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"Research Note"

GREEN AND AMPT INFILTRATION EQUATION: COMPARISON OF TWO ANALYTIC DIRECT METHODS*

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Abstract– The explicit solutions of the Green and Ampt infiltration equation, by developing a modified Adomian Decomposition Method (MADM) and Variational Iteration Method (VIM) solution with an exact solution, are compared. These solutions were compared with the exact implicit solution and a good trend between our approaches and the exact solution was found. The best result is gained by only the first three terms in the series solutions of MADM. As the number of terms increases, the error in the lower bound of the computational domain grows and the upper bound converges after the first three terms of MADM. On the other hand, variational iteration converges after the first two terms. Both methods addressing this phenomenon seem quite promising, and by considering only a few terms the results are accurate, making both methods fast and efficient.

Keywords– Green-Ampt infiltration equation, modified adomian decomposition method variational iteration method

1. INTRODUCTION

There are two schools of thought within the soil-water infiltration issue. One is known as the Hortonian mechanism and the other as the Dunne Mechanism Freeze [1]. The Green and Ampt [2] equation, Eq. (1), which belongs to the former class, is one of the most popular models and practical relations to explain infiltration rate of time evolution in cumulative infiltration depth and rate. Its implementation is relatively easy; it is physically based and it has been used in the following applications: transient rainfall conditions [3], time-varying depth of ponding [4], and soils with altering hydraulic conductivity [5-6]. With regard to implicit behavior and nonlinearity of the equation, to use the equation, one has to employ an iterative procedure to calculate the time, *t*, for different values of the cumulative infiltration, *F*, and then the infiltration rate, *f*, can be derived from *F*. So, the cumulative infiltration versus infiltration rate curve may be constructed with regard to a specific soil and rainfall intensity. The Green-Ampt equation explains the infiltration mechanism by the following equation:

$$f(t) = K\left(\frac{|\psi_f|(n-\theta_i)}{F(t)} + 1\right) \qquad F(t_p) = b \tag{1}$$

where *f(t)* is infiltration rate, *F(t)* is cumulative infiltration depth (*f(t)=dF(t)/t*), *K* is saturated soil conductivity, ψ_f is pressure head at the wetting front, θ_i is initial soil volumetric water content, *n* is the soil porosity, and *b* is the cumulative infiltration in the ponding time. Equation (1) has an implicit closed form solution as follows: [1]

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$$K(t-t_p)=F(t)-F(t_p)+\left|\psi_f\right|(n-\theta_i)\ln\left(\frac{F(t_p)+\left|\psi_f\right|(n-\theta_i)}{F(t)+\left|\psi_f\right|(n-\theta_i)}\right) \tag{2}$$

where t_p is time to pond, and $F(t_p)$ is corresponding cumulative infiltration head. The Variational Iteration Method (VIM) and the Adomian Decomposition Method (ADM), as fast convergence approaches to achieve the analytic solution of nonlinear PDE's or ODE's, have been applied to solve this non-linear differential equation. In this regard, Serranos' approach [7-8] is based on the simplification of the implicit non-linear exact solution of the Green-Ampt equation by using ADM. In the current article, two approaches, MADM and VIM, are compared with the exact solution to solve the Green-Ampt equation.

2. ANALYSIS

There are many techniques such as perturbation methods, the Cole-Hope transformation, the Variational Iteration Method, etc. available to solve the nonlinear partial differential equations or make them easier to solve [8-10]. The ADM was first proposed by G. Adomian in the 1980's [11].

He noted that the ADM can be employed in a direct manner without any need to transform formula or bilinear forms. By using the recursive integration formula, in these methods, the approximation will be closer to the exact solution. Considering the nonlinear operator

$$O_u = g \tag{3}$$

where O_u represents a general form of the nonlinear ordinary operator including parts L linear, N nonlinear, and R as the remainder of the linear and nonlinear terms, g is a known function and u is the exact solution of Eq. (3). Respectively equal to the linear term, $u(\beta)=\alpha$ is considered as an initial condition,

$$\begin{aligned} &Lu+R_y+N_y=g \\ &L^{-1}(L_u)=-L^{-1}(R_y)-L^{-1}(N_y)+L^{-1}(g) \end{aligned} \qquad u(\beta)=\alpha \tag{4}$$

In the ADM, the N_y can be simplified by series' terms ($\phi_n=\sum_{i=0}^{n}u_i, u=\sum_{i=0}^{\infty}u_i$), where

$$\begin{aligned} &u_0=u(\beta)+L^{-1}(g) \\ &u_i=-L^{-1}(R_{u_{i-1}})-L^{-1}(A_{n-1}) \\ &N_y=\sum_{n=0}^{\infty}A_n(u_i) \qquad\qquad i=1,2\cdots n \\ &A_n=\frac{1}{(n)!}\left(\frac{d^n}{d\lambda^n}\right)N_y(u(\lambda))\Big|_{\lambda=0} \end{aligned} \tag{5}$$

N_y, the nonlinear function in Eq. (4), can be replaced by A_n (for more details of A_n see the Adomian book) [9]. There are some modifications to improve this approximated solution [12-16].

