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EXPLICIT MINIMUM FUEL INTERCEPT STRATEGY FOR HIGH-ORDER DYNAMICS*

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Abstract– An optimal fuel guidance strategy with acceleration limit is derived for interception of maneuvering targets. The guidance/control system is assumed as a linear time-varying arbitrary-order and is identical in each channel. An approximate model for drag acceleration is also considered in the solution. The optimal fuel strategy has some detrimental effects in most practical situations, if it is used entirely to the end. This strategy can be used at the beginning of the guidance phase for large initial heading errors in order to quickly reduce it with minimum effort. An explicit guidance law is then developed for minimum and nonminimum phase autopilots in order to modify the undesirable effects of the optimal fuel guidance law.

Keywords– Intercept guidance, minimum fuel, acceleration limit

1. INTRODUCTION

The optimal control theory has been used to derive modern guidance laws with improved performance. This is in an attempt to replace the classical variants of proportional navigation (PN). Most optimal guidance laws (OGLs) have been derived from linear-quadratic optimal control theory to obtain feedback-form solutions, and many of them take target maneuver into account to cope with highly maneuvering target. Optimal control theory assumes the target future maneuver to be completely defined. The feedback nature of the pursuer guidance law allows the pursuer to correct for inaccurate predictions of target maneuvers and unmodelled dynamics [1-4]. The requirement has led to the development of OGLs from perfect to time-varying high-order autopilots including nonminimum phase control systems [5-8].

The desired performance index for a guidance law in the exoatmosphere is usually the amount of fuel consumption required for corrective maneuvers. The lateral divert, the integral of the absolute value of commanded acceleration imparted to an interceptor during intercept, is directly related to the amount of fuel required by the interceptor [2]. If the lateral divert were minimized, the solution would be mathematically intractable [2], especially with trajectory constraints. In very simple cases the solution of an optimal fuel control problem can be obtained in closed forms [1, 9-11].

Explicit guidance laws (EGLs) may be developed to deal with the objective function of lateral divert. The original idea of the explicit guidance laws comes from Cherry, who developed a zero-miss guidance from a given time history of acceleration command for a space vehicle with perfect dynamics [12]. The method has been extended for midcourse/terminal guidance laws with autopilot dynamics and against maneuvering targets [8, 13]. In addition, applying the thrust and drag effects in the governing equation in the presence of autopilot dynamics and gravitational acceleration is of high interest in the field of guidance theory and application. An approximate model and its analytical solution for zero-effort miss have been presented in Ref. [14].

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In this work, an optimal fuel strategy is developed by considering autopilot dynamics and an approximate model of drag, utilized in Ref. [14], in the presence of gravity. In order to solve the problem, the governing equation is transformed and rewritten in terms of zero-effort miss (ZEM). A closed-form optimal fuel solution (or near optimal) is also obtained in terms of ZEM. The optimal fuel guidance law needs to be modified to eliminate the chattering effect. To this end, a modified guidance law is finally presented.

2. ZERO-EFFORT MISS EQUATION

Consider the three-dimensional engagement geometry in which an interceptor M having a velocity $\mathbf{v}_m$ is pursuing a target T with velocity $\mathbf{v}_t$. The interceptor and target positions with respect to an inertial reference $Oxyz$ are $\mathbf{r}_m$ and $\mathbf{r}_t$, respectively (Fig. 1). The relative displacement and velocity are then given by

$$\mathbf{r} = \mathbf{r}_t - \mathbf{r}_m \tag{1.1}$$

$$\mathbf{v} = \mathbf{v}_t - \mathbf{v}_m \tag{1.2}$$

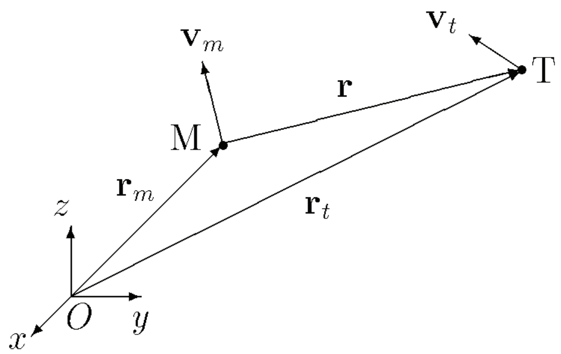

Fig. 1. Engagement geometry

The governing equation of motion for an interceptor as a particle is modeled as [14]

$$\ddot{\mathbf{r}}_m = -c(t)\mathbf{v}_m + \mathbf{a}_c + \mathbf{g}(t) \tag{2}$$

where $\mathbf{a}_c$ is the achieved commanded acceleration vector of the interceptor. The acceleration due to drag is modeled by the term $-c(t)\mathbf{v}_m$, the gravitational acceleration, $\mathbf{g}(t)$, is taken as a vectorial function of time, and any other forces given as functions of time may also be included in the term $\mathbf{g}(t)$.

By integrating Eq. (2) twice with respect to time, interceptor position is obtained as

$$\mathbf{r}_m(t) = \mathbf{r}_{m_0} + \beta(t,t_0)\,\mathbf{v}_{m_0} + \int_{t_0}^{t} \beta(t,t')[\mathbf{a}_c(t') + \mathbf{g}(t')]\,dt' \tag{3}$$

where

$$\beta(t,t_0) = \int_{t_0}^{t} e^{-\int_{t_0}^{\eta} c(\tau)d\tau}\,d\eta \tag{4}$$

Assuming that the interceptor autopilot is identical for each axis leads to three identical solutions for x, y, and z axes. Therefore, the problem is solved for one axis (i.e., y axis), that is, $\ddot{y}_m = -c(t)v_m + g_y(t) + a_{yc}$ where $v_m = \dot{y}_m$ and the subscript "y" represents the component along the y axis. The dynamics of the nth order autopilot in the y channel can be shown in a partitioned form as [7]

$$\begin{bmatrix} \dot{a}_{yc} \\ \dot{\mathbf{q}}_y \end{bmatrix} = \begin{bmatrix} a_{11}(t) & \mathbf{a}_{12}(t) \\ \mathbf{a}_{21}(t) & \mathbf{a}_{22}(t) \end{bmatrix} \begin{bmatrix} a_{yc} \\ \mathbf{q}_y \end{bmatrix} + \begin{bmatrix} b_1(t) \\ \mathbf{b}_2(t) \end{bmatrix} u_y \tag{5}$$

where a_{yc} is the first state variable and $\mathbf{q}_y$ elements are the remaining $(n-1)$ state variables; a_{11} and b_1 are scalars; $\mathbf{a}_{21}$, $\mathbf{a}_{12}^T$, and $\mathbf{b}_2$ are $(n-1)\times 1$ vectors; $\mathbf{a}_{22}$ is a $(n-1)\times(n-1)$ matrix; and u_y is the commanded acceleration along the y axis. The fundamental matrix of the autopilot, if obtainable, can be expressed as

$$\mathbf{\Phi}^A(t,t_0) = \begin{bmatrix} \Phi_{11}^A(t,t_0) & \mathbf{\Phi}_{12}^A(t,t_0) \\ \mathbf{\Phi}_{21}^A(t,t_0) & \mathbf{\Phi}_{22}^A(t,t_0) \end{bmatrix} \tag{6}$$

