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ANALYTICAL DISCRETE OPTIMIZATION*

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Abstract– Although for many years scientists and engineers have been faced with the problem of optimizing discrete functions, no general analytical method has been proposed for multidimensional discrete optimization. All the methods that have already been reported in the literature are algorithmic. In one dimensional discrete optimization, no general analytical method has yet been proposed to find all the local minima and maxima of multimodal one dimensional discrete functions either. In this paper, for the first time, a general analytical method has been proposed for solving multidimensional and multimodal discrete optimization problems. Here, first we will introduce the method and present its mathematical proof, and then confirm its validity through some examples and computer simulations.

Keywords– Discrete optimization, multimodal optimization

1. INTRODUCTION

Nowadays, optimization is one of the most interesting branches of science that is useful in many applied problems in engineering, mathematics, computer science, physics, and so on. One of the most important branches of optimization is discrete optimization (or integer optimization) which deals with the methods of optimizing functions with integer variables. No general method has yet been proposed to analytically evaluate the minima and maxima of multidimensional discrete functions. All the known methods have been developed based on algorithmic approaches such as integer programming (IP) and searching, which are very complex and need a high order of computations [1-12]. In [13] an analytical method has been proposed for maximizing unimodal one dimensional discrete functions, however the author emphasized that the extension of his method to a multidimensional case would not be valid. In this paper we will introduce an analytical approach that is preferable to the known existing ones. Using this method, the local minima and maxima of the discrete functions would be evaluated by a single analytical formula, even if the function has infinite maxima and minima. This leads to an enormous reduction of the computational load, particularly in multidimensional cases, without using any suboptimal approach. The other advantage of using this analytical approach is in solving classic discrete optimization problems very easily without using any induction approach. If the discrete function has infinite minima and maxima distributed in an unconstrained domain, this approach is able to catch them analytically, where it is not possible to do so by using the IP and searching approaches because an unconstrained domain contains an infinite number of points.

In this paper, first we will introduce the methodology and present its mathematical proof, and then confirm its validity through some examples and computer simulations.

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2. ANALYTICAL APPROACH

a) One dimensional discrete optimization

Definition of local minimum and maximum for discrete functions:
We say n^* is a local minimum for the discrete function $f[n]$ if n^*, n^*-1 and n^*+1 are in the domain of $f[n]$ and we have:
$f[n^*]\le f[n^*-1]$ and $f[n^*]\le f[n^*+1]$.
We say n^* is a local maximum for the discrete function $f[n]$ if n^*, n^*-1 and n^*+1 are in the domain of $f[n]$ and we have:
$f[n^*]\ge f[n^*-1]$ and $f[n^*]\ge f[n^*+1]$.

Theorem 1:
Suppose that $f[n]$ is a discrete function defined on $D\subset Z$ and $f_r(x)$ is a continuous real function defined on $D_r\subset R$ so that $D\subset D_r$, and at each integer point such as n' we have $f_r(n')=f[n']$.
Let $x_1,x_2,...,x_m$ be all the solutions of the equation $f_r(x)-f_r(x-1)=0$. The set A consisting of the integer parts of all x_i's ($i=1, 2, \ldots, m$) plus all (x_i-1)'s when x_i is integer contains all the local minima and maxima of $f[n]$.

Proof:
Assume that n^* is a local minimum of $f[n]$. We have:

$$\begin{aligned}&\mathrm{f}[\mathrm{n}^*]\le \mathrm{f}[\mathrm{n}^*-1]\Rightarrow \mathrm{f_r}(\mathrm{n}^*)\le \mathrm{f_r}(\mathrm{n}^*-1)\\&\Rightarrow \mathrm{f_r}(\mathrm{n}^*)-\mathrm{f_r}(\mathrm{n}^*-1)\le 0\end{aligned}\tag{1.1}$$

$$\begin{aligned}&\mathrm{f}[\mathrm{n}^*]\le \mathrm{f}[\mathrm{n}^*+1]\Rightarrow \mathrm{f_r}(\mathrm{n}^*)\le \mathrm{f_r}(\mathrm{n}^*+1)\\&\Rightarrow \mathrm{f_r}(\mathrm{n}^*+1)-\mathrm{f_r}(\mathrm{n}^*)\ge 0\end{aligned}\tag{1.2}$$

Since $f_r(x)$ is a continuous function, $f_r(x-1)$ is also a continuous function and therefore $g(x)\overset{\Delta}{=}f_r(x)-f_r(x-1)$ is a continuous function. According to (1.1) and (1.2) we have $g(n^*)\le 0$ and $g(n^*+1)\ge 0$. Since $g(x)$ is a continuous function it has at least one root in the interval $[n^*,n^*+1]$. Thus, the equation $f_r(x)-f_r(x-1)=0$ has at least one solution in the interval $[n^*,n^*+1]$. If the solution(s) is (are) in the interval $[n^*,n^*+1)$ then the integer part of the solution(s) would be n^*, and if the solution is equal to n^*+1 then n^*+1 and $(n^*+1)-1=n^*$ would be in the set A. Thus, in each case the set A contains the local minimum n^*.

Assume that n^* is a local maximum of $f[n]$. We have:

$$\begin{aligned}&\mathrm{f}[\mathrm{n}^*]\ge \mathrm{f}[\mathrm{n}^*-1]\Rightarrow \mathrm{f_r}(\mathrm{n}^*)\ge \mathrm{f_r}(\mathrm{n}^*-1)\\&\Rightarrow \mathrm{f_r}(\mathrm{n}^*)-\mathrm{f_r}(\mathrm{n}^*-1)\ge 0\end{aligned}\tag{2.1}$$

$$\begin{aligned}&\mathrm{f}[\mathrm{n}^*]\ge \mathrm{f}[\mathrm{n}^*+1]\Rightarrow \mathrm{f_r}(\mathrm{n}^*)\ge \mathrm{f_r}(\mathrm{n}^*+1)\\&\Rightarrow \mathrm{f_r}(\mathrm{n}^*+1)-\mathrm{f_r}(\mathrm{n}^*)\le 0\end{aligned}\tag{2.2}$$

According to (2.1) and (2.2) we have $g(n^*)\ge 0$ and $g(n^*+1)\le 0$, thus the continuous function $g(x)\overset{\Delta}{=}f_r(x)-f_r(x-1)$ has at least one root in the interval $[n^*,n^*+1]$. Therefore, the equation $f_r(x)-f_r(x-1)=0$ has at least one solution in the interval $[n^*,n^*+1]$. If the solution(s) is (are) in the interval $[n^*,n^*+1)$ then the integer part of the solution(s) would be n^* and if the solution is equal to n^*+1, then n^*+1 and $(n^*+1)-1=n^*$ would be in the set A. Therefore in each case the set A contains the local maximum n^*.

