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EFFECT OF HALL'S CURRENT FOR STOKES' PROBLEM FOR A THIRD GRADE FLUID IN THE CASE OF SUCTION*

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Abstract– Unsteady incompressible unidirectional third-grade fluid past an infinite porous wall is considered in the presence of Hall current. The plate at the lower boundary y=0 is executing sinusoidal oscillations in its own plane with superimposed blowing or suction. The governing equation (representing the velocity field) is modeled and described by a third order non-linear partial differential equation.

Keywords– Non-Newtonian fluid, oscillating and accelerating boundary, blowing/suction, perturbation technique, Hall effects

1. INTRODUCTION

In an ionized gas where the density is low and/or the magnetic field is very strong, the conductivity normal to the free spiraling of electrons and ions around the magnetic lines of force before suffering collisions induces a current in a direction normal to both the electric and magnetic fields. This phenomenon is called the Hall effect [1-5]. The study of magnetohydrodynamic flows with Hall currents has important engineering and industrial applications in problems of magnetohydrodynamics generators and of Hall accelerators, as well as inflight magnetohydrodynamics.

In the few past decades there has been significant work on flows of non-Newtonian fluids, not only because of their non-linearity which occur in the inertial part, but also in the surface forces of the governing equations [6-10]. On the other hand, it is well known that the rheological properties of many fluids are not well modeled by Navier-Stokes equations.

The shear thinning and thickening phenomena is a comprehensive description of the properties of viscoelastic fluids. Although the second-grade fluid model is able to predict the normal stress differences which are characteristic of non-Newtonian fluids, it does not take the shear thinning and thickening phenomena that third-grade fluids describe. Keeping these analyses in mind, the model in the present work is a third-grade fluid and the flow is bounded by the lower plate which is oscillating sinusoidally in time, whereas the fluid is infinite in the other direction.

In this paper, we discuss the effects of Hall currents on the unsteady flow of an electrically conducting non-Newtonian (third-grade) fluid. The fluid considered is of third-grade, which makes the governing equation a non-linear third-order partial differential equation. For such a fluid, equations are modeled and solved by the method used in [6-8].

2. FORMULATION OF THE PROBLEM

The basic equations governing the motion of a homogeneous incompressible third-grade fluid are

$$\nabla \cdot \mathbf{V} = 0, \tag{1}$$

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$$\rho \frac{d\mathbf{V}}{dt} = \rho\mathbf{b} + \mathbf{J}\times\mathbf{B} + \mathrm{div}\mathbf{T}, \tag{2}$$

$$\nabla\cdot\mathbf{B} = 0,\ \nabla\times\mathbf{B} = \mu_m\mathbf{J},\ \nabla\times\mathbf{E} = 0 \tag{3}$$

and the Cauchy stress $\mathbf{T}$ for an incompressible third-grade fluid is given by [11]

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_1\mathbf{A}_3 + \beta_2(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1) + \beta_3(\mathrm{tr}\mathbf{A}_1^2)\mathbf{A}_1, \tag{4}$$

where $\mathbf{V}$ is the velocity vector, d/dt signifies mobile operator, $-p\mathbf{I}$ the spherical stress due to the constraint of incompressibility $(\nabla\cdot\mathbf{V}=0)$, μ the coefficient of viscosity, $\mathbf{b}$ the body force per unit mass, α_1, α_2, β_1, β_2 and β_3 are material constants, ρ is the density, $\mathbf{J}$ is the current density, $\mathbf{B}$ is the total magnetic field, μ_m the magnetic permeability and $\mathbf{E}$ the total electric field current. Making reference to Cowling [11], when the strength of the magnetic field is high, the generalized Ohm's law is modified to include the Hall current so that

$$\mathbf{J} + \frac{\omega_e\tau_e}{\mathbf{B}_0}(\mathbf{J}\times\mathbf{B}) = \sigma\left[\mathbf{E} + \mathbf{V}\times\mathbf{B} + \frac{1}{en_e}\nabla p_e\right] \tag{5}$$

in which ω_e is the cyclotron frequency, τ_e is the electron collision time, σ is the electrical conductivity, e is the electron charge and p_e is the electron pressure. The ion-slip and thermoelectric effects are not included in (5). Further, it is assumed that $\omega_e\tau_e \sim O(1)$ and $\omega_i\tau_i << 1$, where ω_i and τ_i are the cyclotron frequency and collision time for ions respectively.