where Φ_{11}^A is scalar; $\mathbf{\Phi}_{21}^A$ and $\mathbf{\Phi}_{12}^{A^T}$ are $(n-1)\times 1$ vectors; and $\mathbf{\Phi}_{22}^A$ is a $(n-1)\times(n-1)$ matrix. Consequently, the system state equation, $\dot{\mathbf{X}} = A(t)\mathbf{X} + B(t)\mathbf{U} + \mathbf{G}(t)$, for the y channel is

$$\frac{d}{dt}\begin{bmatrix} y_m \\ v_m \\ a_{yc} \\ \mathbf{q}_y \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \mathbf{0} \\ 0 & -c(t) & 1 & \mathbf{0} \\ 0 & 0 & a_{11}(t) & \mathbf{a}_{12}(t) \\ \mathbf{0} & \mathbf{0} & \mathbf{a}_{21}(t) & \mathbf{a}_{22}(t) \end{bmatrix} \begin{bmatrix} y_m \\ v_m \\ a_{yc} \\ \mathbf{q}_y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ b_1(t) \\ \mathbf{b}_2(t) \end{bmatrix} u_y + \begin{bmatrix} 0 \\ g_y(t) \\ 0 \\ \mathbf{0} \end{bmatrix} \tag{7}$$

where $\mathbf{0}$'s are zero matrices of appropriate dimensions.

We are to reach the desired final position $\mathbf{r}_{m_f}^*$. The zero-effort miss, **ZEM**(t), is the distance that the interceptor would miss its target position if the interceptor made no acceleration command after the time t. For the mentioned system, ZEM for the y channel can be obtained as [14]

$$\begin{aligned} \mathrm{ZEM}_y = y_{m_f}^* - y_m - v_m\beta(t_f,t) - \int_t^{t_f} \beta(t_f,\xi) g_y(\xi) d\xi \\ -\begin{bmatrix} \int_t^{t_f} \beta(t_f,\xi)\Phi_{11}^A(\xi,t)d\xi & \int_t^{t_f} \beta(t_f,\xi)\mathbf{\Phi}_{12}^A(\xi,t)d\xi \end{bmatrix} \begin{bmatrix} a_{yc} \\ \mathbf{q}_y \end{bmatrix} \end{aligned} \tag{8}$$

The preceding relation is utilized as a change of variable in the optimization problem [see the Appendix].

For identical autopilot channels and predetermined miss distance $\mathbf{m}_p$ we have [14]

$$\mathbf{m}_p - \mathbf{ZEM}(t) = -\int_t^{t_f} \alpha(\xi)\mathbf{u}(\xi)d\xi \tag{9}$$

$$\mathbf{Z\dot{E}M}(t) = -\alpha(t)\mathbf{u} \tag{10}$$

where

$$\alpha(t) = \int_t^{t_f} \beta(t_f,\xi)[b_1(t)\Phi_{11}^A(\xi,t) + \mathbf{\Phi}_{12}^A(\xi,t)\mathbf{b}_2(t)]d\xi \tag{11}$$

For the y channel or in the linearized kinematics, Eq. (10) is written as

$$\mathrm{Z\dot{E}M}_y(t) = -\alpha(t)u_y \tag{12}$$

which integrates into

$$\mathrm{ZEM}_y(t_f) - \mathrm{ZEM}_y(t_0) = -\int_{t_0}^{t_f} \alpha(t)u_y(t)\,dt \tag{13}$$

Note that α (t) is also a function of specified final time t_f, which for simplicity is dropped from its argument.

1. Special cases

a) Moving targets

If we are to intercept a moving target, the desired interceptor final position must be the target final position, that is,

$$\mathbf{r}^*_{m_f} = \mathbf{r}_t(t_f) \tag{14}$$

Several models may be used for estimating the target future maneuvers such as constant jerk, constant acceleration, constant normal acceleration, etc. These models are also applicable here.

b) PN

The zero-lag zero-effort miss relation in the absence of drag and gravity effects against a nonmaneuvering target simplifies to

$$\mathbf{ZEM} = \mathbf{r} + \mathbf{v}\, t_g \tag{15}$$

where t_g is time-to-go until intercept ($t_g = t_f - t$). The optimal form of PN is

$$\mathbf{u} = \frac{N'}{t_g^2}(\mathbf{r} + \mathbf{v}\, t_g) = \frac{N'}{t_g^2}\mathbf{ZEM} \tag{16}$$

with an effective navigation ratio of 3 ($N' = 3$). This form of acceleration command minimizes $J = \int_{t_0}^{t_f} u^2 dt$. The solution for J and lateral divert $\Delta V = \int_{t_0}^{t_f} |\mathbf{u}(t)|\, dt$ (t_f is specified) are given by [15]

$$J = \frac{N'^2}{(2N' - 3)t_f^3}\ \mathrm{ZEM}_0 \tag{17}$$

$$\Delta V = \frac{N'}{(N' - 1)t_f}\ \mathrm{ZEM}_0 \tag{18}$$

As expected, the minimum value of J is for $N' = 3$. The lateral divert becomes infinity for $N' \le 1$ and J becomes infinity for $N' \le 1.5$. Increasing N' from 1 makes the lateral divert requirements smaller. As N' goes to infinity, ΔV tends to ZEM_0 / t_f. An infinite value of effective navigation ratio is equivalent to an impulsive command that zeros out the initial heading error instantly.

c) Time-invariant coefficients

In the case that A and B are time-invariant, also $c(t) = const.$, the relation for $\alpha(t)$ simplifies to [16]

$$\alpha(t) = L^{-1}\left[\frac{1}{s(s+c)}\frac{\tilde{a}_{yc}(s)}{\tilde{u}_y(s)}\right]\Bigg|_{t_f - t} \tag{19}$$

where s is the Laplace variable; L^{-1} is the inverse Laplace transform operator; and $\tilde{a}_{yc}(s)/\tilde{u}_y(s) = b_1 \tilde{\Phi}^A_{11}(s) + \tilde{\mathbf{\Phi}}^A_{12}(s)\,\mathbf{b}_2$ is the autopilot transfer function for the y channel.

