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Bending and buckling of functionally graded Poisson's ratio nanoscale beam based on nonlocal theory

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Abstract

Functionally graded Poisson's ratio structures have been developed for critical protection. In this paper, the static bending and buckling of FGPR nanoscale beam are studied based on the nonlocal Timoshenko beam model, in which both Young's modulus and Poisson's ratio are assumed to vary continuously in the thickness direction. By utilizing total potential energy principle, equilibrium equations are derived. In the numerical results, beam models with different material properties are introduced, and the effects of the nonlocal parameter, aspect ratio and the Poisson's ratio on the deflection and buckling are discussed.

Keywords: Functionally graded; Poisson's ratio; nanoscale beam; nonlocal theory

1. Introduction

Functionally graded materials (FGM), in which the property and microstructure vary from one kind of material to the other as function of position along thickness of structure to achieve a required function, have been widely applied in many different engineering fields. Functionally graded Poisson's ratio (FGPR) structures are one novel material. The concept of FGPR material/structure is derived by a combination of FGM design methodology and optimal cellular structure configuration, which exhibited improved impact protection.

The static and dynamic characteristics of FGM structures have been studied by some researchers. The mechanical behavior of structures with FGMs has been explored with several theories and models. Both the classical and non-classical continuum theory was used to predict the behavior of micro scaled mechanical structures such as micro sensors or actuator, in which several theories are adopted as the higher order shear deformation theory, the strain gradient theory, the modified couple stress theory, based on local and nonlocal elasticity theory.

Kadoli et al. introduced the higher order shear deformation theory to study the FGM beams under ambient temperature, and the static equilibrium equation in finite element form is presented by using the principle of stationary potential energy (Kadoli et al. 2008). The results suggested that the deflections, stresses and the location of the neutral surface are higher depending on power law index. Xia et al. investigated the static bending, postbuckling and free vibration of nonlinear microbeams based on non-classical continuum mechanics by introducing material length scale parameters. The results showed that the size effects and nonlinearity are important for the design of microscale devices (Xia et al. 2010). Kahrobaiyan et al. developed a size-dependent FG beam model based on the strain gradient theory, which is capable of capturing the size-effect in micro-scaled structures. It found that the behavior of microbeams is a function of the ratios of the thickness and length of the microbeam to its length scale parameters (Kahrobaiyan et al. 2012). Salamat-talab et al. studied third-order shear deformation FG micro beam by using modified couple stress theory and the governing equations are derived by applying Hamilton's principle (Salamat-talab et al. 2012). Asghari et al. analytically investigated the size-dependent static and vibration behavior of FGMs micro-beams based on the modified couple stress theory in the elastic range. The results suggested that the use of non-classic theories seems to be essential for the analysis of micro-beams (Asghari et al. 2010). Nateghi et al. presented the buckling analysis of FGM microbeam based on modified couple stress theory, in which three different beam theories are considered to study the effect of shear deformations. The numerical results showed that size dependency of FGM microbeams differs from isotropic homogeneous micro beams as it is a function of power index of material

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distribution (Nateghi et al. 2012). Zhang et al. developed a novel size-dependent FGM curved microbeam model based on the strain gradient elasticity theory and *n* shear deformation theory. The results indicated that size effect leads to an increase in microbeam stiffness (Zhang et al. 2013). Li et al. presented a size-dependent model for bilayered microbeams based on the strain gradient elasticity theory, and found that the deflection and total axial stress of beam and locations of zero-strain and zero-stress axes exhibit obvious size effect (Li et al. 2014). Tajalli et al. studied the FGMs Timoshenko beams based on the strain gradient theory, in which a nonclassical continuum theory able to capture the size-effect in micro-scaled structures (Tajalli et al. 2013). Nie et al. investigated the plane stress problem of an orthotropic FG beam by using the displacement function approach. The influences of different graded models on the stress and displacement fields are illustrated (Nie et al. 2013).