The kinematics tensors $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$, defined in (4), are the first three Rivlin-Ericksen tensors defined through [12]

$$\mathbf{A}_n = \frac{d\mathbf{A}_{n-1}}{dt} + \mathbf{A}_{n-1}(\mathrm{grad}\mathbf{V}) + (\mathrm{grad}\mathbf{V})^{\mathrm{T}}\mathbf{A}_{n-1},\ n>1, \tag{6}$$

$$\mathbf{A}_1 = (\mathrm{grad}\mathbf{V}) + (\mathrm{grad}\mathbf{V})^{\mathrm{T}}, \tag{7}$$

where grad denotes the gradient operator and T the transpose. It is proved by Fosdick and Rajagopal [13] that if third grade fluid is to satisfy equations of motion which are compatible with Clausius-Duhem inequality and the assumption that the fluid be locally at rest, then the material constants in (4) must follow the following conditions

$$\mu \ge 0,\ \alpha_1 \ge 0,\ \beta_1 = \beta_2 = 0,\ \beta_3 \ge 0,\ |\alpha_1 + \alpha_2| \le \sqrt{24\mu\beta_3}. \tag{8}$$

In the present analysis we are concerned with the fluid which obeys the restrictions given in (8). When (4), satisfying (8), is substituted in (2) and making use of (1) we obtain

$$\begin{aligned}\rho\frac{d\mathbf{V}}{dt} = &-\mathrm{grad}p + \mu\nabla^2\mathbf{V} + (\alpha_1+\alpha_2)\mathrm{div}\mathbf{A}_1^2 + \alpha_1[\nabla^2\mathbf{V}_t + \nabla^2(\nabla\times\mathbf{V})\times\mathbf{V}\\ &+\mathrm{grad}(\mathbf{V}\cdot\nabla^2\mathbf{V}) + \frac{1}{4}\mathrm{tr}\mathbf{A}_1^2] + \beta_3\mathbf{A}_1\mathrm{grad}(\mathrm{tr}\mathbf{A}_1^2) + \beta_3(\mathrm{tr}\mathbf{A}_1^2)\nabla^2\mathbf{V}\\ &-\frac{\sigma B_0^2}{1-im}\mathbf{V} + \rho\mathbf{b},\end{aligned} \tag{9}$$

where subscript t denotes the partial derivative with respect to time t, $m=\omega_e\tau_e$ is the Hall parameter and ∇^2 is the Laplacian operator.

We consider the unsteady flow generated in a semi-infinite expanse of a third grade fluid bounded by an infinite porous plate. The fluid is at rest for $t<0$ and for $t>0$; the plate is oscillating sinusoidally at $y=0$. For the problem under consideration we write the velocity and the boundary conditions as follows:

$$\mathbf{V}=[u(y,t),-V_0,0], \tag{10}$$

$$\begin{gathered} u(0,t)=U(t), \\ u(y,t)\to 0 \text{ as } y\to\infty. \end{gathered} \tag{11}$$

with simultaneous suction or blowing: $V_0>0$ corresponds to the case of suction and $V_0<0$ indicates blowing.

Substituting (11) into the balance of linear momentum (9) and using the fact that the fluid is incompressible and there are no body forces, we obtain

$$\begin{gathered} \rho\left(\frac{\partial u}{\partial t}-V_0\frac{\partial u}{\partial y}\right)=-\frac{\partial \hat{p}}{\partial x}+\mu\frac{\partial^2 u}{\partial y^2}+\alpha_1\left(\frac{\partial^3 u}{\partial y^2\partial t}-V_0\frac{\partial^3 u}{\partial y^3}\right) \\ +6\beta_3\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2}-\frac{\sigma B_0^2}{1-im}u, \end{gathered} \tag{12}$$

$$0=-\frac{\partial \hat{p}}{\partial y}=-\frac{\partial \hat{p}}{\partial z}, \tag{13}$$

where

$$\hat{p}=p-\left(2\alpha_1+\alpha_2\right)\left(\frac{\partial u}{\partial y}\right)^2, \tag{14}$$

is the modified pressure. In the case when there is no pressure imposed we get

$$\frac{\partial u}{\partial t}-V_0\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}+\frac{\alpha_1}{\rho}\left(\frac{\partial^3 u}{\partial y^2\partial t}-V_0\frac{\partial^3 u}{\partial y^3}\right)+\frac{6\beta_3}{\rho}\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2}-\frac{\sigma B_0^2}{\rho(1-im)}u, \tag{15}$$

where $\nu=\mu/\rho$ is the kinematic coeffecient of viscosity.