Hence, we conclude that the set A contains all local maxima and minima of $f[n]$.

Note: In theorem 1, in the case that some of the solutions of $f_r(x)-f_r(x-1)=0$, as x_i, are integer we have:

$$\begin{aligned}&f_r(x_i)-f_r(x_i-1)=0\\&\Rightarrow f[x_i]-f[x_i-1]=0\\&\Rightarrow f[x_i]=f[x_i-1]\end{aligned}$$

Thus, if x_i is a maximum (minimum) of $f[n]$, then x_i-1 is equivalently another maximum (minimum) of $f[n]$ with the same value.

Note: Suppose $f[n]$ has global minimum or maximum. If the set A in theorem 1 consists of only one number as n_1^*, then n_1^* is the global minimum or global maximum of $f[n]$. If the set A consists of only two successive numbers as n_1^*-1 and n_1^*, for which $f[n_1^*]=f[n_1^*-1]$, then n_1^*-1 and n_1^* are the global minima or global maxima of $f[n]$.

Note: When the discrete function $f[n]$ is given by a mathematical formula we can use the same formula for $f_r(x)$ by substituting discrete variable n with real variable x whenever the resulting $f_r(x)$ is a continuous function. In this case we name $f_r(x)$ as $f(x)$. For example, if $f[n]=n^2$ we can choose $f_r(x)=f(x)=x^2$.

We use the notation:

$$\underset{f_r}{dif}(f[n])=f_r(x)-f_r(x-1)$$

But when $f_r(x)$ has the same formula as $f[n]$, we drop the index f_r in the above notation and use the notation:

$$dif(f[n])=f(x)-f(x-1)$$

1. Properties of the operator dif: When $f_r(x)=f(x)$ has the same formula as $f[n]$, we can state the following properties for the operator *dif*.

1- $dif(C)=0$ when C is a constant

Proof:

$f[n]=C$, $f(x)=C$

$\Rightarrow dif(C)=C-C=0$

2- $dif(an)=a$ when a is a constant

Proof:

$f[n]=an$, $f(x)=ax$

$\Rightarrow dif(an)=ax-a(x-1)=a$

3- Linearity:

$dif(a_1f_1[n]+a_2f_2[n])=a_1dif(f_1[n])+a_2dif(f_2[n])$

where a_1 and a_2 are constants

Proof:

$$\begin{aligned}&f[n]=a_1f_1[n]+a_2f_2[n]\\&f(x)=a_1f_1(x)+a_2f_2(x)\\&\Rightarrow dif(f[n])=a_1f_1(x)+a_2f_2(x)-(a_1f_1(x-1)+a_2f_2(x-1))\\&=a_1(f_1(x)-f_1(x-1))+a_2(f_2(x)-f_2(x-1))\\&=a_1dif(f_1[n])+a_2dif(f_2[n])\end{aligned}$$

Example 1:

Find the maximum(s) of the function $f[n]=\binom{k}{n}$, where the domain of f is $\{n \mid n\in Z,\ 0\le n\le k\}$.

Since $n!=\Gamma(n+1)$ we have:

$$f[n]=\binom{k}{n}=\frac{k!}{n!(k-n)!}=\frac{\Gamma(k+1)}{\Gamma(n+1)\Gamma(k+1-n)}$$

The function $f_r(x)=\dfrac{\Gamma(k+1)}{\Gamma(x+1)\Gamma(k+1-x)}$ is a continuous real function for $0\le x\le k$ because $\Gamma(x)$ is a continuous function for $x>0$ and we have $\Gamma(x+1)\neq 0$ and $\Gamma(k+1-x)\neq 0$ for $0\le x\le k$. At integer points such as $x=n'$ we have $f_r(n')=f[n']$. Using theorem 1 to find the maximum(s) of $f[n]$ we have:

$$\begin{aligned}
&f_r(x)-f_r(x-1)=0\\
&\Rightarrow \frac{\Gamma(k+1)}{\Gamma(x+1)\Gamma(k+1-x)}-\frac{\Gamma(k+1)}{\Gamma(x)\Gamma(k+1-x+1)}=0\\
&\Rightarrow \frac{\Gamma(k+1)\{\Gamma(x)\Gamma(k+1-x+1)-\Gamma(x+1)\Gamma(k+1-x)\}}{\Gamma(x+1)\Gamma(k+1-x)\Gamma(x)\Gamma(k+1-x+1)}=0\\
&\Rightarrow \Gamma(x)\Gamma(k+1-x+1)=\Gamma(x+1)\Gamma(k+1-x)
\end{aligned}$$

Using the property $\Gamma(x+1)=x\Gamma(x)$ we can rewrite the above relation in the form:

$$\Gamma(x)(k+1-x)\Gamma(k+1-x)=x\Gamma(x)\Gamma(k+1-x)$$

Since $\Gamma(x+1)\neq 0$ and $\Gamma(k+1-x)\neq 0$ for $0\le x\le k$ we can divide both sides by these terms and obtain:

$$\begin{aligned}
&(k+1-x)=x\\
&\Rightarrow x=\frac{k+1}{2}
\end{aligned}$$

According to theorem 1 the maximum(s) of the discrete function $f[n]$ can be found by:

$$n^*=\begin{cases}[\frac{k+1}{2}] & \text{if } \frac{k+1}{2} \text{ is not integer}\\ \frac{k+1}{2} \text{ and } \frac{k+1}{2}-1 & \text{if } \frac{k+1}{2} \text{ is integer}\end{cases}$$

Where [.] denotes the integer part function and n^* denotes the maximum(s) of $f[n]$.