The next used novel approach, VIM, is proposed by He [17]. This method, as the previous method, presents a series solution to describe the equation behavior, both of which are highly dependent on the operation function of the differential equation. Applying the variational calculus and recursive integration in order to reduce errors in the computational range, this method improves its approximated solution. In other words, this method improves the solution by using the evolutionary integration algorithm. To show the basic concept of the VIM, the following partial differential equation is considered:

$$L_y + N_y = g \tag{6}$$

where L_y is a linear term, N_y is equal to the nonlinear term, and g is an inhomogeneous term. So, the correcting functional is constructed as shown below [17-18],

$$u_{n+1} = u_n + \int_0^x \lambda\left(L_{u_n}(\zeta) + N_{\tilde{u}_n}(\zeta) - g(\zeta)\right)d\zeta \tag{7}$$

where u_0, λ are respectively an initial condition and lagrangian multiplier, n addresses the nth approximation term, which is assumed as a restriction, and x is an independent variable, which in Eq. (1) is defined as time [17]. Here is an example to show the ability and accuracy of these methods. In this example, the rainfall intensity is 1.2 cm/hr and the soil specifications are cited from Chow [19]. In order to use ADM and VIM, some simplifications are applied to Eq. (1)

$$f(t) = \frac{a}{F(t)} + K \qquad (a = K\left|\psi_f\right|\left(n - \theta_i\right)) \tag{8}$$

In Table 1, three parametric terms of ADM and VIM are presented. Figures 1-3 represent the first three ADM approximations of the equation (ϕ_1, ϕ_2, ϕ_3). It may be considered that by increasing the order of ADM approximation, the absolute error near the initial value increases, and if the time of infiltration approaches large values, excellent convergence appears. To achieve a better result, one should reduce the oscillation and detergency by applying the [2/2] order of Pade approximation to ϕ_2 (Modified ADM). The result is shown in Figs. 4 and 5; errors reduce in a manner in which the Pade approximation is completely fitted to ϕ_1. After applying VIM on the simplified Eq. (1), it is converted to series solution with different convergence as shown (Table 1). In order to satisfy the stationary condition of Eq. (8), λ_n will produce a complex exponential function in terms of F_n, t, and t_p, so a high order integral can not be simply computed iteratively, therefore we assume λ_n as -1 [17,20-21], but the solution by this assumption converges less rapidly than the exact form. The behavior of solutions for the first two and three terms of VIM is shown in Figs. 6 and 7. This method converges well to the exact solution (except in the neighborhood of initial values).

Table 1. First three parametric terms of ADM and VIM solutions

ADM	VIM
$F_0(t) = c + K(t)$	$F_0 = b$
$F_1(t) = \dfrac{a\ln(c+Kt)}{2K}$	$F_1(t) = b + (\dfrac{a}{b} + K)t$
$F_2(t) = \dfrac{a^2(1+\ln(c+Kt))}{2k(c+Kt)}$	$F_2(t) = b + Kt + ab\dfrac{\ln(b^2+(a+bK)t)}{(a+bK)}$
⋮	⋮
F(t) is the initial condition($F(t_p) = b$) and ($c = b - Kt_p$)	

3. SUMMARY AND CONCLUSIONS

An explicit solution to the Green and Ampt infiltration equation was derived by developing ADM and its modified version versus VIM. VIM and ADM both are other types of series' solutions used to solve linear or nonlinear PDEs accurately. A few first terms of series expressions of the mentioned Hortonian equation for infiltration rate were proposed. The solutions of both methods were compared with the exact solution. It was found that the ADM has the least error when there are only two terms. The inclusion of additional terms in the ADM series increases the error, especially in the neighborhood of initial values. However, using Pade approximation [2/2] reduces the error in a manner which modified ϕ_2 matches ϕ_1. The second method, VIM, also produces satisfactory results with different approximated series' terms. Adding more terms in this method, in contrast with ADM, decreases error and converges rapidly. Both methods address this phenomenon very well and, considering only a few terms, result in an accurate solution, making both methods fast and efficient. The accuracy of the first three terms of the MADM and VIM series solution seem adequate for most practical calculations, versus its implicit exact solution.