Defining the dimensionless parameters

$$\begin{gathered} \bar{\alpha}_1=\frac{V_0^2}{\rho\nu^2}\alpha_1,\ y=\frac{\nu}{V_0}\bar{y},\ t=\frac{\nu}{V_0^2}\bar{t},\ u=V_0\bar{u}, \\ \varepsilon=\frac{6\beta_3}{\rho\nu^3}V_0^4,\ U(t)=V_0\bar{U}(\bar{t}),\ \phi=\frac{\sigma B_0^2}{\rho(1-im)}\frac{\nu}{V_0^2}, \end{gathered} \tag{16}$$

the boundary value problem becomes

$$\frac{\partial\bar{u}}{\partial\bar{t}}-\frac{\partial\bar{u}}{\partial\bar{y}}=\frac{\partial^2\bar{u}}{\partial\bar{y}^2}+\bar{\alpha}_1\left(\frac{\partial^3\bar{u}}{\partial\bar{y}^2\partial\bar{t}}-\frac{\partial^3\bar{u}}{\partial\bar{y}^3}\right)+\varepsilon\left(\frac{\partial\bar{u}}{\partial\bar{y}}\right)^2\frac{\partial^2\bar{u}}{\partial\bar{y}^2}-\phi\bar{u}, \tag{17}$$

$$\bar{u}=\bar{U}(\bar{t}) \text{ at } \bar{y}=0,\ \bar{u}\to 0 \text{ as } \bar{y}\to\infty. \tag{18}$$

3. SOLUTION OF THE PROBLEM

Since in the case of non-Newtonian fluid the order of equations of motion is higher than the Navier-Stokes equations, the adherence boundary condition is insufficient to determine the solution completely [14, 15]. In order to overcome this difficulty Beard and Walters [16], in their study of incompressible fluid of second order, proposed a method. They suggested a perturbation approach in which the velocity and the pressure field were expanded in terms of a small parameter. Though this approximation reduces the order of the equation, it treats the singular perturbation problem as a regular perturbation problem. Therefore, $\bar{u}$ can be expanded in powers of ε as follows:

$$\bar{u}(\bar{y},\bar{t};\varepsilon) = \bar{u}_0(\bar{y},\bar{t}) + \varepsilon \bar{u}_1(\bar{y},\bar{t}) + \varepsilon^2 \bar{u}_2(\bar{y},\bar{t}) + \cdots. \tag{19}$$

Substituting (19) into (17) and the boundary conditions (18), and then equating equal powers of ε, we obtain the following systems:

Zeroth order system

$$\frac{\partial \bar{u}_0}{\partial \bar{t}} - \frac{\partial \bar{u}_0}{\partial \bar{y}} = \frac{\partial^2 \bar{u}_0}{\partial \bar{y}^2} + \bar{\alpha}_1 \left(\frac{\partial^3 \bar{u}_0}{\partial \bar{y}^2 \partial \bar{t}} - \frac{\partial^2 \bar{u}_0}{\partial \bar{y}^3} \right) - \phi \bar{u}_0 \tag{20}$$

$$\bar{u}_0 = \bar{U}(\bar{t}) \text{ at } \bar{y} = 0, \ \bar{u}_0 \to 0 \text{ as } \bar{y} \to \infty. \tag{21}$$

First order system

$$\frac{\partial \bar{u}_1}{\partial \bar{t}} - \frac{\partial \bar{u}_1}{\partial \bar{y}} = \frac{\partial^2 \bar{u}_1}{\partial \bar{y}^2} + \bar{\alpha}_1 \left(\frac{\partial^3 \bar{u}_1}{\partial \bar{y}^2 \partial \bar{t}} - \frac{\partial^3 \bar{u}_1}{\partial \bar{y}^3} \right) + \left(\frac{\partial \bar{u}_0}{\partial y} \right)^2 \frac{\partial^2 \bar{u}_0}{\partial \bar{y}^2} - \phi \bar{u}_1 \tag{22}$$

$$\bar{u}_1 = 0 \text{ at } \bar{y} = 0, \ \bar{u}_1 \to 0 \text{ as } \bar{y} \to \infty. \tag{23}$$

These systems are solved by employing the method used by Rajagopal [8] and Hinch [17]. Now introducing the similarity transformation

$$\eta = \bar{y}, \ \bar{u}_0 = f_0(\eta) e^{\gamma \bar{t}}, \ \bar{u}_1 = f_1(\eta) e^{3\gamma \bar{t}}, \tag{24}$$

thus we can write Eqs. (20) and (21) in the following manner:

$$\bar{\alpha}_1 f_0''' - (1 + \gamma \bar{\alpha}_1) f_0'' - f_0' + (\gamma + \phi) f_0 = 0, \tag{25}$$

$$f_0(\eta) = 1 \text{ at } \eta = 0, \ f_0(\eta) \to 0 \text{ as } \eta \to \infty, \tag{26}$$

where

$$\bar{U}(\bar{t}) = e^{\gamma \bar{t}}.$$

Similarly, the Eqs. (22) and (23) can be written as

$$\bar{\alpha}_1 f_1''' - (1 + 3\bar{\alpha}_1 \gamma) f_1'' - f_1' + (3\gamma + \phi) f_1 = (f_0')^2 f_0'', \tag{27}$$

$$f_1(\eta) = 0 \text{ at } \eta = 0, \ f_1(\eta) \to 0 \text{ as } \eta \to \infty. \tag{28}$$