Simsek and Yurtcu studied the static bending, buckling and vibration properties of FG microbeams by using the local, nonlocal Timoshenko and Euler-Bernoulli beam models, higher order beam theory based on the modified couple stress theory, in which the material properties of FGM are estimated to vary in the thickness direction through the Mori-Tanaka homogenization technique. The influences of the volume fraction index, material properties, length scale parameter, the aspect ratio, nonlocal parameter and the Poisson's ratio on static, buckling and vibration responses of the FG microbeam (Simsek and Yurtcu, 2013). Emam presented a unified model for the nonlocal response of nanobeams in buckling and postbuckling states, in which the equations of equilibrium are obtained by using the principle of virtual work. It presented the variation of the critical buckling load and the amplitude of buckling with the nonlocal parameter and the length-to-height ratio for simply supported and clamped-clamped nanobeams (Emam, 2013). Eltaher et al. studied the size dependent static buckling behavior of FGM nanobeams based on the nonlocal continuum model, which is described by the differential constitutive model of Eringen. The results suggested that in the material distribution profile, nonlocal effects are crucial in the behavior of the nanobeams (Eltaher et al. 2013). Aydogdu studied the bending, buckling and free vibration of nanobeams by using a generalized nonlocal beam theory and effects of nonlocality and length of beam are discussed in detail (Aydogdu, 2009).

In recent years, the concept of a functionally graded Poisson's ratio (FGPR) material has been developed for a critical protection which can concentrate material into areas to improve impact mitigation and crew protection. The Poisson's ratio is defined as the ratio of the transverse contraction strain to the longitudinal extension strain in a simple tension condition. For the isotropic materials, the Poisson's ratio is bounded by two theoretical limits, greater than -1 and less than or equal to 0.5. By introducing the graded Poisson's ratio in materials, many applications have been designed in various fields of strain sensors, shock and sound absorbers, and smart textiles.

This paper discusses static bending and buckling of FGPR nanobeam, in which the nonlocal Timoshenko beam model is utilized to capture the nonlocal effects and the Poisson's ratio on the mechanical behavior of the nanoscale beam. The minimum total potential energy principle is used to derive the equilibrium equations, which are solved analytically for nanobeam subjected to a point load. In the numerical results, the effects of the nonlocal parameter, aspect ratio and the Poisson's ratio on the deflection and buckling are discussed.

2. Nonlocal nanoscale beam

In Eringen's nonlocal elasticity theory the stress state at a reference point in the body is regarded to be dependent not only on the strain state at this point but also on the strain state at all points of the body, which is in accordance with atomic theory of lattice dynamics and experimental observations on phonon dispersion.

$$\begin{aligned} &\sigma_{kj,j}=0 \\ &\sigma_{kj}(x)=\int_V \alpha(|x-x'|,\xi)C_{kjil}\varepsilon_{il}(x')\mathrm{d}V(x'), \qquad \forall x\in V \qquad (1) \\ &\varepsilon_{kj}=\frac{1}{2}(u_{k,j}+u_{j,k}) \end{aligned}$$

σ_{kj} and ε_{kj} are the stress and strain tensors, C_{kjil} is the elastic modulus tensor in classical isotropic elasticity, u_k is the displacement vector, $\alpha(|x-x'|,\xi)$ is the nonlocal modulus or attenuation function incorporating the nonlocal effects into the constitutive equations. $|x-x'|$ is the Euclidean distance and $\xi = {e_0 a}/{l}$, where a is an internal characteristic length, l is an external characteristic length, e_0 is a nonlocal scaling parameter, which has been assumed as a constant appropriate to each material.

For homogeneous and isotropic elastic solids, the constitutive equation of nonlocal elasticity can be given as

$$(1-(e_0 a)^2\nabla^2)\boldsymbol{\sigma}=\mathbf{C_0}:\boldsymbol{\varepsilon} \qquad (2)$$

where $\mathbf{C}_0$ is the elastic stiffness matrix of classical isotropic elasticity, $\boldsymbol{\sigma}$ is the nonlocal stress tensor at x, $\boldsymbol{\varepsilon}$ is the strain tensor at any point x' in the body.