3. OPTIMAL FUEL GUIDANCE LAW

The problem is to obtain the zero-miss optimal intercept guidance law that minimizes the integral of the absolute value of the commanded acceleration subject to state equation (7). One approach to solve the problem is through the reduced form (12) of the state equation (7) and restating the optimization problem. For simplicity, the linearized kinematics is dealt with. Minimization of the cost function $\Delta V = \int_{t_0}^{t_f} |u_y(t)|\,dt$ (t_f is specified) is to be performed subject to the linear differential constraint (12) and input constraint $|u_y| \le U_{\text{sat}}$ with the final condition $\text{ZEM}_y(t_f)=0$. The initial value is $\text{ZEM}_y(t_0) = \text{ZEM}_{y_0} \neq 0$. Note that the optimal input will be zero if $\text{ZEM}_{y_0}=0$. The Hamiltonian of the reduced system is written as

$$H = |u_y| - \alpha(t)\, p\, u_y \tag{20}$$

where p is the costate variable. It turns out that the costate variable must be a constant in order to satisfy one of the necessary conditions of optimality, that is,

$$\dot{p} = -\frac{\partial H}{\partial \text{ZEM}_y} = 0 \;\rightarrow\; p = const. \tag{21}$$

The optimal u_y is obtained as

$$u_y = \begin{cases} U_{\text{sat}}\,\text{sgn}(\alpha p) & \text{for } |\alpha(t)p| > 1 \\ 0 & \text{for } |\alpha(t)p| < 1 \\ \text{undetermined but bounded} & \text{for } |\alpha(t)p| = 1 \end{cases} \tag{22}$$

where $\text{sgn}(\cdot)$ is the signum function. Equation (13) for $\text{ZEM}_y(t_f) = 0$ is written as

$$\text{ZEM}_y(t_0) = \int_{t_0}^{t_f} \alpha(t)\, u_y(t)\, dt \tag{23}$$

Substituting Eq. (22) for u_y into the preceding relation yields,

$$\text{ZEM}_{y_0} = \int_{t_0}^{t_f} \alpha(t) \begin{Bmatrix} U_{\text{sat}}\,\text{sgn}(\alpha p) & \text{for } |\alpha(t)p| > 1 \\ 0 & \text{for } |\alpha(t)p| < 1 \\ \text{undetermined but bounded} & \text{for } |\alpha(t)p| = 1 \end{Bmatrix} dt \tag{24}$$

Assuming that $\alpha(t)$ is continuous and $|\alpha(t)| = c_1$ (c_1 is an arbitrary constant) has finite solutions, we have no singular interval. We may have only singular points for which the controls are undetermined. Since the input is bounded, these points do not affect the value of the integral. We can thus write

$$\text{ZEM}_{y_0} = U_{\text{sat}} \sum_{i=1}^{n_s} \int_{t_{s_i}^+}^{t_{b_i}^-} \alpha(t)\, \text{sgn}[\alpha(t)p]\, dt \tag{25}$$

where $t_{s_i}^+$ and $t_{b_i}^-$ are the beginning and ending times of steering periods for $|\alpha(t)p| > 1$, respectively; and n_s is the number of steering periods. Equation (25) can be rewritten as

$$\text{ZEM}_{y_0} = U_{\text{sat}}\, \text{sgn}(p) \sum_{i=1}^{n_s} \int_{t_{s_i}^+}^{t_{b_i}^-} |\alpha(t)|\, dt \tag{26}$$

The preceding relation implies that $\text{sgn}(p) = \text{sgn}(\text{ZEM}_{y_0})$. Therefore, we will have

$$\sum_{i=1}^{n_s} \int_{t_{s_i}^+}^{t_{b_i}^-} |\alpha(t)|\, dt - \frac{|\text{ZEM}_{y_0}|}{U_{\text{sat}}} = 0 \tag{27}$$

Using the following inequality:

$$\sum_{i=1}^{n_s}\int_{t_{s_i}^+}^{t_{b_i}^-}|\alpha(t)|\,dt \le \int_{t_0}^{t_f}|\alpha(t)|\,dt \tag{28}$$

the intercept criterion is obtained as

$$U_{\text{sat}} \ge \frac{|\,\text{ZEM}_{y_0}|}{\int_{t_0}^{t_f}|\alpha(t)|\,dt} \tag{29}$$

The optimal fuel guidance can then be written as

$$u_y = \begin{cases} U_{\text{sat}}\ \text{sgn}[\text{ZEM}_{y_0}\alpha(t)] & \text{for } |\alpha(t)p|>1 \\ 0 & \text{for } |\alpha(t)p|<1 \\ \text{undetermined but bounded for } |\alpha(t)p|=1 \end{cases} \tag{30}$$

a) Determination of Costate

The following observations are of importance in this regard:

- Equation (27) implies that there is, at least, one time interval in which $|\alpha(t)p|>1$ or $u_y = U_{\text{sat}}\ \text{sgn}(\text{ZEM}_{y_0}\alpha)$.
- If $\int_{t_0}^{t_f}|\alpha(t)|\,dt > |\,\text{ZEM}_{y_0}|/U_{\text{sat}}$, at least one time interval exists in which $|\alpha(t)p|<1$ or $u_y = 0$. Since $\alpha(t)$ is a continuous function, there is, at least, one point $t = t_s$ such that $|\alpha(t)p|=1$.
- The optimal input is $\pm U_{\text{sat}}$ in the time intervals for which $|\alpha(t)|>|\alpha(t_s)|$.
- The optimal input is zero in the time intervals for which $|\alpha(t)|<|\alpha(t_s)|$.
- Considering a given time t_1, if $|\alpha(t_1)|>|\alpha(t_s)|$ then the optimal input will be $\pm U_{\text{sat}}$. Also, if $|\alpha(t_1)|<|\alpha(t_s)|$, the optimal input will be zero.

Hence, the following steps may be taken in order to find p:

1- Draw the plot of $|\alpha(t)|$ versus time (Fig. 2).

2- Draw a line parallel to the time axis. Assign the intersect points of the line with the plot of $|\alpha(t)|$.

3- Find the corresponding time for each intersect point. Assign the time intervals in which the $|\alpha(t)|$ graph is above the horizontal line. Name these intervals in a way that begins with t_{s_i} and ends with t_{b_i}. Denote the number i for which $t_{s_i} < t_{s_{i+1}}$ and $t_{b_i} < t_{b_{i+1}}$.

4- Calculate the following expression:

$$I = \sum_{i=1}^{n_s}\int_{t_{s_i}^+}^{t_{b_i}^-}|\alpha(t)|\,dt - \frac{|\,\text{ZEM}_{y_0}|}{U_{\text{sat}}} \tag{31}$$

If $I>0$ we must shift up the horizontal line and repeat the steps 2-4 till $I=0$. If $I<0$ we must shift down the horizontal line until $I=0$.

5- The value of $|\alpha(t)|$ for the horizontal line that makes $I=0$ is equal to $1/|p|$. We also know that the costate is co-sign with ZEM_{y_0}.

If $\alpha(t)$ is continuous, $|\alpha(t)|=c_1$ has only finite solutions and inequality (29) holds, the solution will be unique. If $|\alpha(t)|$ is constant over at least one time interval, the problem may have one or infinitely many solutions, depending on the initial condition.