For the solution of (25), the complimentary function must satisfy

$$\bar{\alpha}_1 m^3 - (1 + \bar{\alpha}_1 \gamma) m^2 - m + (\gamma + \phi) = 0, \tag{29}$$

For small value of $\overline{\alpha}_1$, the roots of Eq. (29) can be obtained by perturbation expansion method. For that m can be expressed as

$$m = \frac{c_{-1}}{\overline{\alpha}_1} + c_0 + c_1\overline{\alpha}_1 + c_2\overline{\alpha}_1^2 + \cdots. \tag{30}$$

Now using Eq. (30) in (29) and comparing the like powers of $\overline{\alpha}_1$ we get

$$\overline{\alpha}_1^{-2} \ : \ c_{-1}^3 - c_{-1}^2 = 0, \tag{31}$$

$$\overline{\alpha}_1^{-1} \ : \ 2c_{-1}c_0^2 - c_0c_{-1}^2 - c_{-1}^2\gamma - 2c_{-1}c_0 - c_{-1} = 0, \tag{32}$$

$$\overline{\alpha}_1^{0} \ : \ 3c_0^2c_{-1} + 3c_{-1}^2c_1 - 2c_{-1}c_1 - c_0^2 - c_0 + \gamma + \phi - 2c_{-1}c_0 = 0, \tag{33}$$

$$\overline{\alpha}_1 \ : \ 3c_{-1}^2c_2 + 6c_{-1}c_0c_1 - 2c_{-1}c_2 - 2c_0c_1 + c_0^3 - c_1 - \gamma(c_0^2 + 2c_{-1}c_1) = 0, \tag{34}$$

$$\overline{\alpha}_1^{2} \ : \ 3c_0^2c_1 + 6c_{-1}c_0c_2 + 3c_{-1}c_1^2 - 2c_0c_2 - c_1^2 - c_2 - \gamma(2c_0c_1 + 2c_{-1}c_2) = 0. \tag{35}$$

From Eq. (31) we get

$$c_{-1} = 0, 0, 1. \tag{36}$$

The corresponding roots for the three values of c_{-1} are given by

$$\begin{aligned} m_1 &\approx c_0 + c_1\overline{\alpha}_1 + c_2\overline{\alpha}_1^2, \\ m_2 &\approx \widehat{c}_0 + \widehat{c}_1\overline{\alpha}_1 + \widehat{c}_2\overline{\alpha}_1^2, \\ m_3 &\approx \overline{\alpha}_1^{-1} + \widehat{c}_0 + \widehat{c}_1\overline{\alpha}_1 + \widehat{c}_2\overline{\alpha}_1^2, \end{aligned} \tag{37}$$

$$c_0 = -\left(\frac{1+\sqrt{1-4(\gamma+\phi)}}{2}\right), \ c_1 = \frac{c_0^3 - c_0^2\gamma}{2c_0+1}, \quad c_2 = \frac{3c_0^2c_1 - 2c_0c_1\gamma - c_1^2}{2c_0+1},$$

$$\widehat{c}_0 = -\left(\frac{1-\sqrt{1-4(\gamma+\phi)}}{2}\right), \ \widehat{c}_1 = \frac{\widehat{c}_0^3 - \widehat{c}_0^2\gamma}{2\widehat{c}_0+1}, \quad \widehat{c}_2 = \frac{3\widehat{c}_0^2\widehat{c}_1 - 2\widehat{c}_0\widehat{c}_1\gamma - \widehat{c}_1^2}{2\widehat{c}_0+1},$$

$$\widehat{c}_0 = 1+\gamma, \ \widehat{c}_1 = \widehat{c}_0 - 2\widehat{c}_0^2 - \gamma(1-2\widehat{c}_0) - \phi, \quad \widehat{c}_2 = \widehat{c}_1 + \gamma(2\widehat{c}_1 - \widehat{c}_0^2) - \widehat{c}_0^3 - 4\widehat{c}_0\widehat{c}_1.$$

