Water wave scattering by a surface discontinuity over a uniform porous bottom

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Abstract

Scattering of an incoming wave train by a discontinuity at the upper surface of the ocean over a uniform porous bottom of finite depth is considered here in the framework of linearized water wave theory. By employing residue calculus method, the reflection and transmission coefficient are obtained. These coefficients are computed numerically for different values of the porosity parameter G and these results are displayed in a number of Figs.

Keywords: Wave scattering; surface discontinuity; porous bottom; energy identity; residue calculus method; reflection and transmission coefficient

1. Introduction

The problem of water wave scattering by obstacles situated at the bottom in a finite depth water has a long history in the literature. Propagation of long wave along free surface over a sudden change in depth was discussed by Lamb [1], Stoker [2], Kreisel [3], Davies [4] who were the earliest contributors in this context. There are several important studies on water wave scattering that have been made in the framework of linearized theory over the last few decades by many scientists.

Water wave scattering also occurs by the presence of a discontinuity at the upper surface of water. A discontinuity in the upper surface or elsewhere may occur when there is a difference in wave number of the incoming waves of certain frequency from a sudden change in the constant width of the region. So there will be two different boundary conditions on either side of the ocean. The upper surface of the ocean may be covered by two vast sheets of floating ice plate or mat of different thickness or materials, broken ice covers, semi-infinite floating dock etc. The presence of obstacles or materials of different area densities or properties leads to a change in the boundary condition due to the difference in wave numbers. Evans and Linton [5] considered the problem of water wave scattering by a surface discontinuity in uniform finite depth water. By employing residue calculus technique they obtained the reflection and transmission coefficients for finite depth water. Recently Mandal

and De [6] considered the problem of water wave scattering by a small undulation at the bottom in the presence of upper surface discontinuity, and obtained the energy coefficients using perturbation method and Green's integral theorem.

Problem of water wave interaction over a porous bottom or structure has also been the focus of several researchers in recent times. If the bottom is composed of some specific type of porous materials (rigid or non rigid), the effect of porosity on the hydrodynamic coefficients is another important aspect to study. A porous sea bottom or structure may be rigid or nonrigid. The only difference is that a rigid or impermeable type porous structure or bottom does not allow the fluid to penetrate into it. Water wave interaction with the porous sea bed was studied earlier by Chakrabarti [7], Mase [8], Silva [9], Jeng [10] and many others. The flow of fluid into the porous media or the presence of porous materials in the bottom leads to different phenomena like energy dissipation, wave damping or decaying of wave height reaching towards the coast etc.

In the present paper, we consider the problem of scattering of an incoming wave train in the presence of a discontinuity at the upper surface of the ocean. The upper surface is assumed to be covered by two vast floating inertial surfaces of different materials and area densities. So there will be a difference in wave number of the incoming wave train due to the presence of the two different kinds of inertial surfaces at the upper surface of the ocean. The water is of uniform finite depth and the bottom is

composed of some specific kind of rigid porous material which is characterized by a known porosity parameter G. In the mathematical formulation we find that, there will be two different boundary conditions on either side of the upper surface of the ocean. Here the incoming wave train is partially reflected and partially transmitted through the ocean. The main object of this paper is to determine the reflection and transmission coefficients and investigate the effect of porosity on these hydrodynamic coefficients. Here, we employ the method of residue calculus of the complex variable theory(cf. [11]) to determine the reflection and transmission coefficient. Evans and Linton [5] also followed the same technique to obtain these hydrodynamic coefficients for uniform non porous bottom of finite depth. Here, the effect of porosity on the reflection and transmission coefficients is investigated numerically and these results are depicted in a few Figs. against the wave number of the incident wave. The effect of dimensionless porosity parameter on the hydrodynamic coefficients and phase values are also shown here graphically by taking different values of the dimensionless porosity parameter(Gh). These obtained results are explained by using the energy identity relation which is reformulated here in the presence of porous bottom.