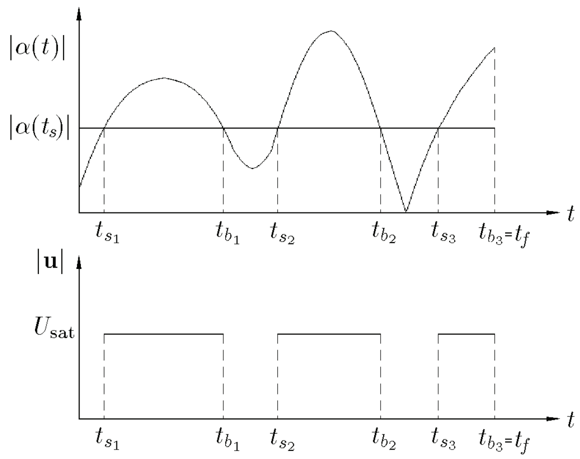

Fig. 2. Optimal fuel strategy

b) Three-dimensional solution

Minimization of the cost function $\Delta V = \int_{t_0}^{t_f} |\mathbf{u}(t)|\, dt$ (t_f is specified) is to be performed subject to the linear differential constraint (10), and input constraint $|\mathbf{u}| \le U_{\text{sat}}$ with the final condition $\mathbf{ZFM}(t_f) = \mathbf{0}$. The initial value is $\mathbf{ZFM}(t_0) = \mathbf{ZFM}_0 \neq \mathbf{0}$. Equation (10) is written in the form of

$$\dot{\text{ZEM}}_i = -\alpha(t)\,|\mathbf{u}|\cos\beta_i\ , \quad i = x, y, z \tag{32}$$

where β_i is the angle of $\mathbf{u}$ direction with respect to the i axis and $\cos^2\beta_x + \cos^2\beta_y + \cos^2\beta_z = 1$. Therefore,

$$\mathbf{u} = |\mathbf{u}|(\cos\beta_x\,\mathbf{e}_x + \cos\beta_y\,\mathbf{e}_y + \cos\beta_z\,\mathbf{e}_z) \tag{33}$$

where $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$ are the unit vectors along x, y, and z axes.

The Hamiltonian is written as

$$H = |\mathbf{u}|[1 - \gamma\,\alpha(t)] \tag{34}$$

where

$$\gamma = p_1\cos\beta_x + p_2\cos\beta_y + p_3\cos\beta_z \tag{35}$$

and p_1, p_2, and p_3 are costates. Since the states ZEM_i's do not appear in the Hamiltonian, the costates become constant. In this formulation, we have three control inputs, i.e., bounded input of $|\mathbf{u}|$ and the two of β_i's which are not bounded. Therefore, differentiating H with respect to β_y and β_z results in

$$p_2 = \cos\beta_y / \cos\beta_x\ , \quad p_3 = \cos\beta_z / \cos\beta_x \tag{36}$$

We can thus write

$$\cos^2\beta_x = \frac{1}{1 + p_2^2 + p_3^2}\ , \quad \cos^2\beta_y = \frac{p_2^2}{1 + p_2^2 + p_3^2}\ , \quad \cos^2\beta_z = \frac{p_3^2}{1 + p_2^2 + p_3^2} \tag{37}$$

$$\gamma = \cos\beta_x(1 + p_2^2 + p_3^2) = 1/\cos\beta_x \tag{38}$$

Therefore, $|\cos\beta_i|$'s and $|\gamma|$ are constant.

Substituting for $\cos\beta_y$ and $\cos\beta_z$ from Eq. (36) into Eq. (33) yields

$$\mathbf{u} = |\mathbf{u}|\cos\beta_x(\mathbf{e}_x + p_2\,\mathbf{e}_y + p_3\,\mathbf{e}_z) \tag{39}$$

Since the costates are constant, the preceding relation implies that either the direction of $\mathbf{u}$ is constant and/or reverses.

By substituting Eq. (39) into the following relation:

$$\mathbf{ZEM}(t_0) = \int_{t_0}^{t_f} \alpha(t)\mathbf{u}(t)dt \tag{40}$$

we arrive at

$$\mathbf{ZEM}(t_0) = -(\mathbf{e}_x + p_2\,\mathbf{e}_y + p_3\,\mathbf{e}_z)\int_{t_0}^{t_f} |\,\mathbf{u}(t)\,| \cos\beta_x(t)\alpha(t)dt \tag{41}$$

It follows that the unit vector of $\mathbf{u}$ becomes $\pm\mathbf{i}_{\mathrm{ZEM}_0}$ where $\mathbf{i}_{\mathrm{ZEM}}$ is the unit vector along the zero-effort miss ($\mathbf{i}_{\mathrm{ZEM}} = \mathbf{ZEM}/|\,\mathbf{ZEM}\,|$). Therefore, Eq. (10) can be reduced to a scalar form. Through a rotation of coordinates in a way that the x axis coincides with the initial ZEM direction, we will have $\beta_x = 0$ or 180 deg and $p_2 = p_3 = 0$. Thus Eq. (39) reduces to

$$\mathbf{u} = s\,|\,\mathbf{u}\,|\,\mathbf{i}_{\mathrm{ZEM}_0}\,,\ \ s = \pm 1 \tag{42}$$

By changing the control variable $u = s\,|\,\mathbf{u}\,|$ $(-U_{\mathrm{sat}} \le u \le U_{\mathrm{sat}})$ we will have $\dot{\mathbf{ZEM}} = -\alpha(t)\,u\,\mathbf{i}_{\mathrm{ZEM}_0}$ or $\dot{\mathrm{ZEM}} = -\alpha(t)\,u$. This leads to a solution similar to the one-dimensional case. Hence, the optimal control is written as [16, 17]

$$\mathbf{u} = \mathbf{i}_{\mathrm{ZEM}_0} \begin{cases} U_{\mathrm{sat}}\,\mathrm{sgn}[\alpha(t)] & \text{for } |\,\alpha(t)\lambda\,| > 1 \\ 0 & \text{for } |\,\alpha(t)\lambda\,| < 1 \\ \text{undetermined value but bounded for } |\,\alpha(t)\lambda\,| = 1 \end{cases} \tag{43}$$

Determination of λ is the same as that of p, i.e., steps 1-5 must be followed. The value of $|\,\alpha(t)\,|$ for the horizontal line that makes $I = 0$, is equal to $1/|\,\lambda\,|$. Note that we must drop the subscript y in Eq. (31), thus arriving at the following intercept criterion for any time:

$$U_{\mathrm{sat}} \ge \frac{|\,\mathbf{ZEM}\,|}{\int_t^{t_f} |\,\alpha(t)\,|\,dt} \tag{44}$$

The preceding relation is useful for closed-form representation.