The solution of the differential equation in system (29) is given by

$$f_0(\eta) = A_1e^{-m_1\eta} + A_2e^{m_2\eta} + A_3e^{m_3\eta}. \tag{38}$$

Now using (i) the physical condition that the velocity reduces to the Newtonian case when $\alpha_1 \to 0$ and (ii) the boundary condition in (29) as $\eta \to 0,$ the solutions corresponding to the roots m_2 and m_2 are neglected. Thus from Eq. (38) we have

$$f_0(\eta) = A_1e^{-m_1\eta}. \tag{39}$$

Now using the boundary condition at $\eta = 0$ from (26) into Eq. (39) we obtain

$$f_0(\eta) = e^{-(c_0 + c_1\overline{\alpha}_1 + c_2\overline{\alpha}_1^2)\eta}.$$

Similarly, the solutions of systems (27) to (28) are respectively given by using

$$f_0^{'}(\eta)=-m_1e^{-m_1\eta},$$

$$f_0^{''}(\eta)=m_1^2e^{-m_1\eta},$$

then (27) takes the form

$$\overline{\alpha}_1 f_1^{'''}-(1+3\gamma\overline{\alpha}_1)f_1^{''}-f_1^{'}+3(\gamma+\phi)f_1=m_1^4e^{-3m_1\eta}. \tag{40}$$

For the complimentary solution $f_{1c}(\eta)$, we have

$$\overline{\alpha}_1 m^3-(1+3\gamma\overline{\alpha}_1)m^2-m+3(\gamma+\phi)=0. \tag{41}$$

Now

$$\widehat{m}=\frac{d_{-1}}{\overline{\alpha}_1}+d_0+d_1\overline{\alpha}_1+d_2\overline{\alpha}_1^2,$$

then in (41) we get

$$\widehat{m}_1\approx d_0+d_1\overline{\alpha}_1+d_2\overline{\alpha}_1^2,\ \text{where, } d_0=\frac{-1-\sqrt{1+4(\gamma+\phi)}}{2},$$

$$\widehat{m}_2\approx \widehat{d}_0+\widehat{d}_1\overline{\alpha}_1+\widehat{d}_2\overline{\alpha}_1^2,\ \text{where, } \widehat{d}_0=\frac{-1+\sqrt{1+4(\gamma+\phi)}}{2},$$

$$\widehat{m}_3\approx \overline{\alpha}_1^{-1}+\widehat{d}_0+\widehat{d}_1\overline{\alpha}_1+\widehat{d}_2\overline{\alpha}_1^2,\ \text{where, } \widehat{d}_0=1+3\gamma.$$

Therefore, we have

$$f_{1c}(\eta)=A_4e^{-\widehat{m}_1\eta}+A_5e^{\widehat{m}_2\eta}. \tag{42}$$

For particular solution $f_{1p}(\eta)$, we have

$$f_{1p}(\eta)=\frac{m_1^4e^{-3m_1\eta}}{-27\overline{\alpha}_1m_1^3-9(1+3\gamma\overline{\alpha}_1)m_1^2+3m_1+3\gamma+\phi}. \tag{43}$$

From (42) and (43) we can write

$$f_1(\eta)=A_4e^{-\widehat{m}_1\eta}+A_5e^{\widehat{m}_2\eta}+\frac{m_1^4e^{-3m_1\eta}}{-27\overline{\alpha}_1m_1^3-9(1+3\gamma\overline{\alpha}_1)m_1^2+3m_1+3\gamma+\phi}. \tag{44}$$

Using (27) and (28) we have

$$f_1(\eta)=\frac{m_1^4[e^{-3m_1\eta}-e^{-\widehat{m}_1\eta}]}{-27\overline{\alpha}_1m_1^3-9(1+3\gamma\overline{\alpha}_1)m_1^2+3m_1+3\gamma+\phi}. \tag{45}$$

Thus, we have

$$\bar{u}_0=e^{-m_1\eta+\gamma\bar{t}}, \tag{46}$$

$$\bar{u}_1=\frac{m_1^4\left[e^{-3m_1\eta}-e^{-\widehat{m}_1\eta}\right]e^{3\gamma\bar{t}}}{-27\bar{\alpha}_1m_1^3-9[1+3\overline{\alpha}_1\gamma]m_1^2+3m_1+3\gamma+\phi}. \tag{47}$$

Since γ may be both real and imaginary, we discuss both cases.

Case 1.