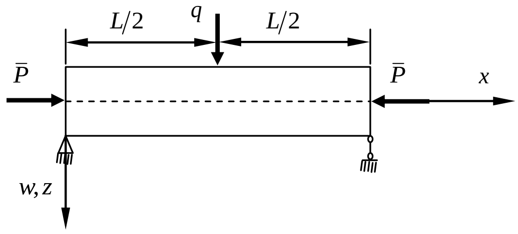

Fig. 1. Functionally graded Poisson's ratio nanoscale beam

3. Functionally graded Poisson's ratio materials

In recent years, the concept of a functionally graded Poisson's ratio material has been developed, which combined the computational design methodology and innovative structural-material concept as shown in Fig. 1. Many cellular materials and structures have been introduced due to their improved properties, such as enhanced shock resistance, fracture toughness, indentation and shear modulus. Functionally graded Poisson's ratio materials/structures is a further extension of FGM material which can be achieved by modifying the microstructures of the material in the thickness direction as shown in Figs. 2-4.

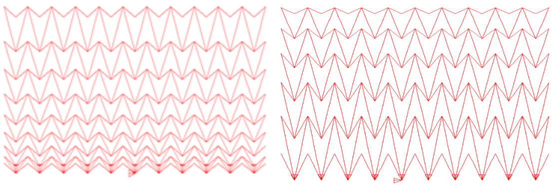

Fig. 2. Double arrowhead FGM structure

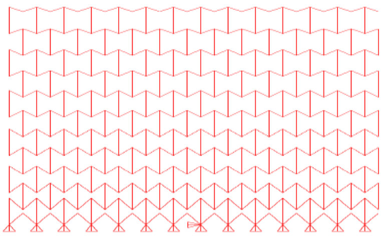

Fig. 3. Re-entrant FGM structure

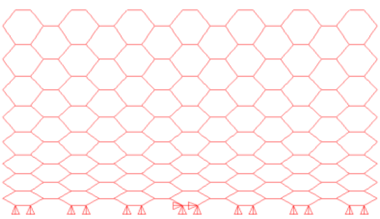

Fig. 4. Hexagonal honeycomb FGM structure

4. Functionally graded Poisson's ratio nanoscale beam model

In this study, the effective material properties of the FGPR nanoscale beam vary continuously in the thickness direction, including Young's modulus E, Poisson's ratio ν, shear modulus G and mass density ρ.

According to the classical rule of mixture, the effective material properties can be estimated as

$$E(z)=(E_b-E_t)\left(\frac{z}{h}+\frac{1}{2}\right)^k+E_t \tag{3}$$

$$\nu(z)=(\nu_b-\nu_t)\left(\frac{z}{h}+\frac{1}{2}\right)^k+\nu_t \tag{4}$$

where the subscripts b and t denote bottom and top surfaces of the nanoscale beam, k is the non-negative variable parameter which indicates the material variation profile through the thickness of the beam as shown in Fig. 5.

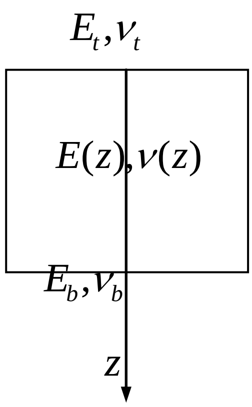

Fig. 5. Variation of material properties through the thickness direction of FGM beam