2. Mathematical formulation of the problem

We consider the two dimensional motion in case of uniform finite depth water. A rectangular cartesian coordinate system is chosen in which y-axis is taken vertically downwards into the fluid region. The discontinuity is taken at the origin by assuming that the upper surface of the ocean is covered by two vast floating inertial surfaces of different materials and of different densities $E_1\rho$ and $E_2\rho$ respectively. The ocean bottom is composed of some specific kinds of porous materials characterized by the porosity parameter G. Let a train of surface water wave from negative infinity be incident on the line of discontinuity of density of inertial surfaces, which is partially reflected and partially transmitted along the ocean surface. Assuming that the fluid flow is irrotational and the motion is simple harmonic in time t with angular frequency ω, it can be described by a velocity potential $\psi(x, y, t) = Re\{\phi(x, y)e^{-i\omega t}\}$, where $\phi(x, y)$ satisfies the two dimensional Laplace equation:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \quad in\ the\ fluid\ region, \quad (1)$$

with the upper surface boundary conditions

$$K_1\phi + \frac{\partial \phi}{\partial y} = 0 \qquad on\ y = 0,\ x < 0, \quad (2)$$

$$K_2\phi + \frac{\partial \phi}{\partial y} = 0 \qquad on\ y = 0,\ x > 0. \quad (3)$$

This produces a discontinuity in the upper surface boundary condition at the point (0,0), where $K_1 = \frac{K}{1-E_1K}$, $K_2 = \frac{K}{1-E_2K}$, $E_1, E_2 < \frac{g}{\sigma^2}$ and $K = \frac{\sigma^2}{g}$ (g is the acceleration due to gravity).

The edge condition is given by

$$r^{1/2}\nabla\phi = O(1) \quad as \quad r = \{x^2 + y^2\}^{1/2} \to 0, \quad (4)$$

and the bottom boundary condition

$$\frac{\partial \phi}{\partial y} = G\phi \qquad on\ y = h. \quad (5)$$

The far field behavior of $\phi(x, y)$ is described by

$$\phi(x,y) \to \begin{cases} Te^{is_0 x}\psi_0^2(y) & as\ x \to +\infty, \\ (e^{ik_0 x} + Re^{-ik_0 x})\psi_0^1(y) & as\ x \to -\infty, \end{cases} \quad (6)$$

where

$$\psi_0^1(y) = N_0^1\left(\cosh k_0(h - y) - \frac{G}{k_0}\sinh k_0(h - y)\right),$$

$$\psi_0^2(y) = N_0^2\left(\cosh s_0(h - y) - \frac{G}{s_0}\sinh s_0(h - y)\right)$$

and

$$N_0^1 = \frac{2k_0^{\frac{3}{2}}}{\sqrt{2k_0(G - G^2 h + k_0^2 h) - 2Gk_0\cosh 2k_0 h + (k_0^2 + G^2)\sinh 2k_0 h}}$$

$$N_0^2 = \frac{2s_0^{\frac{3}{2}}}{\sqrt{2s_0(G - G^2 h + s_0^2 h) - 2Gs_0\cosh 2s_0 h + (s_0^2 + G^2)\sinh 2s_0 h}}$$

Here, $e^{ik_0 x}\psi_0^1(y)$ represents the incident wave field, R and T are respectively the unknown reflection and transmission coefficients(complex) to be determined. k_0 and s_0 are the real positive roots (cf. McIver [12]) of the following two transcendental equations in terms of λ:

$$\left(\lambda + \frac{K_1 G}{\lambda}\right)\tanh\lambda h = K_1 + G,$$

$$\left(\lambda + \frac{K_2 G}{\lambda}\right)\tanh\lambda h = K_2 + G.$$

3. Surface discontinuity: Energy identity relation

A discontinuity in the upper surface boundary condition may occur due to a sudden change in wave number of the incoming wave train. This difference in the wave number could arise due to change in the constant width of the region or sudden change in the boundary condition. The energy identity($R^2 + T^2 = 1$) is not followed due to the presence of discontinuity at the upper

surface. However, the modified energy identity relation has been formulated using Green's integral theorem by Evans and Linton [5]. Here, we reproduce the same energy identity in case of a uniform porous bottom. Two distinct types of solutions can be considered, describing waves incident from either $x \to -\infty$ and $x \to +\infty$ respectively, and these waves are partially reflected and partially transmitted from $x = 0$.