If $\alpha(t) \ge 0$ is a strictly monotonically decreasing function of time, and the intercept criterion holds, we will have

$$\int_{t_0}^{t_{b_1}^-} |\,\alpha(t)\,|\,dt = \frac{|\,\mathbf{ZEM}_0\,|}{U_{\mathrm{sat}}}, \qquad \Delta V = U_{\mathrm{sat}}\,t_{b_1} \tag{45}$$

where $t_{b_1} \le t_f$ is calculated by the preceding relation and we have $|\,\lambda\,| = 1/|\,\alpha(t_{b_1})\,|$. Therefore, the solution is

$$\mathbf{u} = \begin{cases} U_{\mathrm{sat}}\,\mathbf{i}_{\mathrm{ZEM}_0} & \text{for } t < t_{b_1} \\ 0 & \text{for } t > t_{b_1} \end{cases} \tag{46}$$

Integrating Eq. (10) and substituting for $\mathbf{u}$ from the preceding relation yields

$$\mathbf{ZEM}(t) = \mathbf{ZEM}(t_0) - U_{\mathrm{sat}}\,\mathbf{i}_{\mathrm{ZEM}_0}\int_{t_0}^{t} |\,\alpha(\xi)\,|\,d\xi \quad \text{for } t < t_{b_1} \tag{47}$$

Equation (47) implies that $\mathbf{i}_{\mathrm{ZEM}}$ remains constant. We can thus write

$$|\mathbf{ZEM}(t_0)|=|\mathbf{ZEM}(t)|+U_{\text{sat}}\int_{t_0}^{t}|\alpha(\xi)|\,d\xi \quad \text{for } t<t_{b_1} \tag{48}$$

Substituting the preceding relation into Eq. (45) results in

$$\int_{t}^{t_{b_1}^-}|\alpha(\xi)|\,d\xi=\frac{|\mathbf{ZEM}(t)|}{U_{\text{sat}}} \quad \text{for } t<t_{b_1} \tag{49}$$

Therefore, when $|\mathbf{ZEM}(t)|\neq 0$ the steering period continues until $|\mathbf{ZEM}|=0$. Thus, the closed-loop form is given by [18]

$$\mathbf{u}=\begin{cases} U_{\text{sat}}\,\mathbf{i}_{\text{ZEM}} & \text{for } |\mathbf{ZEM}|>0 \\ 0 & \text{for } |\mathbf{ZEM}|=0 \end{cases} \tag{50}$$

It should be noted that the explicit relation has been obtained in the case that $\alpha(t)\geq 0$ is a strictly monotonically decreasing function of time. Using Eq. (49) the switching time t_{b_1} can be calculated through numerical methods; however, there is no need to calculate t_{b_1} in order to obtain closed-form optimal control. The term $\text{ZEM}=0$ is the switching curve. For $|\mathbf{ZEM}|\neq 0$ we have one switching point and the control is "bang-off". In this case, the solution is also minimum time to zero out the zero-effort miss with specified final time t_f.

a) Perfect autopilot

For a perfect guidance/control system and $c(t)=0$, we have $\alpha(t)=t_g$ which is a non-negative and strictly monotonically decreasing function of time. Therefore, we can use Eqs. (45-50). Solving Eqs. (45) and (49) for t_{b_1} yields ($t_0=0$):

$$\frac{t_{b_1}}{t_f}=1-\sqrt{1-\frac{2|\mathbf{ZEM}_0|}{U_{\text{sat}}\,t_f^2}} \tag{51}$$

or

$$\frac{t_{bg_1}}{t_g}=1-\sqrt{1-\frac{2|\mathbf{ZEM}|}{U_{\text{sat}}\,t_g^2}} \tag{52}$$

where $t_{bg_1}=t_{b_1}-t$. The intercept criterion is simply obtained as $U_{\text{sat}}\geq 2|\mathbf{ZEM}_0|/t_f^2$.

The lateral divert can be found as

$$\frac{\Delta V}{U_{\text{sat}}\,t_f}=1-\sqrt{1-\frac{2|\mathbf{ZEM}_0|}{U_{\text{sat}}\,t_f^2}} \tag{53}$$

In the case of nonmaneuvering targets, $\mathbf{g}(t)=\mathbf{0}$, and $t_f=r_0/v_{c_0}$ (assume $v_{c_0}>0$), we have

$$\frac{\Delta V}{r_0}=\frac{U_{\text{sat}}}{v_{c_0}}\left[1-\sqrt{1-\frac{2v_{c_0}\Omega_0}{U_{\text{sat}}}}\right] \tag{54}$$

where r_0, v_{c_0}, and Ω_0 are the initial interceptor-target relative position, closing velocity, and line-of-sight (LOS) angular velocity normal to LOS, respectively. The intercept criterion is given by $U_{\text{sat}}\geq 2v_{c_0}\Omega_0$.

b) First-order autopilot

For a single-lag autopilot with time constant T, and for $c(t) = 0$, we have $\alpha(t) = T(e^{-\tau} + \tau - 1)$ where $\tau = t_g / T$. This function is a non-negative and strictly monotonically decreasing function of time. Thus, the closed-form optimal control follows Eq. (50). Figure 3 shows the behavior of $\alpha(t)$ for different values of time constants.

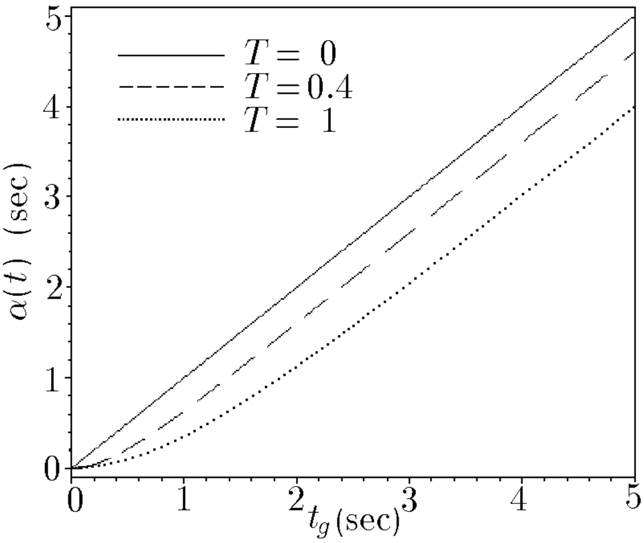

Fig. 3. The value of $\alpha(t)$ for single-lag autopilots

Using Eq. (49) one can write ($t_0 = 0$):

$$T\int_{t}^{t_{b_1}} [e^{-\tau(\xi)} + \tau(\xi) - 1]\, d\xi = \frac{|\,\mathbf{ZEM}\,|}{U_{\text{sat}}} \tag{55}$$

which leads to a transcendental algebraic equation for t_{bg_1}.

c) Nonminimum phase autopilot

As an example, consider a nonminimum phase interceptor whose airframe and control model are described by a third-order transfer function as follows [7]:

$$\frac{\tilde{a}_{m_{yc}}(s)}{\tilde{u}_{m_y}(s)} = \frac{1 + T_z s}{(1+0.11s)(1+0.17s)(1+0.24s)} \tag{56}$$

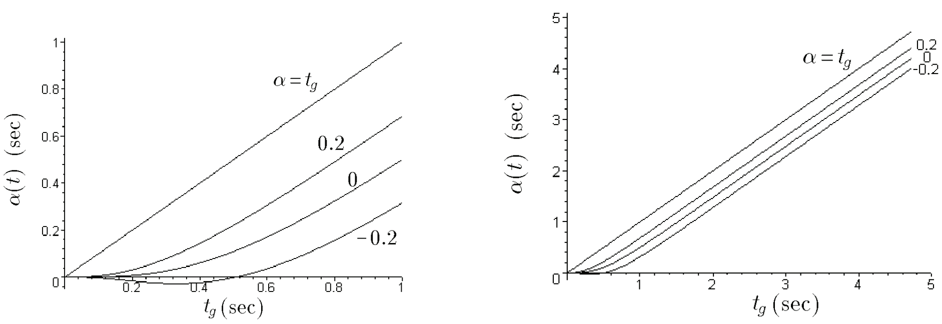

Fig. 4. The behavior of $\alpha(t)$ for different values of T_z

The behavior of $\alpha(t)$ with $c = 0$ for different values of $T_z = 0, \pm 0.2$ is depicted in Fig. 4. As shown in the figure, the value of $\alpha(t)$ becomes negative for small values of time-to-go when $T_z = -0.2$.