Based on the Timoshenko beam theory, the displacement fields of the beam

$$u(x,z)=u_0(x)-z\phi(x) \tag{5}$$

$$v(x,z)=0 \tag{6}$$

$$w(x,z)=w_0(x) \tag{7}$$

where u_0 and w_0 are the axial and the transverse displacement on the neutral axis, ϕ is the total bending rotation of the cross-sections on the neutral axis. Then, the strains of the Timoshenko beam theory can be obtained as

$$\varepsilon_{xx}=\varepsilon_{xx}^{0}-z\kappa_{x},\ \varepsilon_{xx}^{0}=\frac{\mathrm{d}u}{\mathrm{d}x},\ \kappa_{x}=\frac{\mathrm{d}\phi}{\mathrm{d}x},$$
$$\varepsilon_{xz}=\frac{\mathrm{d}w}{\mathrm{d}x}-\phi,\ \varepsilon_{xy}=\varepsilon_{yy}=\varepsilon_{yz}=\varepsilon_{zz}=0 \tag{8}$$

Based on the principle of total potential energy, the equilibrium equations can be derived. The first variations of the strain energy and work done by the external applied force are given by

$$\delta U=\int_{V}\left(\sigma_{xx}\delta\varepsilon_{xx}+\sigma_{xz}\delta\varepsilon_{xz}\right)\mathrm{d}V \tag{9}$$

$$\delta W=\int_{0}^{L}\bar{P}\left(\frac{\mathrm{d}\delta u}{\mathrm{d}x}+\frac{\mathrm{d}w}{\mathrm{d}x}\frac{\mathrm{d}\delta w}{\mathrm{d}x}\right)\mathrm{d}x+\int_{0}^{L}q\delta w\mathrm{d}x \tag{10}$$

where U is the strain energy, $\bar{P}$ is the axial compressive force, q is the transverse component of the body forces per unit length, L is the length of the beam.

By performing the integrating processes, the virtual strain energy can be written as

$$\begin{Bmatrix}M_1\\M_2\\M_3\end{Bmatrix}=\int_A\begin{Bmatrix}\sigma_{xx}\\k_s\sigma_{xz}\\z\sigma_{xx}\end{Bmatrix}\mathrm{d}A=\begin{Bmatrix}A_1\dfrac{\partial u}{\partial x}-B_1\dfrac{\partial\phi}{\partial x}\\k_sA_2\left(\dfrac{\partial w}{\partial x}-\phi\right)\\B_1\dfrac{\partial u}{\partial x}-C_1\dfrac{\partial\phi}{\partial x}\end{Bmatrix} \tag{11}$$

where k_s is the shear correction factor and

$$\begin{Bmatrix}A_1\\B_1\\C_1\\A_2\end{Bmatrix}=\begin{Bmatrix}\int_A E(z)\mathrm{d}A\\\int_A zE(z)\mathrm{d}A\\\int_A z^2E(z)\mathrm{d}A\\\int_A \nu(z)\mathrm{d}A\end{Bmatrix} \tag{12}$$

According to the minimum total potential energy principle, the first variation of the total potential energy must be zero

$$\delta\prod=\delta\left(U-W\right)=0 \tag{13}$$

where $\prod$ is the total potential energy.

The equilibrium equations of the FGPR nanoscale beam in terms of the displacements can be obtained as

$$A_1\frac{\partial^2u}{\partial x^2}-B_1\frac{\partial^2\phi}{\partial x^2}=0 \tag{14}$$

$$k_sA_2\left(\frac{\partial^2w}{\partial x^2}-\frac{\partial\phi}{\partial x}\right)-\bar{P}\frac{\partial^2w}{\partial x^2}+\bar{P}(e_0a)^2\frac{\partial^4w}{\partial x^4}=(e_0a)^2\frac{\mathrm{d}^2w}{\mathrm{d}x^2}-q \tag{15}$$

$$B_1\frac{\partial^2u}{\partial x^2}-C_1\frac{\partial^2\phi}{\partial x^2}-k_sA_2\left(\frac{\partial w}{\partial x}-\phi\right)=0 \tag{16}$$

In this study, the boundary conditions of a simply-supported Timoshenko nanoscale beam are considered. The displacement fields are assumed as follow:

$$u(x)=\sum_{n=1}^{N}U_n\cos\alpha x \tag{17}$$

$$w(x)=\sum_{n=1}^{N}W_n\sin\alpha x \tag{18}$$

$$\phi(x)=\sum_{n=1}^{N}G_n\cos\alpha x \tag{19}$$

where U_n, W_n, G_n are the unknown Fourier coefficients to be determined and $\alpha=n\pi/L$.