$\frac{k+1}{2}$ is integer if and only if k is an odd number, thus:

$$n^*=\begin{cases}[\frac{k+1}{2}]=[\frac{k}{2}+\frac{1}{2}] & \text{for k even}\\ \frac{k+1}{2} \text{ and } \frac{k-1}{2} & \text{for k odd}\end{cases}$$

In the case of k being even, $\frac{k}{2}$ is integer, hence:

$$n^*=\begin{cases}\frac{k}{2} & \text{for k even}\\ \frac{k+1}{2} \text{ and } \frac{k-1}{2} & \text{for k odd}\end{cases}$$

which is the expected solution.

Example 2:

Find the maximum(s) of the Bernoulli distribution function $f[n]=\binom{k}{n}p^n(1-p)^{k-n}$, where k is a positive integer constant and the domain of f is $\{n \mid n \in Z,\ 0 \le n \le k\}$ and p is a real constant and $0<p<1$.

Since $n! = \Gamma(n+1)$ we have:

$$f[n]=\frac{k!}{n!(k-n)!}p^n(1-p)^{k-n}$$
$$=\frac{\Gamma(k+1)}{\Gamma(n+1)\Gamma(k+1-n)}p^n(1-p)^{k-n}$$

Similar to example 1, we can realize that the function $f_r(x)=\frac{\Gamma(k+1)}{\Gamma(x+1)\Gamma(k+1-x)}p^x(1-p)^{k-x}$ is a continuous real function for $0 \le x \le k$. At integer points such as $x=n'$ we have $f_r(n')=f[n']$. Using theorem 1 to find maximum(s) of $f[n]$ we have:

$$f_r(x)-f_r(x-1)=0$$
$$\Rightarrow \frac{\Gamma(k+1)}{\Gamma(x+1)\Gamma(k+1-x)}p^x(1-p)^{k-x}$$
$$-\frac{\Gamma(k+1)}{\Gamma(x)\Gamma(k+1-x+1)}p^{x-1}(1-p)^{k-x+1}=0$$
$$\Rightarrow \Gamma(k+1)p^x(1-p)^{k-x}\{\Gamma(x)\Gamma(k+1-x+1)$$
$$-\Gamma(x+1)\Gamma(k+1-x)p^{-1}(1-p)\}=0$$

Since $\Gamma(\mathrm{k}+1)p^x(1-p)^{k-x} \neq 0$ we have:

$$\Gamma(x)\Gamma(k+1-x+1)=\Gamma(x+1)\Gamma(k+1-x)p^{-1}(1-p)$$

Using the property $\Gamma(x+1)=x\Gamma(x)$ we can rewrite the above relation in the following form:

$$\Gamma(x)\{(k+1-x)\Gamma(k+1-x)\}$$
$$=\{x\Gamma(x)\}\Gamma(k+1-x)p^{-1}(1-p)$$

Since $\Gamma(x)\neq 0$ and $\Gamma(k+1-x)\neq 0$ for $0 \le x \le k$ we can divide both sides by these terms and obtain:

$$(k+1-x)=xp^{-1}(1-p)$$
$$\Rightarrow x\{p^{-1}(1-p)+1\}=k+1$$
$$\Rightarrow x(p^{-1})=k+1 \Rightarrow x=p(k+1)$$

According to theorem 1 the maximum(s) of the discrete function $f[n]$ can be obtained as:

$$n^*=\begin{cases}[p(k+1)] & \text{if } p(k+1) \text{ is not integer}\\ p(k+1) \text{ and } p(k+1)-1 & \text{if } p(k+1) \text{ is integer}\end{cases}$$

Where [.] denotes the integer part function and n^* denotes the maximum(s) of $f[n]$.

Example 3:

Find local maxima and minima of the function $f[n]=\sin(n/3)$.

The function $f_r(x)=\sin(x/3)$ is a continuous real function and at integer points such as $x=n'$ we have $f_r(n')=f[n']$. Using theorem 1 to find local maxima and minima of $f[n]$ we have:

$$dif(f[n])=f_r(x)-f_r(x-1)=0$$
$$\Rightarrow \sin(x/3)=\sin((x-1)/3)$$
$$\Rightarrow x=1/2+3(k-1/2)\pi \qquad ,k\in Z$$

According to theorem 1, the local maxima and minima of the discrete function $f[n]$ can be obtained from:

$$n^* = [1/2 + 3(k-1/2)\pi] \qquad , k \in Z$$

Where [.] denotes the integer part function and n^* shows the local maxima and minima of $f[n]$. Note that the term $1/2+3(k-1/2)\pi$ is always non-integer because π is irrational and the result of adding or multiplying an irrational number by a nonzero rational number is irrational, and hence, non-integer.

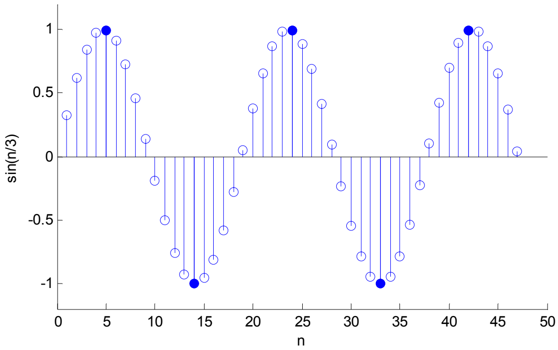


Fig. 1. Plot of sin(n/3) versus n. The filled points represent maxima and minima of sin(n/3) for k=1,2,…,5

2. Necessary and Sufficient condition for local minima and maxima: In few cases, the set A obtained from theorem 1 may contain some additional points rather than maxima and minima of $f[n]$. Each of the two following conditions are sufficient conditions for $f_r(x)$ so that the set A contains only the minima and maxima of $f[n]$.Thus, if each of the two following conditions are satisfied, theorem 1 gives a *necessary and sufficient* condition for local minima and maxima of $f[n]$.

1- For every interval $(m, m+1)$ for which m is an integer number and $f[m] \neq f[m+1]$ we have $f[m] < f_r(x) < f[m+1]$ or $f[m+1] < f_r(x) < f[m]$ (particular case: $f_r(x)$ can be strictly monotonic in each of the above intervals).

2- $f_r(x)$ is unimodal.