When γ is real, the velocity and skin friction τ_{ω_0} are respectively given by

$$\bar{u}(\bar{y},\bar{t};\varepsilon) = e^{\gamma\bar{t}} f_0(\eta) + \varepsilon e^{3\gamma\bar{t}} f_1(\eta) + ..., \tag{48}$$

$$\tau_{\omega_0} = \rho U_0^2 \left[e^{(\delta+\gamma)\bar{t}} f_0^{'}(0) + \varepsilon e^{3(\delta+\gamma)\bar{t}} f_1^{'}(0) + ... \right], \tag{49}$$

where

$$f_0(\eta) = e^{-m_1\eta},$$

$$f_1(\eta) = \frac{m_1^4 \left[e^{-3m_1\eta} - e^{-\widehat{m}_1\eta} \right]}{-27\bar{\alpha}_1 m_1^3 - 9\left[1 + 3\widehat{\alpha}_1\gamma\right] m_1^2 + 3m_1 + 3\gamma + \phi},$$

$$f_0^{'}(0) = -m_1,$$

$$f_1^{'}(0) = \frac{m_1^4 \left(-3m_1 + \widehat{m}_1 \right)}{-27\bar{\alpha}_1 m_1^3 - 9\left[1 + 3\overline{\alpha}_1\gamma\right] m_1^2 + 3m_1 + 3\gamma + \phi},$$

and prime $\left(' \right)$ denotes the differentiation with respect to the variable η .

Case 2.

When γ is imaginary $(\gamma = i\omega)$, then we have

$$\bar{u}(\bar{y},\bar{t};\varepsilon) = (f_{0R} \cos\omega\bar{t} - f_{0I} \sin\omega\bar{t}) + \varepsilon(f_{1R} \cos 3\omega\bar{t} - f_{1I} \sin 3\omega\bar{t}) + \cdots, \tag{50}$$

$$\tau_{\omega_0} = \rho U_0^2 \begin{bmatrix} (f_{0R}^{'}(0)\cos\omega\bar{t} - f_{0I}^{'}(0)\sin\omega\bar{t}) \\ + \varepsilon(f_{1R}^{'}(0)\cos 3\omega\bar{t} - f_{1I}^{'}(0)\sin 3\omega\bar{t}) + \cdots \end{bmatrix}, \tag{51}$$

where

$$f_0(\eta) = f_{0R}(\eta) + i f_{0I}(\eta), \quad f_1(\eta) = f_{1R}(\eta) + i f_{1I}(\eta),$$

$$f_{0R}(\eta) = e^{-a_1\eta} \cos a_2\eta, \quad f_{0I}(\eta) = -e^{-a_1\eta} \sin a_2\eta,$$

$$f_{1R}(\eta) = A_R \left[e^{-3a_1\eta} \cos 3a_2\eta - e^{-a_1^{'}\eta} \cos a_2^{'}\eta \right]$$
$$- A_I \left[e^{-a_1^{'}\eta} \sin a_2^{'}\eta - e^{-3a_1\eta} \sin 3a_2\eta \right],$$

$$f_{1I}(\eta) = A_I \left[e^{-3a_1\eta} \cos 3a_2\eta - e^{-a_1^{'}\eta} \cos a_2^{'}\eta \right]$$
$$+ A_R \left[e^{-a_1^{'}\eta} \sin a_2^{'}\eta - e^{-3a_1\eta} \sin 3a_2\eta \right],$$

$$f_{0R}^{'}(0) = -a_1, \quad f_{0I}^{'}(0) = -a_2,$$

$$f_{1R}^{'}(0) = A_R \left(a_1^{'} - 3a_1 \right) - A_I \left(a_2^{'} - 3a_2 \right),$$

$$f_{1I}^{'}(0) = A_I \left(a_1^{'} - 3a_1 \right) - A_R \left(a_2^{'} - 3a_2 \right),$$

$$A_R = \frac{\begin{array}{l}(a_1^4+a_2^4-6a_1^2a_2^2)\left[\begin{array}{c}-27\bar{\alpha}_1(a_1^3-3a_1a_2^2)-9\{a_1^2-a_2^2\\ -6a_1a_2\bar{\alpha}_1\,\omega\}+3a_1+\phi\end{array}\right]\\ +(4a_1^3a_2-4a_1a_2^3)\left[\begin{array}{c}-27\bar{\alpha}_1(3a_1^2a_2-a_2^3)-9\{(a_1^2-a_2^2)\\ 3\bar{\alpha}_1\,\omega+2a_1a_2\}+3a_2+\omega\end{array}\right]\end{array}}{\begin{array}{c}\left[\begin{array}{c}-27\bar{\alpha}_1(a_1^3-3a_1a_2^2)-9\{a_1^2-a_2^2\\ -6a_1a_2\bar{\alpha}_1\,\omega\}+3a_1+\phi\end{array}\right]^2\\ +\left[\begin{array}{c}-27\bar{\alpha}_1(3a_1^2a_2-a_2^3)-9\{(a_1^2-a_2^2)3\bar{\alpha}_1\,\omega\\ +2a_1a_2+3a_2+3\omega\end{array}\right]^2\end{array}},$$