For the static bending problem, the applied transverse load q can be expanded in Fourier series as

$$q(x)=\sum_{n=1}^{N}Q_n\sin\alpha x \tag{20}$$

where

$$Q_n=\frac{2}{L}\int_0^L q(x)\sin\alpha x\mathrm{d}x \tag{21}$$

Assuming the FGPR nanoscale beam is subjected to the point load on the midspan of the beam, where the $q(x)=P\delta(x-L/2)$ and the Fourier coefficients can be expressed as

$$Q_n=\frac{2P}{L}\sin\frac{n\pi}{2}\quad n=1,2,3,... \tag{22}$$

Substituting Eqs (17-20) into Eqs (14-16) leads to the following equations

$$\begin{bmatrix}T_{11}&T_{12}&T_{13}\\T_{21}&T_{22}&T_{23}\\T_{31}&T_{32}&T_{33}\end{bmatrix}\begin{Bmatrix}U_n\\W_n\\G_n\end{Bmatrix}=\begin{Bmatrix}0\\(\alpha^2(e_{0a}a)^2+1)Q_n\\0\end{Bmatrix} \tag{23}$$

where

$$T_{11} = \alpha^2 A_1, T_{12} = T_{21} = 0, T_{13} = T_{31} = -\alpha^2 B_1,$$
$$T_{22} = \alpha^2 A_2 k_s, T_{23} = -\alpha A_2 k_s, T_{32} = -\alpha A_2 k_s, \qquad (24)$$
$$T_{33} = \alpha^2 C_1 + A_2 k_s$$

Eq. (23) can be solved to obtain the Fourier coefficients

$$U_n = \frac{2P_0(\alpha^2(e_0a)^2+1)B_1\sin\frac{n\pi}{2}}{\alpha^3 L[(\alpha^2 l^2 + 4k_s)(B_1^2 - A_1C_1) - 4k_s l^2 A_1 A_2]},$$
$$W_n = \frac{2P_0[\alpha^2(A_1C_1 - B_1^2) + k_s A_1 A_2](\alpha^2(e_0a)^2+1)\sin\frac{n\pi}{2}}{\alpha^4 k_s A_2 L(A_1C_1 - B_1^2)}, \qquad (25)$$
$$G_n = \frac{2P_0 A_1(\alpha^2(e_0a)^2+1)\sin\frac{n\pi}{2}}{\alpha^3 L(A_1C_1 - B_1^2)}$$

For the buckling problem, one obtains

$$\begin{bmatrix} T'_{11} & T'_{12} & T'_{13} \\ T'_{21} & T'_{22} & T'_{23} \\ T'_{31} & T'_{32} & T'_{33} \end{bmatrix} \begin{Bmatrix} U_n \\ W_n \\ G_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (26)$$

and

$$T'_{11} = \alpha^2 A_1, T'_{12} = T'_{21} = 0, T'_{13} = T'_{31} = -\alpha^2 B_1,$$
$$T'_{22} = \alpha^2 A_2 k_s - \alpha^2 P_2 - \alpha^4 (e_0a)^2 P_2, \qquad (27)$$
$$T'_{23} = -\alpha A_2 k_s, T'_{32} = -\alpha A_2 k_s, T'_{33} = \alpha^2 C_1 + A_2 k_s$$