Proof:

The solutions of the equation $f_r(x) = f_r(x-1)$ are found by the intersection of different parts of $f_r(x)$ in the consecutive intervals such as intervals $(m, m+1)$ and $(m-1, m)$.

If the condition 1 is satisfied then the equation $f_r(x) = f_r(x-1)$ has a solution only in the intervals $(m, m+1)$, for which m is a local minimum or maximum and hence, the set A contains only the local minima and maxima of $f[n]$.

In the case that $f_r(x)$ is unimodal, suppose that the equation $f_r(x) = f_r(x-1)$ has more than one solution as $x_1, x_2, \ldots$. Without loss of generality, we assume that $x_1 < x_2$. Since $f_r(x)$ is continuous, it has at least a maximum or minimum in the interval $(x_1 - 1, x_1)$ because $f_r(x_1) = f_r(x_1 - 1)$. Similarly, $f_r(x)$ has a maximum or minimum in the interval $(x_2 - 1, x_2)$. If the intervals $(x_1 - 1, x_1)$ and $(x_2 - 1, x_2)$ do not overlap, then $f_r(x)$ will have two local maximum or minimum, which is a contradiction as $f_r(x)$ is assumed to be unimodal. If $(x_1 - 1, x_1)$ and $(x_2 - 1, x_2)$ overlap, we may have three cases:

a) $f_r(x_2) = f_r(x_2 - 1) < f_r(x_1) = f_r(x_1 - 1)$
b) $f_r(x_2) = f_r(x_2 - 1) > f_r(x_1) = f_r(x_1 - 1)$
c) $f_r(x_2) = f_r(x_2 - 1) = f_r(x_1) = f_r(x_1 - 1)$

Since $x_1 - 1 < x_2 - 1 < x_1 < x_2$ (due to overlap of intervals) and $f_r(x)$ is continuous, then in each of the above cases $f_r(x)$ would be multimodal in the interval $[x_1 - 1, x_2]$ as shown in figure 2 which is again a contradiction.

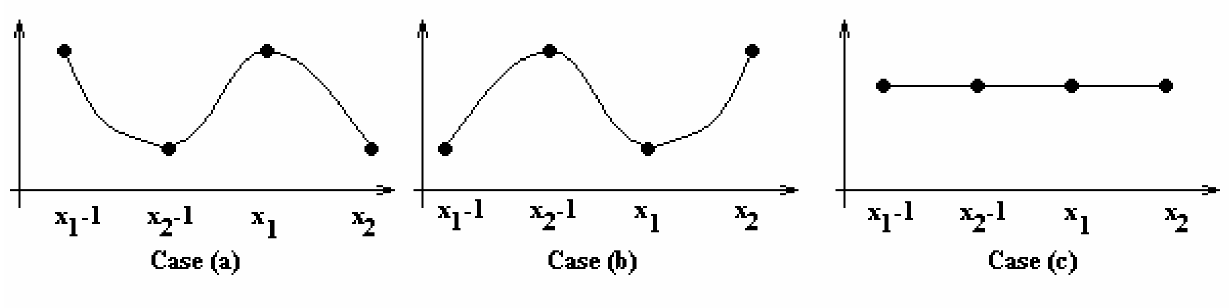


Fig. 2. Typical diagram of $f_r(x)$ in the cases a, b, and c

3. Linear Interpolation: In some cases $f[n]$ does not have an equivalent function in the real domain with the same formula. For example, $f[n]=\sum_{k=1}^{n}(-1)^k/k$. In these cases we can find $f_r(x)$ by interpolating $f[n]$ on real points. In linear interpolation we connect successive points of f[n] by a line. This method is very simple and has some useful properties.

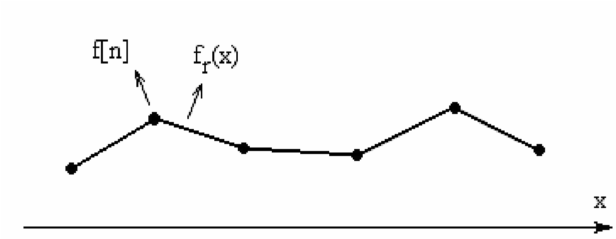


Fig. 3. Typical linear interpolation

In the interval $[n,n+1]$, $f_r(x)$ can be found by the following formula:

$$f_r(x)=(f[n+1]-f[n])(x-n)+f[n]$$

To apply theorem 1 we have:

$$f_r(x)-f_r(x-1)=(f[n+1]-2f[n]+f[n-1])(x-n)+f[n]-f[n-1] \quad ,n\le x\le n+1$$

$$f_r(x)-f_r(x-1)=0 \Leftrightarrow \begin{cases} x=-\dfrac{f[n]-f[n-1]}{f[n+1]-2f[n]+f[n-1]}+n & \text{if } f[n+1]-2f[n]+f[n-1]\ne 0 \\ f[n]-f[n-1]=0 & \text{otherwise} \end{cases}$$

Since $n\le x\le n+1$ we have:

$$f_r(x)-f_r(x-1)=0 \Leftrightarrow \begin{cases} 0\le -\dfrac{f[n]-f[n-1]}{f[n+1]-2f[n]+f[n-1]}\le 1 & \text{if } f[n+1]-2f[n]+f[n-1]\ne 0 \\ f[n]-f[n-1]=0 & \text{otherwise} \end{cases} \tag{3}$$

Lemma: n^* is a local minimum or maximum of the discrete function $f[n]$ if and only if:

$$\begin{cases} 0\le -\dfrac{f[n^*]-f[n^*-1]}{f[n+1]-2f[n]+f[n^*-1]}\le 1 & \text{if } f[n^*+1]-2f[n^*]+f[n^*-1]\ne 0 \\ f[n^*]-f[n^*-1]=0 & \text{otherwise} \end{cases} \tag{4}$$

Proof: Using (3) in theorem 1 it ends up at (4). Since in linear interpolation $f_r(x)$ is monotonic for all intervals $(m,m+1)$ for which m is an integer number, theorem 1 gives a necessary and sufficient condition. In the cases that the solution of $f_r(x)-f_r(x-1)=0$ is integer, we have $f[n^*]-f[n^*-1]=0$, which satisfies both parts of (4).