$$A_I = \frac{\begin{array}{l}(4a_1^3a_2-4a_1a_2^3)\left[\begin{array}{c}-27\bar{\alpha}_1(a_1^3-3a_1a_2^2)-9\{a_1^2-a_2^2\\ -6a_1a_2\bar{\alpha}_1\,\omega\}+3a_1+\phi\end{array}\right]\\ -(a_1^4+a_2^4-6a_1^2a_2^2)\left[\begin{array}{c}-27\alpha_1(3a_1^2a_2-a_2^3)-9\{(a_1^2-a_2^2)\\ 3\bar{\alpha}_1\,\omega+2a_1a_2\}+3a_2+\omega\end{array}\right]\end{array}}{\begin{array}{c}\left[\begin{array}{c}-27\bar{\alpha}_1(a_1^3-3a_1a_2^2)-9\{a_1^2-a_2^2\\ -6a_1a_2\bar{\alpha}_1\,\omega\}+3a_1+\phi\end{array}\right]^2\\ +\left[\begin{array}{c}-27\bar{\alpha}_1(3a_1^2a_2-a_2^3)-9\{(a_1^2-a_2^2)3\bar{\alpha}_1\,\omega\\ +2a_1a_2+3a_2+3\omega\end{array}\right]^2\end{array}},$$

$$a_1 = c_{0R}+c_{1R}\,\bar{\alpha}_1+c_{2R}\,\bar{\alpha}_1^2,\quad a_2 = c_{0I}+c_{1I}\,\bar{\alpha}_1+c_{2I}\,\bar{\alpha}_1^2,$$

$$a_3 = d_{0R}+d_{1R}\,\bar{\alpha}_1+d_{2R}\,\bar{\alpha}_1^2,\quad a_4 = d_{0I}+d_{1I}\,\bar{\alpha}_1+d_{2I}\,\bar{\alpha}_1^2,$$

$$c_{0R} = -\frac{1}{2}-\frac{1}{2}e_1,\quad c_{0I} = -\frac{1}{2}e_2,$$

$$c_{1R} = \frac{\left[\begin{array}{c}c_{0R}^3-3c_{0R}c_{0I}^2-6c_{0R}^2c_{0I}^2-6\omega c_{0R}c_{0I}+2c_{0R}^4+6\omega c_{0R}^2c_{0I}\\ +6c_{0R}^2c_{0I}^2-2c_{0I}^4-6\omega c_{0I}^4\end{array}\right]}{\left(1+2c_{0R}\right)^2+4c_{0I}^2},$$

$$c_{1I} = \frac{\left[\begin{array}{c}2c_{0I}c_{0R}^3-8c_{0R}c_{0I}^3+6\omega c_{0R}c_{0I}^2+3c_{0R}^2c_{0I}^3-c_{0I}^3-3\omega c_{0R}^2\\ -3\omega c_{0I}^2+6c_{0I}^3c_{0R}^3-6\omega c_{0R}^3\end{array}\right]}{\left(1+2c_{0R}\right)^2+4c_{0I}^2},$$

$$c_{2R} = \frac{\begin{array}{c}\left[\begin{array}{c}3c_{0R}^2c_{1R}-3c_{0I}^2c_{1R}-6c_{0R}c_{0I}c_{1I}+6\omega c_{0R}c_{1I}\\ +6\omega c_{0I}c_{1R}-c_{1R}^2+c_{1I}^2\end{array}\right]\left(2c_{0R}^2-2c_{0I}^2+1\right)\\ \left[\begin{array}{c}c_{0R}^2c_{1I}-c_{0I}^2c_{1I}-2c_{0R}c_{0I}c_{1R}-6\omega c_{0R}c_{1R}\\ +6\omega c_{0I}c_{1I}-2c_{1R}c_{1I}\end{array}\right]4c_{0R}c_{0I}\end{array}}{\left(2c_{0R}^2-2c_{0I}^2+1\right)^2+16c_{0I}^2c_{0I}^2}$$