When time-to-go is large enough, an equivalent time constant, T_{eq}, can be utilized for minimum and nonminimum phase interceptors. Here, the equivalent time constant is the sum of the time constants of the denominator of a transfer function minus the sum of the time constants of the numerator. For the transfer function of Eq. (56), the equivalent time constant is $T_{\mathrm{eq}} = 0.11 + 0.17 + 0.24 - T_z$, thus, $T_{\mathrm{eq}} = 0.72$ sec for $T_z = -0.2$. The values of $\alpha(t)$ with $c = 0$ for this type of transfer function and its equivalent first-order will be similar when $t_g > 2.5$ sec as shown in Fig. 5. Therefore, the closed-form optimal solution for nonminimum phase interceptor can be derived for most of the flight if the total flight time is large enough. Consider an $\alpha(t)$ like the nonminimum phase behavior of Fig. 4. The minimum value of $\alpha(t)$ (in the negative part) is denoted by $\alpha_{\min}$. The time for which $\alpha(t) = |\alpha_{\min}|$ in the positive part of $\alpha(t)$ curve is denoted by T_s. Therefore, $\alpha(T_s) = |\alpha_{\min}|$. If

$$|\mathbf{ZEM}_0| < U_{\mathrm{sat}} \int_{t_0}^{t_f - T_s} |\alpha(t)|\, dt \tag{57}$$

or

$$|\mathbf{ZEM}| < U_{\mathrm{sat}} \int_{t}^{t_f - T_s} |\alpha(\xi)|\, d\xi \quad \text{for } t < t_f - T_s \tag{58}$$

then Eq. (50) or the following equation will be optimal, even for the mentioned nonminimum phase autopilot.

$$\mathbf{u} = U_{\mathrm{sat}}\, \mathbf{i}_{\mathrm{ZEM}} \quad \text{for } t_g > T_s \ \& \ |\mathbf{ZEM}| > 0 \tag{59}$$

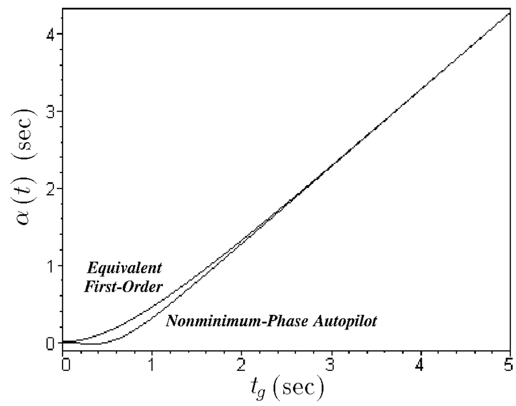

Fig. 5. A typical nonminimum phase autopilot and its equivalent first-order

4. EXPLICIT GUIDANCE LAW

Suppose that the desired maneuvering acceleration profile of an interceptor, $f^*(t) = f(t)/f(t_0)$, has been chosen by the guidance designer as a piecewise continuous function of time and proportional to the initial zero-effort miss, that is,

$$\mathbf{u} = f(t)\, \mathbf{ZEM}(t_0) \tag{60}$$

It is assumed that $f(t_0) \neq 0$. The closed-loop form of the commanded acceleration with a given $f^*(t)$ and without acceleration limit is obtained for $\alpha(t) f(t) \geq 0$ as [16]

$$\mathbf{u} = \frac{f^*(t)\, \mathbf{ZEM}}{\left(\frac{\mathrm{m}_p}{\mathrm{ZEM}_0 - \mathrm{m}_p}\right) \int_{t_0}^{t_f} \alpha(t) f^*(t) dt + \int_{t}^{t_f} \alpha(\xi) f^*(\xi) d\xi} \tag{61}$$

In the preceding relation, m_p is the predetermined miss distance which is chosen nonzero to avoid the guidance gain approaching infinity when t_g tends toward zero ($\mathrm{m}_p < \mathrm{ZEM}_0$). The relation for $\Delta V = \int_0^{t_f} |\mathbf{u}(t)|\,dt$, can then be found as [16]

$$\Delta V = \frac{\int_0^{t_f} |f(t)|\,dt}{\int_0^{t_f} \alpha(t) f(t)\,dt}\,\mathrm{ZEM}_0 \quad \text{for } t_0 = 0 \,\&\, \mathrm{m}_p = 0 \tag{62}$$

If $\alpha(t), f(t) \geq 0$ and $\alpha(t)$ has an absolute maximum at the time $t = t_m$ in the interval $[0, t_f]$, we will find that the impulse function for $f(t)$ at $t = t_m$ minimizes the lateral divert requirement, i.e., $\Delta V = \mathrm{ZEM}_0 / \alpha(t_m)$. The proof of this assertion is as follows:
Eq. (62) can be put into the form

$$\frac{\Delta V}{\mathrm{ZEM}_0} = \frac{1}{|\alpha(t_m)| - Q} \tag{63}$$

where

$$Q = \frac{\int_0^{t_f} [|\alpha(t_m)| - \alpha(t)] f(t)\,dt}{\int_0^{t_f} |f(t)|\,dt} \tag{64}$$

Since $0 \leq Q \leq |\alpha(t_m)|$, ΔV is minimized when $Q = 0$. The impulse function for $f(t)$ at $t = t_m$ causes $Q = 0$.

a) Explicit Guidance Law with Acceleration Limit

Consider the case in which the commanded acceleration is chosen according to Eq. (61) with $f^*(t)$ as a continuous function of time. In practice, the commanded acceleration may be saturated. Therefore, the commanded acceleration is given by

$$\mathbf{u} = \begin{cases} U_{\text{sat}} \operatorname{sgn}[\alpha(t)]\,\mathbf{i}_{\text{ZEM}} & \text{for } U_{\text{sat}} < |\Lambda\,\mathbf{ZEM}| \\ \Lambda\,\mathbf{ZEM} & \text{for } U_{\text{sat}} \geq |\Lambda\,\mathbf{ZEM}| \end{cases} \tag{65}$$

where

$$\Lambda = \frac{f^*(t)}{\left(\frac{\mathrm{m}_{\text{old}}}{\mathrm{ZEM}_0 - \mathrm{m}_{\text{old}}}\right)\int_{t_0}^{t_f} \alpha(t) f^*(t)\,dt + \int_t^{t_f} \alpha(\xi) f^*(\xi)\,d\xi} \tag{66}$$

and m_{old} is the predetermined miss distance without acceleration limit. The differential equation for zero-effort miss is written as

$$\begin{cases} \dot{\mathbf{ZEM}} = -|\alpha(t)|\,U_{\text{sat}}\,\mathbf{i}_{\text{ZEM}} & \text{for } U_{\text{sat}} < |\Lambda\,\mathbf{ZEM}| \\ \dot{\mathbf{ZEM}} + \Lambda\,\alpha(t)\mathbf{ZEM} = \mathbf{0} & \text{for } U_{\text{sat}} \geq |\Lambda\,\mathbf{ZEM}| \end{cases} \tag{67}$$