Example 4: Find local minima and maxima of $f[n]=\sum_{k=1}^{n}\frac{(-1)^k}{k}$.

Using the lemma we have:

$$f[n]-f[n-1]=\frac{(-1)^n}{n}$$

$$0\le -\frac{f[n]-f[n-1]}{f[n+1]-2f[n]+f[n-1]}\le 1 \Rightarrow 0\le -\frac{\frac{(-1)^n}{n}}{\frac{(-1)^{n+1}}{n+1}-\frac{(-1)^n}{n}}\le 1$$

$$\Rightarrow 0\le \frac{n+1}{2n+1}\le 1 \;\; and \;\; n\ne 0 \Rightarrow n>0$$

Thus, for every $n>0$, $f[n]$ has a local minimum or maximum. Since $f[n]-f[n-1]=\frac{(-1)^n}{n}$, the even n's are local maxima and the odd n's are local minima.

Example 5: Find local minima and maxima of $f[n]=(a)_n=a(a+1)...(a+n-1)$.
Using the lemma we have:

$$0\le -\frac{f[n]-f[n-1]}{f[n+1]-2f[n]+f[n-1]}\le 1$$

$$0\le -\frac{a(a+1)...(a+n-3)(a+n-2)^2}{a(a+1)...(a+n-3)(a+n-2)\{(a+n)(a+n-1)-2(a+n-1)+1\}}\le 1$$

Thus, each local minima or maxima of $f[n]$ as n^* should satisfy one of the following relations:

$$a(a+1)...(a+n^*-3)(a+n^*-2)^2=0$$

or

$$0\le -\frac{(a+n^*-2)}{(a+n^*-1)(a+n^*-2)+1}\le 1$$

The solutions will simply be found by solving the above equality and inequality.

b) Multidimensional Discrete Optimization

Definition of local minimum and maximum for discrete multidimensional functions:
We say the vector $\underline{n}^*=(n_1^*,n_2^*,...,n_k^*)$ is a local minimum of k-dimensional discrete function $f[n_1,n_2,...,n_k]$ with the domain D_f if for all $1\le i\le k$ we have:

$$\underline{n}^*=(n_1^*,n_2^*,...,n_k^*)\in D_f\;,$$

$$\underline{n}_i^{*+}=(n_1^*,n_2^*,...,\underbrace{n_i^*+1}_{i'th\ element},...,n_k^*)\in D_f\;,$$

$$\underline{n}_i^{*-}=(n_1^*,n_2^*,...,\underbrace{n_i^*-1}_{i'th\ element},...,n_k^*)\in D_f\;,$$

$$f[\underline{n}_i^{*+}]\ge f[\underline{n}^*] \;\text{ and}$$

$$f[\underline{n}_i^{*-}]\ge f[\underline{n}^*].$$

i.e. $\underline{n}^*$ in all dimensions of $f[n_1,n_2,...,n_k]$ should be a local minimum.
We say the vector $\underline{n}^*=(n_1^*,n_2^*,...,n_k^*)$ is a local maximum of k-dimensional discrete function $f[n_1,n_2,...,n_k]$ with the domain D_f if for all $1\le i\le k$ we have:

$$\underline{n}^* = (n_1^*, n_2^*, ..., n_k^*) \in D_f ,$$
$$\underline{n}_i^{*+} = (n_1^*, n_2^*, ..., \underbrace{n_i^* + 1}_{i'th\ element}, ..., n_k^*) \in D_f ,$$
$$\underline{n}_i^{*-} = (n_1^*, n_2^*, ..., \underbrace{n_i^* - 1}_{i'th\ element}, ..., n_k^*) \in D_f ,$$
$$f[\underline{n}_i^{*+}] \le f[\underline{n}^*] \text{ and}$$
$$f[\underline{n}_i^{*-}] \le f[\underline{n}^*].$$

i.e. $\underline{n}^*$ in all dimensions of $f[n_1, n_2,..., n_k]$ should be a local maximum.

Thus, to find the local minima and maxima of a discrete multidimensional function we can apply theorem 1 to each of its dimensions, if it is feasible. A simple visualization of the above discussion is illustrated in Fig. 4 where theorem 1 has been applied to each of its dimensions. In theorem 2 we elaborate on this subject.

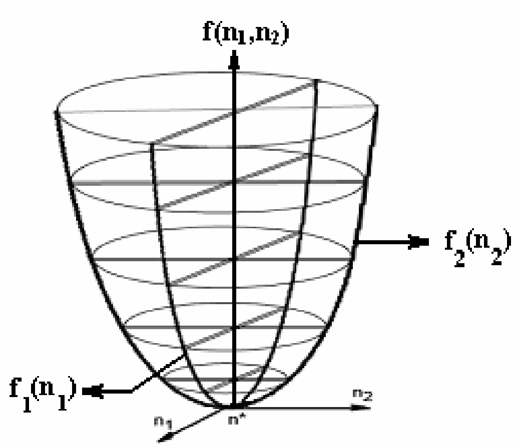


Fig. 4. n* is minimum of $f(n_1,n_2)$ and in the dimensions n_1 and n_2 , n_1^* and n_2^* are minima of $f_1(n_1)$ and $f_2(n_2)$ respectively

Theorem 2:

Suppose that $f[n_1, n_2,..., n_k]$ is a k-dimensional discrete function defined on $D \subset Z^k$ and $f_r(x_1, x_2,..., x_k)$ is a continuous k-dimensional real function defined on $D_r \subset R^k$ so that $D \subset D_r$, and at each integer point such as $\underline{n}' = (n'_1, n'_2,..., n'_k)$ we have $f_r(\underline{n}') = f[\underline{n}']$. Consider the following k equations:

$$f_r(x_1,..., x_i,..., x_k) - f_r(x_1,..., x_i - 1,..., x_k) = 0 \quad ; 1 \le i \le k \tag{5}$$

If in the i th equation we can explicitly express x_i in terms of other x_j 's, $(j \ne i)$ for all $1 \le i \le k$, i.e. if we can write the above equations in the following form (in which $g_i(.)$ is a function of all x_j 's except x_i):

$$\begin{aligned} x_1 &= g_1(x_2,..., x_k) \\ x_2 &= g_2(x_1,..., x_k) \\ &\vdots \\ x_k &= g_k(x_1,..., x_{k-1}) \end{aligned} \tag{6}$$

Then using the functions $g_i(.)$ in (6) and changing variables $x_1, x_2,..., x_k$ to integer variables $n_1, n_2,..., n_k$, we create the following system of k equations:

$$\begin{aligned} n_1 &= [g_1(n_2,...,n_k)] \\ n_2 &= [g_1(n_1,...,n_k)] \\ &\vdots \\ n_k &= [g_k(n_1,...,n_{k-1})] \end{aligned} \tag{7}$$

Where $[g_i(.)]$ denotes the integer part of $g_i(.)$.

