(\bigwedge_{j=1}^n f(a_j')\right) \wedge \\ \left(\bigwedge_{j=1}^n f(b_j')\right) \end{pmatrix} \vee \tfrac{1-k}{2}$$
$$\geq \begin{pmatrix} \left(\bigwedge_{j=1}^n f(xa_2q_jx)\right) \wedge \left(\bigwedge_{j=1}^n f(xq_j'a_1x)\right) \wedge \\ \left(\bigwedge_{j=1}^n f(xa_1q_jx)\right) \wedge \left(\bigwedge_{j=1}^n f(xq_j'a_2x)\right) \end{pmatrix} \vee$$
$$\begin{pmatrix} \left(\bigwedge_{i=1}^m f(xa_1p_ix)\right) \wedge \left(\bigwedge_{i=1}^m f(xp_i'a_1x)\right) \wedge \\ \left(\bigwedge_{i=1}^m f(xa_2p_ix)\right) \wedge \left(\bigwedge_{i=1}^m f(xp_i'a_2x)\right) \end{pmatrix} \vee \tfrac{1-k}{2}$$
$$\geq \min\{f(x), \tfrac{1-k}{2}\} \vee \tfrac{1-k}{2} = f(x) \vee \tfrac{1-k}{2}.$$

This implies that $f \otimes^{\frac{1-k}{2}} f \geq f \vee \frac{1-k}{2}$.

On the other hand, if if $x + \Sigma_{i=1}^m a_ib_i + z = \Sigma_{j=1}^n a_j'b_j' + z$, we have $f(x) \vee \frac{1-k}{2} \geq \min\{f(\Sigma_{i=1}^m a_ib_i), f(\Sigma_{j=1}^n a_j'b_j'), \frac{1-k}{2}\} \wedge \frac{1-k}{2}$ by $(5c')$

$$\geq \min\{\left(\bigwedge_{i=1}^m f(a_ib_i)\right), \left(\bigwedge_{i=1}^m f(a_j'b_j')\right), \tfrac{1-k}{2}\}$$
$$\geq \min\left\{\begin{matrix} \left(\bigwedge_{i=1}^m f(a_i) \wedge f(b_i)\right), \\ \left(\bigwedge_{i=1}^m f(a_j') \wedge f(b_j')\right), \tfrac{1-k}{2} \end{matrix}\right\}$$

(because f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-bi-ideal of R.)

Thus

$$(f \otimes^{\frac{1-k}{2}} f)(x)$$
$$= \bigvee_{x+\sum_{i=1}^m a_ib_i + z = \sum_{j=1}^n a_j'b_j' + z} \begin{pmatrix} \left(\bigwedge_{i=1}^m f(a_i)\right) \wedge \\ \left(\bigwedge_{i=1}^m f(b_i)\right) \wedge \\ \left(\bigwedge_{j=1}^n f(a_j')\right) \wedge \\ \left(\bigwedge_{j=1}^n f(b_j')\right) \end{pmatrix} \vee \tfrac{1-k}{2}$$
$$\leq f(x) \vee \tfrac{1-k}{2}.$$

Consequently $f \vee \frac{1-k}{2} = f \otimes^{\frac{1-k}{2}} f$.

$(ii) \Rightarrow (iii)$ This is straightforward because every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-quasi-ideal of R is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-bi-ideal of R.

$(iii) \Rightarrow (i)$ Let Q be an h-quasi-ideal of R. Then by Theorem 6, C_Q is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy h-quasi-ideal of R. Thus by hypothesis $C_Q \vee \frac{1-k}{2} = C_Q \otimes^{\frac{1-k}{2}} C_Q = C_Q \otimes C_Q \vee \frac{1-k}{2} = C_{\overline{Q^2}} \vee \frac{1-k}{2}$

Then it follows $Q = \overline{Q^2}$. Hence by Lemma 7, R is both h-hemiregular and h-intra-hemiregular.

Theorem 12.
The following conditions are equivalent for a hemiring R:

(i) R is both h-hemiregular and h-intra-hemiregular.

(ii) $f \wedge^{\frac{1-k}{2}} g \leq f \otimes^{\frac{1-k}{2}} g$ for all $(\overline{\in}, \overline{\in \vee q_k})$-fuzzy h-bi-ideals f and g of R.

(iii) $f \wedge^{\frac{1-k}{2}} g \leq f \otimes^{\frac{1-k}{2}} g$ for every $(\overline{\in}, \overline{\in \vee q_k})$-fuzzy h-bi-ideal f and every $(\overline{\in}, \overline{\in \vee q_k})$-fuzzy h-quasi-ideals f of R.

(iv) $f \wedge^{\frac{1-k}{2}} g \leq f \otimes^{\frac{1-k}{2}} g$ for every $(\overline{\in}, \overline{\in \vee q_k})$-fuzzy h-quasi-ideal f and every $(\overline{\in}, \overline{\in \vee q_k})$-fuzzy h-bi-ideals f of R.

(v) $f \wedge^{\frac{1-k}{2}} g \leq f \otimes^{\frac{1-k}{2}} g$ for all $(\overline{\in}, \overline{\in \vee q_k})$-fuzzy h-quasi-ideals f and g of R.

Proof: $(i) \Rightarrow (ii)$ By using Theorem 11, proof is straightforward.

$(ii) \Rightarrow (iii) \Rightarrow (v)$ and $(ii) \Rightarrow (iv) \Rightarrow (v)$ are clear.

$(v) \Rightarrow (i)$ Let f be an $(\overline{\in}, \overline{\in \vee q_k})$-fuzzy right h-ideals of R and g be an $(\overline{\in}, \overline{\in \vee q_k})$-fuzzy left h-ideal of R. Then f and g are $(\overline{\in}, \overline{\in \vee q_k})$-fuzzy h-bi-ideal of R. So by hypothesis $f \wedge^{\frac{1-k}{2}} g \leq f \otimes^{\frac{1-k}{2}} g$ but $f \wedge^{\frac{1-k}{2}} g \geq f \otimes^{\frac{1-k}{2}} g$ by Lemma 3. Thus $f \wedge^{\frac{1-k}{2}} g = f \otimes^{\frac{1-k}{2}} g$. Hence by Theorem 7, R is h-hemiregular. On the other hand, by hypothesis we also have $f \wedge^{\frac{1-k}{2}} g \leq f \otimes^{\frac{1-k}{2}} g$. By Lemma 8, R is h-intra-hemiregular.

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