It is easy to show that the direction of $\mathbf{ZEM}$ remains constant. Assume the case in which the commanded acceleration saturates initially and continues until t_s and never goes to saturation again. Therefore,

$$\mathbf{ZEM} = \begin{cases} \mathbf{ZEM}(t_0) - U_{\text{sat}}\,\mathbf{i}_{\text{ZEM}_0} \int_{t_0}^{t} |\alpha(\xi)|\,d\xi & \text{for } t_0 \leq t < t_s \\ \dfrac{h(t)}{h(t_s)}\mathbf{ZEM}(t_s) & \text{for } t_s \leq t \leq t_f \end{cases} \tag{68}$$

where

$$\mathbf{ZEM}(t_s) = \mathbf{ZEM}(t_0) - U_{\text{sat}}\, \mathbf{i}_{\text{ZEM}_0} \int_{t_0}^{t_s} |\alpha(t)|\, dt \tag{69}$$

$$h(t) = \left(\frac{\text{m}_{\text{old}}}{\text{ZEM}_0 - \text{m}_{\text{old}}} \right) \int_{t_0}^{t_f} \alpha(t) f^*(t) dt + \int_{t}^{t_f} \alpha(\xi) f^*(\xi) d\xi \tag{70}$$

The solution for the commanded acceleration is then given by

$$\mathbf{u}(t) = \begin{cases} U_{\text{sat}} \operatorname{sgn}[\alpha(t)]\, \mathbf{i}_{\text{ZEM}_0} & \text{for } t_0 \le t < t_s \\ \dfrac{f^*(t)}{h(t_s)} \mathbf{ZEM}(t_s) & \text{for } t_s \le t \le t_f \end{cases} \tag{71}$$

and the relation for miss distance is obtained as

$$\frac{\text{m}}{\text{ZEM}_0} = \left[1 - \frac{U_{\text{sat}}}{\text{ZEM}_0} \int_{t_0}^{t_s} |\alpha(t)|\, dt \right] \frac{h(t_f)}{h(t_s)} \tag{72}$$

If m_{old} is set to zero and the commanded acceleration goes off the limit, then the miss distance will be zero. If the commanded acceleration does not go off the limit, we will have

$$\frac{\text{m}}{\text{ZEM}_0} = \left[1 - \frac{U_{\text{sat}}}{\text{ZEM}_0} \int_{t_0}^{t_f} |\alpha(t)|\, dt \right] \tag{73}$$

One must be careful to choose the weight coefficient of miss distance in the performance index or to choose a predetermined miss distance when the acceleration saturates. The miss distance increases even if the commanded acceleration goes off the limit.

5. DISCUSSIONS AND SIMULATION RESULTS

The optimal fuel strategy can be used when there is enough time to remove the zero-effort miss till $t = t_f - T_s$. When the initial heading error, ZEM, or angular velocity of line-of-sight is large, we can reduce it to a relatively small value by maximum acceleration command. The optimal fuel strategy is followed by the mentioned explicit guidance, that is,

$$\mathbf{u} = \begin{cases} U_{\text{sat}}\, \mathbf{i}_{\text{ZEM}} & \text{for } t_{bg_1} \ge T_m \\ \text{Eq. (61)} & \text{for } t_{bg_1} < T_m \end{cases} \tag{74}$$

The time margin T_m in the proposed guidance law must be determined by the guidance designer ($T_m > T_s > 0$). This time margin is dependent on missions and the accuracy of estimations. In order to avoid fluctuation between the two parts of the guidance law, only once must the first part of the guidance law be applied. Then the second part will be used even for $t_{bg_1} \ge T_m$. The relation for perfect autopilot may be used to approximate t_{bg_1} when we choose enough time margin T_m. The switching between the two strategies may be done by comparing ZEM to $\text{ZEM}_m = 0.5 U_{\text{sat}} t_g^2$.

We are now to see the effect of T_m on lateral divert. To verify the solution, a computer simulation of engagement and its analytical solution are presented, and a perfect autopilot is, first, considered. A modified guidance law is obtained for $f^*(t) = (t_g / t_f)^m$ as follows ($m > 0$):

$$\mathbf{u} = \begin{cases} U_{\text{sat}}\, \mathbf{i}_{\text{ZEM}} & \text{for } t_{bg_1} \geq T_m \\ \dfrac{(m+2)}{t_g^2}\mathbf{ZEM} & \text{for } t_{bg_1} < T_m \end{cases} \tag{75}$$

where t_{bg_1} is calculated from Eq. (52). The lateral divert requirement for $T_m = 0$ is denoted by ΔV^* ($\Delta V^* = U_{\text{sat}} t_{b_1}$). Increasing the time margin increases the lateral divert requirement. This percent increase can be obtained as

$$\frac{\Delta V - \Delta V^*}{\Delta V^*}\times 100 = \frac{(T_m / t_{b_1})}{m+1}\,\frac{1-\frac{t_{b_1}}{t_f}(1+\frac{m}{2}\frac{T_m}{t_{b_1}})}{1-\frac{t_{b_1}}{t_f}(1-\frac{T_m}{t_{b_1}})}\times 100 \quad \text{for } U_{\text{sat}} > \frac{m+2}{t_f^2}\,\text{ZEM}_0 \left(\text{or } \frac{t_{b_1}}{t_f} < 1-\sqrt{\frac{m}{m+2}}\right) \tag{76}$$

The preceding relation is depicted in Fig. 6 for $t_{b_1} / t_f = 0.25$ and two values of m ($m = 1, 2$). The percent increase for the explicit guidance law is simply obtained by setting $T_m = t_{b_1}$. This value for the explicit guidance law is viewed in the figure at $T_m / t_{b_1} = 1$.