Let $\underline{N}_1, \underline{N}_2, ..., \underline{N}_m$ be all the solutions of the above system of k equations. Where:

$$\underline{N}_j = (n_1, n_2, ..., n_k)_j = (n_{1j}, n_{2j}, ..., n_{kj}) \quad, j = 1,\ 2,\ ...\ ,\ m$$

(note that n_{ij}'s are integer since in the equation $n_i = [g_i(n_1,...,n_k)]$, n_i should be integer).

If the set A_k consists of all $\underline{N}_j$'s plus all the vectors $\underline{N}_j^i = (n_{1j},..., \underbrace{n_{ij}-1}_{i'th\ element}, ..., n_{kj})$, $(1 \le j \le m,\ 1 \le i \le k)$ for which $g_i(\underline{N}_j)$ is an integer number (where $g_i(\underline{N}_j) = g_i(n_{1j}, n_{2j}, ..., n_{lj}, ..., n_{kj})$ and $l \ne i$), then the set A_k contains all local maxima and minima of $f[n_1, n_2, ..., n_k]$.

Proof:

Let the vector $(n_1^*, n_2^*, ..., n_k^*)$ be a local minimum of the function $f[n_1, n_2, ..., n_k]$. According to the definition of local minimum for discrete multidimensional functions for all $1 \le i \le k$ we have:

$$f[(n_1^*, n_2^*, ..., n_i^*+1, ..., n_k^*)] \ge f[n_1^*, n_2^*, ..., n_k^*] \text{ and}$$
$$f[(n_1^*, n_2^*, ..., n_i^*-1, ..., n_k^*)] \ge f[n_1^*, n_2^*, ..., n_k^*].$$

This means that n_i^* is a local minimum for the discrete one-dimensional function:

$$f[n_i] = f[n_1^*, ..., n_i, ..., n_k^*] \quad (1 \le i \le k)$$

In which only n_i is variable. According to theorem 1 the equation:

$$f_r(n_1^*, ..., x_i, ..., n_k^*) - f_r(n_1^*, ..., x_i - 1, ..., n_k^*) = 0 \quad, 1 \le i \le k \tag{8}$$

has at least one solution for x_i in the interval $[n_i^*, n_i^*+1]$. If we can write the Eqs. (5) in the form of Eqs. (6), then we can also write the Eqs. (8) in the form:

$$\begin{aligned} x_1 &= g_1(n_2^*, ..., n_k^*) \\ x_2 &= g_2(n_1^*, ..., n_k^*) \\ &\vdots \\ x_k &= g_k(n_1^*, ..., n_{k-1}^*) \end{aligned}$$

Thus, $x_i = g_i(n_1^*, ..., n_k^*)$ would be the unique solution of the equation $f_r(n_1^*, ..., x_i, ..., n_k^*) - f_r(n_1^*, ..., x_i - 1, ..., n_k^*) = 0$. Thus, according to theorem 1, $x_i = g_i(n_1^*, ..., n_k^*)$ is in the interval $[n_i^*, n_i^*+1]$. If all the $g_i(n_1^*, ..., n_k^*)$'s for $i = 1, 2, \dots, k$ are in the interval $[n_i^*, n_i^*+1)$, the integer parts of $g_i(n_1^*, ..., n_k^*)$'s would be equal to n_i^*. In this case, the solutions of the system of k Eqs. (7) contain $(n_1^*, n_2^*, ..., n_k^*)$. If some of the $g_i(n_1^*, ..., n_k^*)$'s are equal to n_i^*+1 (which is integer), solutions of the system of k Eqs. (7) contain $(n_1^*, ..., n_i^*+1, ..., n_k^*)$ for $i = 1, 2, \dots, k$ and both $(n_1^*, ..., n_i^*+1, ..., n_k^*)$ and $(n_1^*, ..., n_i^*, ..., n_k^*)$ would be in the set A_k.

In each case the set A_k contains $(n_1^*, ..., n_i^*, ..., n_k^*)$. Thus the set A_k always contains all local minima of $f[n_1, n_2, ..., n_k]$. Similarly, using theorem 1 we can show that the set A_k always contains all local maxima of $f[n_1, n_2, ..., n_k]$.

Note: When the discrete function $f[n_1,n_2,...,n_k]$ is given by a mathematical formula we can use the same formula for $f_r(x_1,x_2,...,x_k)$ by substituting the discrete variables $n_1,n_2,...,n_k$ with the real variables $x_1,x_2,...,x_k$ whenever the resulting $f_r(x_1,x_2,...,x_k)$ is a continuous function. For example, if $f[n_1,n_2]=(n_1+n_2)^2$ we can choose $f_r(x_1,x_2)=(x_1+x_2)^2$. We use the notation:

$$\underset{i,f_r}{dif}(f[n_1,n_2,...,n_k]) = f_r(x_1,...,x_i,...,x_k) - f_r(x_1,...,x_i-1,...,x_k)$$

When $f_r(x_1,x_2,...,x_k)$ has the same formula as $f[n_1,n_2,...,n_k]$, we drop the index f_r in the above notation and use the notation:

$$\underset{i}{dif}(f[n_1,n_2,...,n_k]) = f(x_1,...,x_i,...,x_k) - f(x_1,...,x_i-1,...,x_k)$$

Example 6:

In an industry there are some workers in k different units. The wage of each worker in the ith unit is \$ C_i per day and the per day income of the ith unit is given by $A_i(1-2^{-n_i})$, where n_i is the number of workers and A_i shows the maximum accessible income for the ith unit. We would like to obtain the optimal number of workers in each unit in order to maximize the total profit of the industry. We assume that $A_i \ge C_i$ for $i=1,2,...,k$. This means that the maximum income of each unit must be at least equal to the wage of one worker in that unit; a necessary condition to keep the unit profitable, otherwise the unit will not be operative even with one worker.