When the wave train is incident from negative infinity direction,

$$\phi(x,y) \to \begin{cases} Te^{is_0 x}\psi_0^2(y) & as \quad x \to +\infty, \\ (e^{ik_0 x} + Re^{-ik_0 x})\psi_0^1(y) & as \quad x \to -\infty \end{cases} \quad (7)$$

and if the wave train is incident from $x \to +\infty$

$$\chi(x,y) \to \begin{cases} te^{is_0 x}\psi_0^2(y) & as \quad x \to -\infty, \\ (e^{ik_0 x} + re^{-ik_0 x})\psi_0^1(y) & as \quad x \to +\infty. \end{cases} \quad (8)$$

By employing Green's integral theorem for the two functions $\chi(x,y)$ and $\phi(x,y)$, along the contour L bounded by the lines

$$y = 0, -X \le x \le X; x = \pm X, 0 \le y \le h;$$
$$y = h, -X \le x \le X(X > 0),$$

we get,

$$\oint_L (\phi\chi_\eta - \phi_\eta\chi)dl = 0, \quad (9)$$

where η is the outward normal to the line element dl. Now using the upper surface and bottom boundary conditions (2), (3) and (5) respectively, we see that there is no contribution to the integral from the part $0 < y < h$, $x = 0$ and $y = h$, $-X \le x \le X(X > 0)$ of the above contour L. Now using the far field conditions (7)-(8), we get,

$$\alpha|t| = |T|. \quad (10)$$

The following relations can be obtained by choosing the functions $\phi, \bar{\phi}; \bar{\chi}, \chi$ and ϕ, χ in turn, in place of ϕ, χ in (9) respectively.

$$\alpha(1 - |R|^2) = |T|^2, \quad (11)$$

$$\alpha(1 - |r|^2) = |t|^2, \quad (12)$$

$$\alpha|R||r| + |T||t| = 0, \quad (13)$$

where $a = \frac{k_0}{s_0}$. Now eliminating $\frac{|r|}{|t|}$ between (11)-(13), we obtain,

$$|R|^2 + \frac{1}{\alpha}|T|^2 = 1. \quad (14)$$

The above relation holds good in absence of the discontinuity at $x = 0$ i.e

$$|R|^2 + |T|^2 = 1, \quad (15)$$

which is the well known energy identity.

4. Method of solution: Eigenfunction matching technique

We consider the orthogonal depth eigenfunctions for two regions ($x < 0$ and $x > 0$) respectively as

$$\psi_n^1(y) = N_n^1\left(\cos k_n(h-y) - \frac{G}{k_n}\sin k_n(h-y)\right),$$

$$\psi_n^2(y) = N_n^2\left(\cos s_n(h-y) - \frac{G}{s_n}\sin s_n(h-y)\right)$$

where

$$N_n^1 = \frac{2k_n^{\frac{3}{2}}}{\sqrt{2k_n(G - G^2 h + k_n^2 h) - 2Gk_n\cos 2k_n h + (k_n^2 + G^2)\sin k_n h}},$$

$$N_n^2 = \frac{2s_n^{\frac{3}{2}}}{\sqrt{2s_n(G - G^2 h + s_n^2 h) - 2Gs_n\cos 2s_n h + (s_n^2 + G^2)\sin s_n h}}$$

and $k_n, s_n (n = 1,2,3,\dots)$ are given by the following two equations

$$\left(k_n - \frac{K_1 G}{k_n}\right)\tan k_n h + (K_1 + G) = 0,$$

$$\left(s_n - \frac{K_2 G}{s_n}\right)\tan s_n h + (K_2 + G) = 0.$$

The potential function $\phi(x,y)$ can now be expanded for two different regions in terms of orthogonal depth eigenfunctions in the form given by

$$\phi(x,y) = \begin{cases} \sum_{n=0}^\infty B_n e^{-s_n x}\psi_n^2(y) & as \quad x > 0, \\ e^{ik_0 x}\psi_0^1(y) + \sum_{n=0}^\infty A_n e^{k_n x}\psi_n^1(y) & as \quad x < 0, \end{cases} \quad (16)$$

where

$$A_0 = R,$$

$$B_0 = T$$

and $A_n, B_n (n = 1, 2, \dots)$ are the unknown constants.