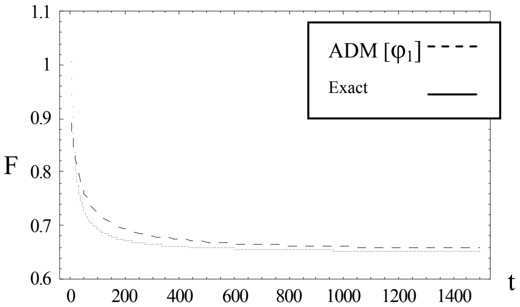

Fig. 1 Comparison of ADM [φ_1] and exact solution

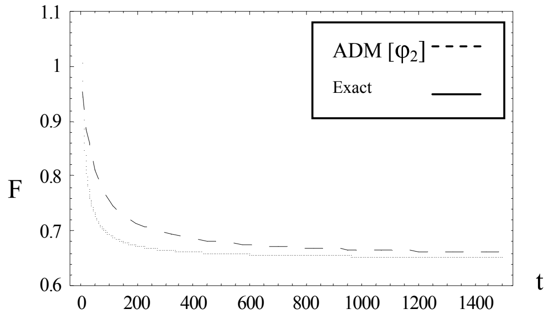

Fig. 2. Comparison of ADM [φ_2] and exact solution

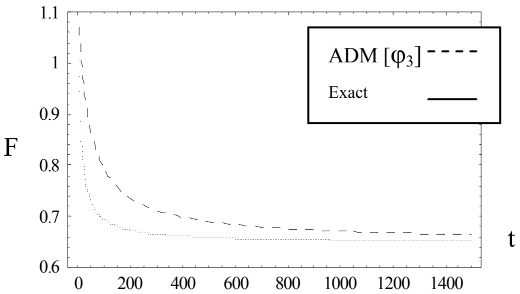

Fig. 3. Comparison of ADM [φ_3] and exact solution

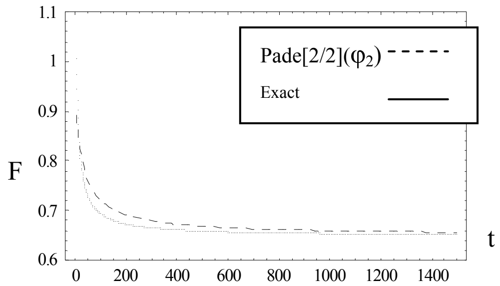

Fig. 4. Comparison of Pade [2/2] (φ_2) and exact solution

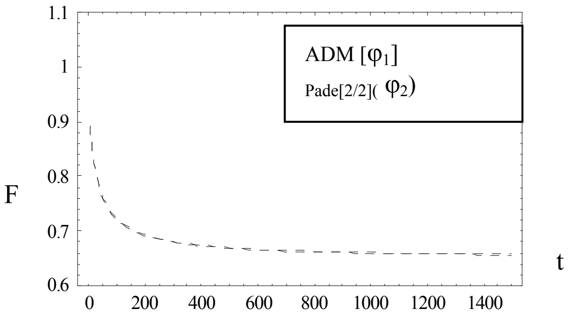

Fig. 5. Comparison of Pade [2/2] (φ_2) and ADM [φ_1]

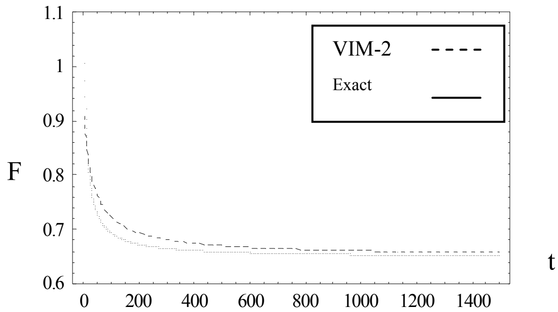

Fig. 6. Comparison of VIM-2 and exact solution

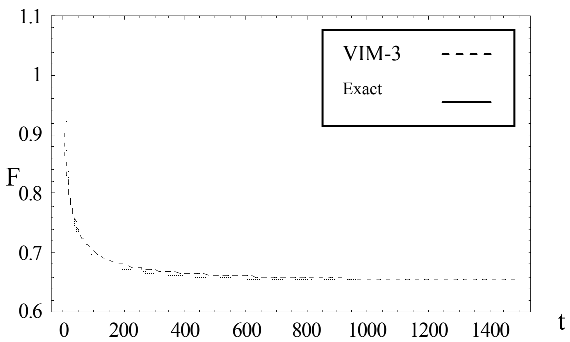

Fig. 7. Comparison of VIM-3 and exact solution

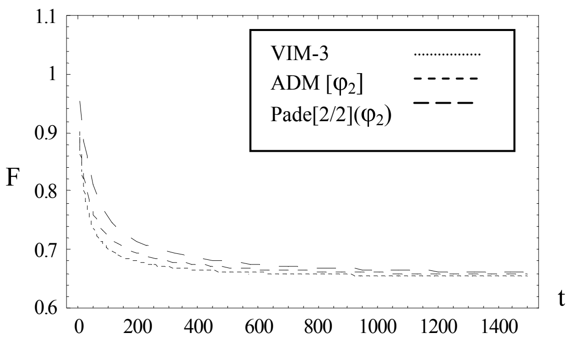

Fig. 8. Comparison of VIM-3, Pade [2/2] (φ_2) and ADM [φ_2]

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