The total profit of the industry can be computed by:

$$f[n_1,n_2,...,n_k] = \sum_{i=1}^{k} A_i(1-2^{-n_i}) - \sum_{i=1}^{k} C_i n_i$$

We have:

$$f(x_1,x_2,...,x_k) = \sum_{i=1}^{k} A_i(1-2^{-x_i}) - \sum_{i=1}^{k} C_i x_i$$

Applying theorem 2 we have:

$$\underset{i}{dif}(f[n_1,n_2,...,n_k]) = 0 \qquad i=1,2,...,k$$

Using properties 1 to 3 of operator dif we have:

$$\underset{i}{dif}(f[n_1,n_2,...,n_k]) = -A_i 2^{-x_i} + A_i 2^{-x_i+1} - C_i = 0$$

$$\Rightarrow A_i 2^{-x_i}(-1+2) = C_i \Rightarrow 2^{-x_i} = \frac{C_i}{A_i}$$

$$\Rightarrow x_i = -\log_2\left(\frac{C_i}{A_i}\right) = \log_2\left(\frac{A_i}{C_i}\right)$$

According to theorem 2 the optimal value for n_i can be obtained as:

$$n_i^* = \begin{cases} [\log_2(\frac{A_i}{C_i})] & \text{if } \log_2(\frac{A_i}{C_i}) \text{ is noninteger} \\ \log_2(\frac{A_i}{C_i}) \text{ and} & \\ \log_2(\frac{A_i}{C_i})-1 & \text{if } \log_2(\frac{A_i}{C_i}) \text{ is integer} \end{cases} \qquad (9)$$

$$i=1,2,...,k$$

Where [.] denotes the integer part function. Since $A_i \ge C_i$ we have:

$$\frac{A_i}{C_i} \geq 1 \Rightarrow \log_2 \frac{A_i}{C_i} \geq 0$$

$$\Rightarrow [\log_2 \frac{A_i}{C_i}] \geq 0$$

Thus, the optimal number of workers is always nonnegative and (9) is always valid. Note that in the case of having $\log_2 \frac{A_i}{C_i} = 0$ and $\log_2 \frac{A_i}{C_i} - 1 = -1$, only $\log_2 \frac{A_i}{C_i} = 0$ would be a valid optimal solution for n_i.

The computer simulation confirms the validity of this result. Figure 5 shows the profit of the industry versus n_1 and n_2 in the case that k=2, A_1=1000, A_2=1200, C_1=10 and C_2=25.
According to this simulation the optimal values for n_1 and n_2 are 6 and 5, respectively. This result agrees with the result obtained in (9).

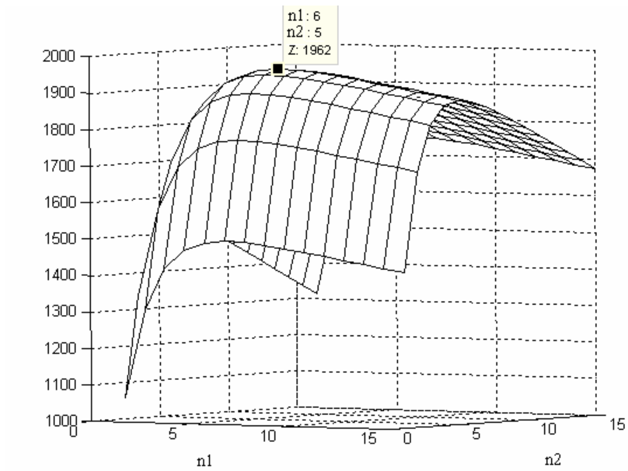


Fig. 5. Profit of the industry versus n_1 and n_2 with k=2, A_1=1000, A_2=1200, C_1=10 and C_2=25. It is obvious that the optimal values for n_1 and n_2 are 6 and 5, respectively.

Example 7:
Find local maxima of the joint probability mass function of the discrete normal random variables, $n_1, n_2, ..., n_k$, given by:

$$P[n_1, n_2, ..., n_k] = ae^{-(N-\mu)^T S(N-\mu)}$$

where $N = [n_1, n_2, ..., n_k]^T$ is the vector of integer random variables, $\mu = [\mu_1, \mu_2, ..., \mu_k]^T$ is the mean vector of N, a is a scalar positive constant and S is a $k \times k$ constant matrix (inverse of covariance matrix of N). We can minimize $(N-\mu)^T S(N-\mu)$ instead of maximizing $ae^{-(N-\mu)^T S(N-\mu)}$. Then we have:

$$(N-\mu)^T S(N-\mu) =$$
$$[\sum_{i=1}^{k} S_{i1}(n_i - \mu_i) \ \sum_{i=1}^{k} S_{i2}(n_i - \mu_i) \ ... \ \sum_{i=1}^{k} S_{ik}(n_i - \mu_i)](N-\mu)$$
$$= \sum_{j=1}^{k}(n_j - \mu_j)\sum_{i=1}^{k} S_{ij}(n_i - \mu_i)$$
$$= \sum_{j=1}^{k}\sum_{i=1}^{k} S_{ij}(n_i - \mu_i)(n_j - \mu_j)$$

Let $x_1, x_2, ..., x_k$ be real variables and $X = [x_1, x_2, ..., x_k]^T$.

$$(X-\mu)^T S(X-\mu) = \sum_{j=1}^{k}\sum_{i=1}^{k} S_{ij}(x_i - \mu_i)(x_j - \mu_j)$$

$$\underset{l}{dif}((N-\mu)^T S(N-\mu)) =$$
$$(X-\mu)^T S(X-\mu) - (\overset{l}{X}-\mu)^T S(\overset{l}{X}-\mu)$$

where $\overset{l}{X} = [x_1, ..., x_l - 1, ..., x_k]^T$.