The matching conditions at $x = 0$ for $\phi(x,y)$ and the orthogonality of the depth eigenfunctions produces the following two systems of linear equations

$$\sum_{n=1}^\infty \frac{V_n}{s_n - k_m} = A\delta_{0m}, \quad (17)$$

$$\sum_{n=1}^\infty \left(\frac{U_n}{k_n - s_m}\right) = N_0^1\left(\cosh k_0(h-y) - \frac{G}{k_0}\sinh k_0(h-y)\right)\left[\frac{R_0}{ik_0 + s_m} - \frac{1}{ik_0 - s_m}\right], \quad (18)$$

where

$$V_n = B_n N_n^2\left(\cos s_n(h-y) - \frac{G}{s_n}\sin s_n(h-y)\right),$$

$$A = -\frac{2ik_0}{(K_2 - K_1)N_0^1\left(\cosh k_0(h-y) - \frac{G}{k_0}\sinh k_0(h-y)\right)}$$

and

$$U_n = A_n N_n^1 \left(\cos k_n(h-y) - \frac{G}{k_n} \sin k_n(h-y) \right), \quad (m,n=1,2,\dots).$$

The unknown constants A_n, $B_n (n=1,2,\dots)$ can be estimated numerically from the above system of linear equations (17) and (18) after truncating the infinite sum up to desired accuracy. The reflection and transmission coefficient can now be determined by using residue calculus method.

5. Reflection and Transmission Coefficient

We consider the following integral

$$J = \oint_{C_N} \frac{f(z)}{z-k_m} dz, \quad (m=0,1,2,\dots).$$

Here the function $f(z)$ has simple poles at $z = s_1, s_2, \dots s_n$, simple zeroes at $z = k_1, k_2, \dots k_m$ and $f(z) \to O(\frac{1}{\sqrt{z}})$ as $z \to \infty$. C_N's are the sequence of circles with radius R_N that increase without bound as $N \to \infty$ avoiding the zeroes of the integrand and all the poles and zeroes are inside of it. Furthermore, C_N must not pass through $(0,0)$.

The Cauchy's integral formula in conjunction with the residue theorem gives

$$f(k_0) = \frac{1}{2\pi i} \oint_{C_N} \frac{f(z)}{z-k_0} dz = \sum_{n=1}^{\infty} \frac{(Res(f(z)|z=s_n)}{s_n-k_0}. \quad (19)$$

Assuming that $f(k_0) = -1$, we find

$$\delta_{0m} = \sum_{n=1}^{\infty} \frac{(Res(f(z)|z=s_n)}{s_n-k_m}. \quad (20)$$

Now comparing (17) with (20), we obtain

$$V_n = ARes(f(z)|z=s_n).$$

Again, we consider the integral

$$I = \oint_{C_N} \frac{f(z)}{z+k_m} dz, (m=0,1,2,\dots), \quad (21)$$

with the same property of the integrand function $f(z)$ as above. The matching conditions at $x=0$ can be combined to give

$$\sum_{n=1}^{\infty} \frac{V_n}{s_n+k_0} = -\frac{2ik_0R}{(K_2-K_1)N_0^1(\cosh k_0(h-y)-\frac{G}{k_0}\sinh k_0(h-y))}. \quad (22)$$

The Cauchy's residue theorem gives for $m=0$ and at $z=-k_0$

$$\sum_{n=1}^{\infty} \frac{V_n}{s_n+k_0} = Af(-k_0). \quad (23)$$

Comparing (22) and (23), we obtain

$$Af(-k_0) = -\frac{2ik_0R}{(K_2-K_1)N_0^1(\cosh k_0(h-y)-\frac{G}{k_0}\sinh k_0(h-y))}. \quad (24)$$