$$c_{2I}=\frac{\begin{bmatrix}3c_{0R}^2c_{1R}-3c_{0I}^2c_{1R}-6c_{0R}c_{0I}c_{1I}+6\omega c_{0R}c_{1I}\\+6\omega c_{0I}c_{1R}-c_{1R}^2+c_{1I}^2\end{bmatrix}(-4c_{0R}c_{0I})-\begin{bmatrix}c_{0R}^2c_{1I}-c_{0I}^2c_{1I}+2c_{0R}c_{0I}c_{1R}-6\omega c_{0R}c_{1R}\\+6\omega c_{0I}c_{1I}-2c_{1R}c_{1I}\end{bmatrix}\left(2c_{0R}^2-2c_{0I}^2+1\right)}{\left(2c_{0R}^2-2c_{0I}^2+1\right)^2+16c_{0I}^2c_{0I}^2}$$

$$d_{0R}=-\frac{1}{2}+\frac{1}{2}e_1,\qquad d_{0I}=\frac{1}{2}e_2$$

$$d_{1R}=\frac{\begin{bmatrix}d_{0R}^3-3d_{0R}d_{0I}^2-6d_{0R}^2d_{0I}^2-6\omega d_{0R}d_{0I}+2d_{0R}^4+6\omega d_{0R}^2d_{0I}\\+6d_{0R}^2d_{0I}^2-2d_{0I}^4-6\omega d_{0I}^4\end{bmatrix}\left(2c_{0R}^2-2c_{0I}^2+1\right)}{\left(1+2d_{0R}\right)^2+4d_{0I}^2}$$

$$d_{1I}=\frac{\begin{bmatrix}2d_{0I}d_{0R}^3-8d_{0R}d_{0I}^3+6\omega d_{0R}d_{0I}^2+3d_{0R}^2d_{0I}^3-d_{0I}^3-3\omega d_{0R}^2\\-3\omega d_{0I}^2+6d_{0I}^3d_{0R}^3-6\omega d_{0R}^3\end{bmatrix}}{\left(1+2d_{0R}\right)^2+4d_{0I}^2}$$

$$d_{2R}=\frac{\begin{bmatrix}3d_{0R}^2d_{1R}-3c_{0I}^2c_{1R}-6d_{0R}d_{0I}d_{1I}+6\omega d_{0R}c_{1I}\\+6\omega d_{0I}d_{1R}-d_{1R}^2+d_{1I}^2\end{bmatrix}\left(2d_{0R}^2-2c_{0I}^2+1\right)-\begin{bmatrix}d_{0R}^2d_{1I}-d_{0I}^2d_{1I}+2d_{0R}d_{0I}d_{1R}-6\omega d_{0R}d_{1R}\\+6\omega d_{0I}d_{1I}-2d_{1R}d_{1I}\end{bmatrix}4d_{0R}d_{0I}}{\left(2d_{0R}^2-2d_{0I}^2+1\right)^2+16d_{0I}^2d_{0I}^2}$$

$$d_{2R}=\frac{\begin{bmatrix}3d_{0R}^2d_{1R}-3d_{0I}^2d_{1R}-6d_{0R}d_{0I}d_{1I}+6\omega d_{0R}d_{1I}\\+6\omega d_{0I}d_{1R}-d_{1R}^2+d_{1I}^2\end{bmatrix}(-4d_{0R}d_{0I})-\begin{bmatrix}d_{0R}^2d_{1I}-d_{0I}^2d_{1I}+2d_{0R}d_{0I}d_{1R}-6\omega d_{0R}d_{1R}\\+6\omega d_{0I}d_{1I}-2d_{1R}d_{1I}\end{bmatrix}\left(2d_{0R}^2-2c_{0I}^2+1\right)}{\left(2d_{0R}^2-2d_{0I}^2+1\right)^2+16d_{0I}^2d_{0I}^2}$$

$$e_1=\left[\frac{(1+4\phi)+\sqrt{(1+4\phi)^2+16\omega^2}}{2}\right]^{\frac{1}{2}},\quad e_2=\frac{2\omega}{\left[\frac{(1+4\phi)+\sqrt{(1+4\phi)^2+16\omega^2}}{2}\right]^{\frac{1}{2}}}.$$

4. CONCLUDING REMARKS

The third order non-linear partial differential equation is solved by the perturbation technique. The solutions are given up to the square of the perturbation parameter. Both cases are discussed (i) when the plate at y=0 is oscillating exponentially in time and (ii) when the the plate at the same location is oscillating sinusoidally in time with frequency ω. The results of second-grade fluid are recovered by taking the third-grade parameter β_3=0. When α_1=β_3=0, we obtain the Newtonian fluid with Hall effects.

Also when $\alpha_1=\beta_3=m=0$, we readily obtain the viscous fluid with Stokes I and II problems [18], depending on whether the boundary at $y=0$ is of an impulsive nature or oscillating in time.

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