Only the terms that contain x_l remain and the other terms vanish:

$$\underset{l}{dif}((N-\mu)^T S(N-\mu)) =$$
$$\sum_{\substack{j=1\\ j\neq l}}^{k} S_{lj}(x_l-\mu_l)(x_j-\mu_j) - \sum_{\substack{j=1\\ j\neq l}}^{k} S_{lj}(x_l-1-\mu_l)(x_j-\mu_j) +$$
$$\sum_{\substack{i=1\\ i\neq l}}^{k} S_{il}(x_i-\mu_i)(x_l-\mu_l) - \sum_{\substack{i=1\\ i\neq l}}^{k} S_{il}(x_i-\mu_i)(x_l-1-\mu_l)$$
$$+S_{ll}(x_l-\mu_l)^2 - S_{ll}(x_l-1-\mu_l)^2$$
$$= \sum_{\substack{j=1\\ j\neq l}}^{k} S_{lj}(x_j-\mu_j) + \sum_{\substack{i=1\\ i\neq l}}^{k} S_{il}(x_i-\mu_i) + 2S_{ll}(x_l-\mu_l) - S_{ll}$$

Combining the two summations in one yields:

$$\underset{l}{dif}((N-\mu)^T S(N-\mu)) = \sum_{\substack{j=1\\ j\neq l}}^{k} (S_{lj}+S_{jl})(x_j-\mu_j) + 2S_{ll}(x_l-\mu_l) - S_{ll}$$
$$\underset{l}{dif}((N-\mu)^T S(N-\mu)) = 0 \qquad (10)$$
$$\Rightarrow x_l = \mu_l + \frac{1}{2} - \sum_{\substack{j=1\\ j\neq l}}^{k} (\frac{S_{lj}+S_{jl}}{2S_{ll}})(x_j-\mu_j)$$

According to theorem 2 and considering only one of the solutions for the case of getting successive equal minima, the minima of $(N-\mu)^T S(N-\mu)$ satisfy the following system of *k* equations:

$$n_l^* = [\mu_l + \frac{1}{2} - \sum_{\substack{j=1\\ j\neq l}}^{k} (\frac{S_{lj}+S_{jl}}{2S_{ll}})(n_j^*-\mu_j)] \qquad (11)$$
$$l = 1, ..., k$$

Where (11) is obtained by changing real variables in (10) to discrete ones and taking integer part. n_l^* denotes optimum value for n_l. If n_l is integer, then $n_l^* - 1$ may also be a solution for n_l.

If $S = \frac{1}{\sigma^2} I$ we have:

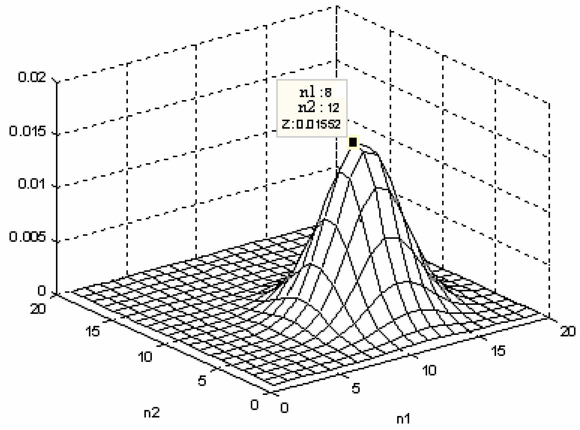
$$n_l^* = [\mu_l + \frac{1}{2}] \qquad , l = 1, ..., k \qquad (12)$$

Where in (11) and (12), [.] denotes the integer part function.

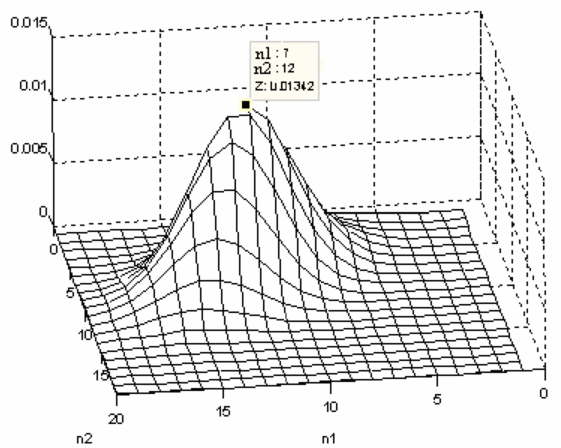
For k=2 and $\mu = [7.6 \quad 12.3]^T$, if $S = 0.1I$, according to (12), the maximum of $ae^{-(N-\mu)^T S(N-\mu)}$ occurs at $n_1^* = 8, n_2^* = 12$. If $S = \begin{bmatrix} 0.1 & -0.05 \\ -0.05 & 0.1 \end{bmatrix}$, substituting the parameters in (11) we have:

$$n_1^* = [1.95 + \frac{1}{2} n_2^*]$$
$$n_2^* = [9 + \frac{1}{2} n_1^*]$$

Only $n_1^* = 7, n_2^* = 12$ satisfy the above equations, thus it is the maximum of $ae^{-(N-\mu)^T S (N-\mu)}$. Although in these two examples the mean vectors are the same, the maxima are different. Computer simulations confirm these results. Figure 6 shows $ae^{-(N-\mu)^T S (N-\mu)}$ versus n_1 and n_2 for the above cases.



(a)



(b)

Fig. 6. a) $ae^{-(N-\mu)^T S (N-\mu)}$ versus n_1 and n_2 for k=2, $\mu = [7.6 \quad 12.3]^T$, and $S = 0.1I$, b) $ae^{-(N-\mu)^T S (N-\mu)}$ versus n_1 and n_2 for k=2, $\mu = [7.6 \quad 12.3]^T$, and $S = \begin{bmatrix} 0.1 & -0.05 \\ -0.05 & 0.1 \end{bmatrix}$

3. CONCLUSION

A novel analytical approach has been presented for the first time to solve multidimensional and multimodal discrete optimization problems. This approach has complete preference over the existing methods. In this approach the local minima and maxima of discrete functions can be evaluated by analytical formulas that enormously reduce the complexity of computations. All the existing methods have been developed based on algorithmic approaches such as integer programming (IP) and exhaustive search method that are very complex, need a high order of computations, and usually lead to suboptimal algorithms.

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