A suitable form of the function $f(z)$ can be taken as

$$f(z) = \frac{k_0}{z} \prod_{n=1}^{\infty} \left[\frac{(1-\frac{z}{k_n})(1-\frac{k_0}{s_n})}{(1-\frac{z}{s_n})(1-\frac{k_0}{k_n})} \right]. \quad (25)$$

At $z = -k_0$, from (25) f(z) gives

$$f(-k_0) = -\prod_{n=1}^{\infty} \left[\frac{\left(1+\frac{k_0}{k_n}\right)\left(1-\frac{k_0}{s_n}\right)}{\left(1+\frac{k_0}{s_n}\right)\left(1-\frac{k_0}{k_n}\right)} \right], \quad (26)$$

and replacing A and $f(-k_0)$ in (23), we obtain

$$R = \frac{k_0-s_0}{k_0+s_0} \prod_{n=1}^{\infty} \left[\frac{(1+\frac{k_o}{k_n})(1-\frac{k_0}{s_n})}{(1+\frac{k_0}{s_n})(1-\frac{k_0}{k_n})} \right].$$

Thus

$$R = \frac{k_0-s_0}{k_0+s_0} e^{2i\alpha}, \quad (27)$$

where

$$\alpha = \sum_{n=1}^{\infty} [\tan^{-1}(\frac{k_0}{s_n}) - \tan^{-1}(\frac{k_0}{k_n})], (n=1,2,\dots).$$

To obtain the transmission coefficient, we consider the following relation

$$V_n = ARes(f(z)|z=s_n). \quad (28)$$

Since we have $B_0 = T, V_n = B_n N_n^2(\cos s_n(h-y) - \frac{G}{s_n}\sin s_n(h-y))$ and hence we obtain,

$$T = \frac{2k_0 P(s_0+k_0)}{(K_2-K_1)N_0^1 N_0^2(\cosh k_0(h-y)-\frac{G}{k_0}\sinh k_0(h-y))(\cosh s_0(h-y)-\frac{G}{s_0}\sinh s_0(h-y))}, \quad (29)$$

where,

$$P = \prod_{n=1}^{\infty} \left[\frac{(1+\frac{s_0}{k_n})(1+\frac{s_0}{s_n})}{(1+\frac{k_0}{k_n})(1+\frac{k_0}{k_n})} \right] \quad (30)$$

An alternative form of T can be obtained by using the relation (14) and the expressions given by (29)-(30) as

$$|T| = \frac{2k_0}{k_0+s_0} \quad (31)$$

and the phase of T is the phase of P. Therefore, we have

$$\frac{\bar{P}}{P} = e^{-2i(\alpha+\beta)},$$

where

$$\beta = \sum_{n=1}^{\infty} \left[\tan^{-1}\left(\frac{s_0}{k_n}\right) - \tan^{-1}\left(\frac{s_0}{s_n}\right) \right], \quad (n=1,2,\dots).$$

6. Numerical results

The forms of the reflection and transmission coefficients given by (27) and (31) are computed numerically against the wave number of the incident wave. The effect of bottom porosity is investigated on the values of the reflection and transmission coefficients by taking different values of the dimensionless porosity parameter Gh. The absolute values of the reflection and transmission coefficients can be found from (27) and (31) and they are given by

$$|R| = \frac{k_0 - s_0}{k_0 + s_0},$$

$$|T| = \frac{2k_0}{k_0 + s_0}.$$
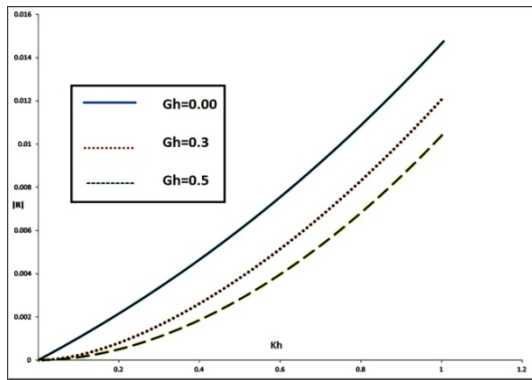
Fig. 1. $|R|$ for different Gh