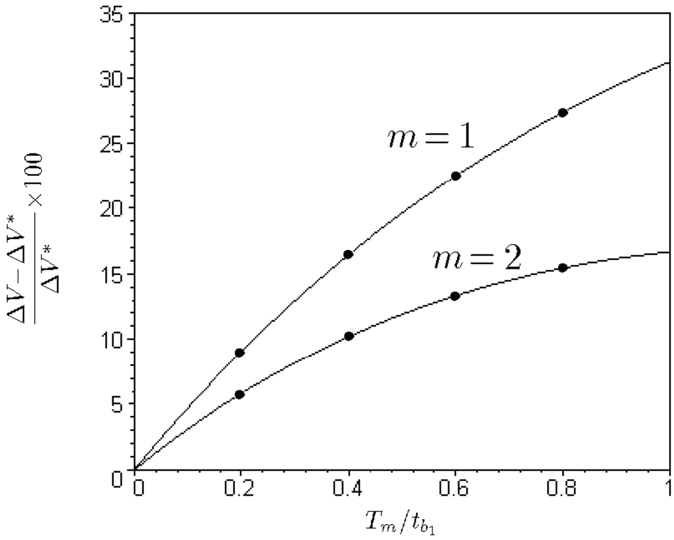

Fig. 6. Percent increase of ΔV vs time margin for $t_{b_1} / t_f = 1/4$

For simulation the initial position and velocity of an interceptor and its target are chosen as $\mathbf{r}_{m_0} = \mathbf{0}$, $\mathbf{v}_{m_0} = [350\ 400\ 50]^T$ m/s, $\mathbf{r}_{t_0} = [6\ 9\ 7]^T$ km, and $\mathbf{v}_{t_0} = [300\ 150\ 0]^T$ m/s. The target maneuvers at $\mathbf{a}_t = [40\ 30\ 0]^T$ m/s^2 during the engagement. The final time is specified and $m = 1, 2$ and $T_m / t_{b_1} = 0.2, 0.4, 0.6, 0.8$. To satisfy the relation of $t_{b_1} / t_f = 0.25$ in Fig. 6, we choose $U_{\text{sat}} = 887.14, 421.28, 326.47$ m/s^2 for $t_f = 8, 12, 14$ sec, respectively. The results are shown in Fig. 6 with circles, verifying the presented analytical solution of the modified guidance strategy of Eq. (75).

As mentioned before, the first part of the modified explicit guidance law may be developed for perfect or the equivalent first-order dynamics for most minimum and nonminimum phase autopilots. Since the behavior of $|\alpha(t)|$ for large values of t_g is nearly similar to that of a perfect system or its equivalent first-order, Eq. (52) may be used for the first part to estimate t_{bg_1} when time-to-go is large. Therefore, approximation by first-order system is suitable when t_g is large enough and $|\alpha(t)|$ is also in the strictly decreasing region. For the second part, the acceleration profile is chosen proportional to $\text{sgn}[\alpha(t)]\,|\alpha(t)|^m$ [16].

The simulation results for perfect and single-lag autopilots are listed in Table 1 using the previous initial conditions and target maneuver strategy. The final time and acceleration limit are taken 14 sec and

$300\ \mathrm{m/s^2}$, respectively. Three guidance strategies are compared, that is, minimum energy guidance (MEG) that minimizes J, explicit guidance law (EGL), and minimum fuel guidance (MFG). As expected, the values of J and ΔV are minimum for minimum energy and minimum fuel guidance laws, respectively, for a given final time.

Table 1. Simulation results for three guidance laws

Guidance types	Guidance/Airframe parameters	ΔV (m/s)	J ($\mathrm{m^2/s^3}$)
MEG	$T=0,\ c=0$ $T=0, c=0.05$ $T=0.4,\ c=0$	1499.8 2245.4 1545.5	214234 455761 233746
EGL ($m=2$)	$T=0,\ c=0$ $T=0, c=0.05$ $T=0.4,\ c=0$	1333.2 2066.9 1372.6	228533 473364 249284
MFG	$T=0,\ c=0$ $T=0, c=0.05$ $T=0.4,\ c=0$	1159.8 (1160.1*) 1924.8 (1925.2*) 1207.8 (1208.1*)	347940 577440 362340

* Analytical solution

Advanced guidance laws need to estimate time-to-go until intercept and interceptor future velocity profile. There are several methods for estimation of t_g and $\beta(t_f, t)$ that can be used for the presented guidance strategy [3, 18-20]. Also, stochastic guidance strategies that minimize miss distance can be used instead of the mentioned explicit guidance for the second part.

6. CONCLUSION

An optimal fuel guidance law with acceleration limit is derived for interception of maneuvering targets. The interceptor is modelled as a particle with a linear time-varying arbitrary-order guidance/control system. An approximate model for drag acceleration is also considered in the solution. The optimal fuel strategy has some detrimental effects in practical applications. The switching time is highly sensitive to uncertainties, seeker noise, and unmodeled dynamics. These in turn can cause the commanded acceleration to fluctuate between its maximum and minimum limits thus destabilizing the autopilot system. To cope with the aforementioned detrimental effects, a modified explicit guidance law is developed for several minimum and nonminimum phase autopilots. The time margin to switching time must be determined by consideration of uncertainties, unmodeled dynamics, and the accuracy of estimations. Sliding mode techniques [21, 22] for avoiding chattering can also be utilized for modification of the optimal fuel law.

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Appendix: Change of variable

Substituting the solution of $\mathbf{a}_c$ into Eq. (3) for the y-channel yields

$$y_m(t) = y_m(t_0) + v_m(t_0)\beta(t,t_0) + \int_{t_0}^{t} \beta(t,\xi) g_y(\xi) d\xi$$
$$+ a_{yc}(t_0)\int_{t_0}^{t} \beta(t,\xi)\Phi_{11}^{A}(\xi,t_0)d\xi + \left[\int_{t_0}^{t} \beta(t,\xi)\mathbf{\Phi}_{12}^{A}(\xi,t_0)d\xi\right]\mathbf{q}_y(t_0)$$

$$+\int_{t_0}^{t} \alpha(t)u_y(\lambda)d\lambda \qquad \text{(A.1)}$$

The preceding relation for the final time t_f and using the final condition $y_m(t_f) = y_m^*(t_f)$ is written as

$$y_m^*(t_f) = y_m(t_0) + v_m(t_0)\beta(t_f,t_0) + \int_{t_0}^{t_f} \beta(t_f,\xi)g_y(\xi)d\xi$$
$$+a_{yc}(t_0)\int_{t_0}^{t_f} \beta(t_f,\xi)\Phi_{11}^A(\xi,t_0)d\xi + \left[\int_{t_0}^{t_f} \beta(t_f,\xi)\mathbf{\Phi}_{12}^A(\xi,t_0)d\xi\right]\mathbf{q}_y(t_0)$$
$$+\int_{t_0}^{t_f} \alpha(t)u_y(\lambda)d\lambda \qquad \text{(A.2)}$$

Rearrangement results in

$$\int_{t_0}^{t_f} \alpha(\lambda)u_y(\lambda)d\lambda = y_m^*(t_f) - y_m(t_0) - v_y(t_0)\beta(t_f,t_0) - \int_{t_0}^{t_f} \beta(t_f,\xi)g_y(\xi)d\xi$$
$$-a_{yc}(t_0)\int_{t_0}^{t_f}\beta(t_f,\xi)\Phi_{11}^A(\xi,t_0)d\xi - \left[\int_{t_0}^{t_f} \beta(t_f,\xi)\mathbf{\Phi}_{12}^A(\xi,t_0)d\xi\right]\mathbf{q}_y(t_0) \qquad \text{(A.3)}$$

The right-hand side of the preceding relation is constant and it is denoted by Z_{y_0}. Hence,

$$\int_{t_0}^{t_f} \alpha(t)u_y(t)dt = Z_{y_0} = const \qquad \text{(A.4)}$$

The problem is now to find u_y in order to minimize $\int_{t_0}^{t_f} |u_y(t)|\,dt$.