The absolute values of these coefficients are depicted graphically in the Fig. 1 and 2 respectively for different values of the dimensionless porosity parameter taken as $Gh = 0.00, 0.3, 0.5$ respectively. In the Fig. 1, it is seen that as the value of the dimensionless porosity parameter Gh increases, the values of $|R|$ decreases rapidly.
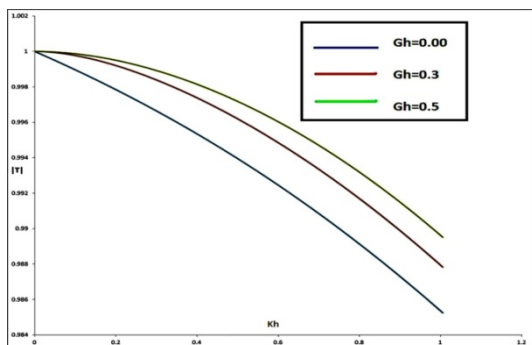
Fig. 2. $|T|$ for different Gh

This may be attributed to the presence of specific porous material at the bottom of the ocean which resists the wave field reflected by the discontinuity at the upper surface. The absolute values of the

reflection coefficient are decreased due to the characteristic of the porous materials at the bottom. The opposite phenomena is observed in the Fig. 2 for the case of $|T|$. As the value of the porosity parameter Gh increases, the values of $|T|$ also increase with Gh. This fact can be explained by the energy identity relation given by (14) whence there is a difference in wave number. Hence the effect of porosity in the ocean bed does not violate the energy identity relation as given by (14) in the presence of upper surface discontinuity. It is also noticeable that, in absence of the upper surface discontinuity, we have from (27) and (31):

$$R = 0,$$

$$T = 1.$$

In the Fig. 3 and 4, the phase values of the reflection and transmission coefficients are shown for $Gh = 0.0, 0.3, 0.5$ respectively. The phase values of the reflection and transmission coefficients can be found from (27) and (31) and they are given by 2α and $(\alpha+\beta)$ respectively, where
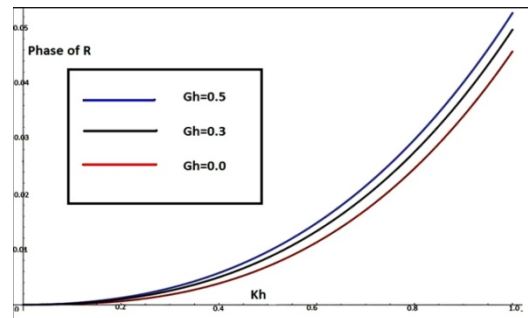
$$\alpha = \left(\frac{360^0}{\pi}\right) \sum_{n=1}^{\infty} \left[\tan^{-1}\left(\frac{k_0}{s_n}\right) - \tan^{-1}\left(\frac{k_0}{k_n}\right)\right],$$
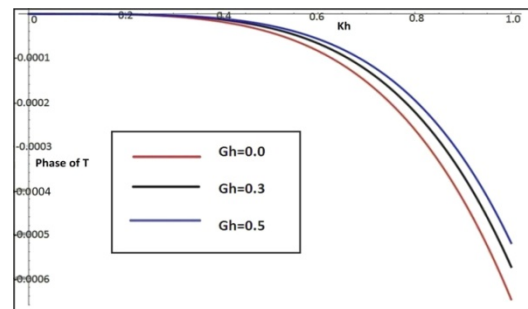$$(n = 1,2,\ldots\ldots),$$

and

$$\beta = \left(\frac{180^0}{\pi}\right) \sum_{n=1}^{\infty} \left[\tan^{-1}\left(\frac{s_0}{k_n}\right) - \tan^{-1}\left(\frac{s_0}{s_n}\right)\right],$$
$$(n = 1,2,\ldots\ldots).$$

Fig. 3. *Phase of R for different* Gh

Fig. 4. *Phase of T for different* Gh

In the Fig. 3, it is observed that the phase values of R are increasing as the values of the porosity parameter Gh increases. But in Fig. 4, the phase values of the transmission coefficient are found to be negative. This can be explained due the presence of discontinuity at the upper surface of the ocean and the reflected wave is opposite to the incident or transmitted wave field.
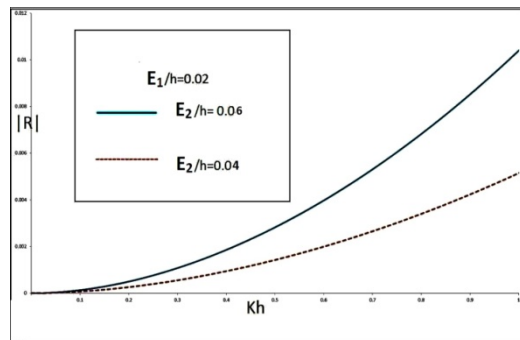

Fig. 5. $|R|$ for different E_2/h

The reflection and transmission coefficients can also be investigated for different area densities of the inertial surfaces at the upper surface of the ocean. Figures 5 and 6 show the values of $|R|$ and $|T|$ for $E_1/h = 0.02, E_2/h = 0.04, 0.06$ and dimensionless porosity parameter is taken as $Gh = 0.3$ in this case. It is noticeable that absolute values of the reflection coefficient increases as the area densities of the inertial surface on the right side $(x > 0)$ E_2/h increases and consequently the values of $|T|$ decreases. As the wave field is incident from negative infinity direction, and the area density of the inertial surface on the right hand side E_2/h increases, the magnitude of the reflection is found to be increased here. This fact can be also be justified by the energy identity relation given by (14). In Fig. 7, the values of $|R|$ are plotted for $E_1/h = 0.01, 0.02$ and $E_2/h = 0.06$ respectively. Since the wave train is incident from negative infinity, the values of $|R|$ will be decreasing with the area densities of the inertial surface on the side $(x < 0)$, as observed in the Fig.
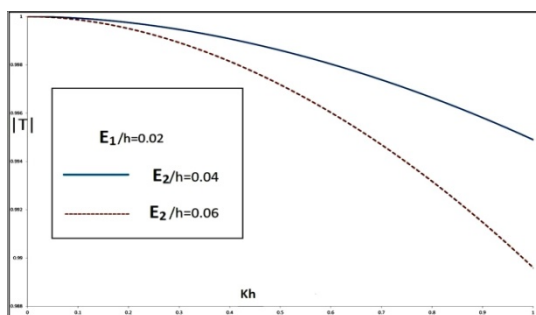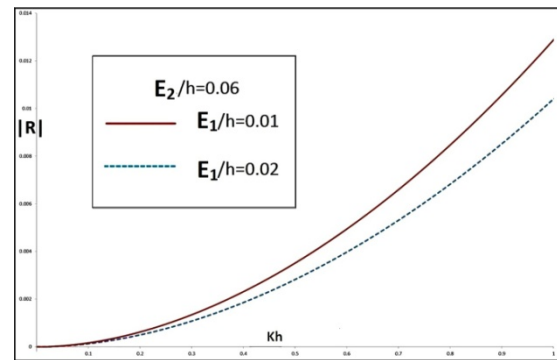

Fig. 6. $|T|$ for different E_2/h

Fig. 7. $|R|$ for different E_1/h

7. Conclusion

A large class of problems involving water wave scattering by obstacles along with porous media has been studied in the literature. The present study is concerned with the scattering of surface wave by a discontinuity at the upper surface in a finite depth water with porous bottom. The residue calculus method of complex variable theory is employed to obtain the reflection and transmission coefficient for the case of a finite depth water. The magnitude and the phase displacement of these hydrodynamic coefficients are computed numerically for different values of the dimensionless porosity parameter. These results are plotted in a number of Figs. for different cases. From the analytical and numerical results, it is observed that the porous bottom has an effect on the absolute values and the phase of the hydrodynamic coefficients.

It may be noted that for convenience the point of discontinuity of the surface boundary condition is chosen at $x = 0$. The discontinuity of the surface boundary condition may be at the another point, and in that case the same method can be applied.

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