Eq. (26) presents an eigenvalue problem with $\lambda_n = P_{cr}(n)$. And the critical buckling load can be obtained as

$$P_2 = P_{cr}(n) = \frac{\alpha^2 k_s A_2 (A_1C_1 - B_1^2)}{(\alpha^2(e_0a)^2+1)(\alpha^2(A_1C_1 - B_1^2) + k_s A_1 A_2)} \qquad (28)$$

5. Numerical results

In the numerical calculations, three different FGPR nanoscale beam models are introduced: isotropic, FG beam with Young's modulus E varies in the thickness direction, FG beam with both Young's modulus E and Poisson's ratio ν vary in the thickness direction, and the following material parameters are used: $E_t = 1\text{TPa}, \nu_t = 0.1, E_b = 0.1\text{TPa}, \nu_b = 0.3$. The shear correction factor is taken as $k_s = 5/6$ for Timoshenko beam theory.

In Fig. 6, the effects of nonlocal parameter on the deflection and buckling load are shown, in which the beam is considered to be isotropic material. It can be seen that the deflection is increased with the aspect ratios L/h, but the buckling load decreased with the aspect ratios. The results show that the deflections vary linearly with the nonlocal parameter, but the buckling loads vary nonlinearly.

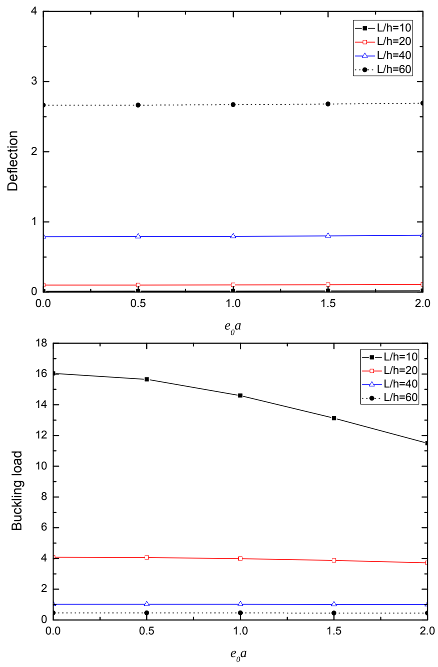

Fig. 6. Effects of nonlocal parameter on the deflection and buckling load of isotropic beam

In most literatures about FG structures, the Poisson's ratio is considered to be constant in the thickness direction. So, the same model is considered first in this part. In Fig. 7, the effects of nonlocal parameter on the deflection and buckling load are exhibited, in which Young's modulus varies continuously in the thickness direction and Poisson's ratio $\nu = 0.3$. The same trends were shown as in Fig. 6.

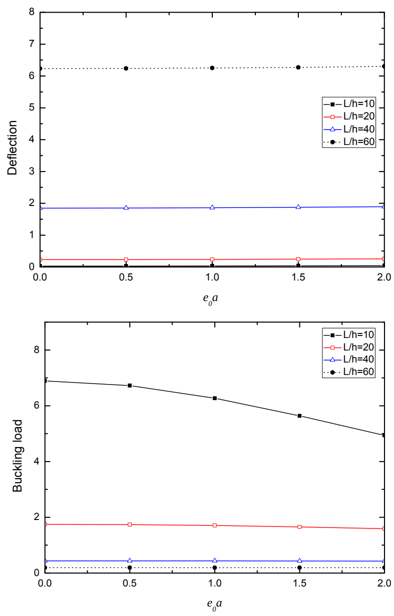

Fig. 7. Effects of nonlocal parameter on the deflection and buckling load of FG beam with constant Poisson's ratio

In Figs. 8-9, the FGPR nanoscale beam is considered and the effects of nonlocal parameter and gradient index on the deflection and buckling load are exhibited, in which both Young's modulus and Poisson's ratio vary continuously in the thickness direction. From Fig. 8, it can be seen that most results decreased due to the variations of gradient of Poisson's ratio in the deflection and buckling load compared with the results in Fig. 7, which suggested that the Poisson's ratio effect has an influence on the FG beam structures and may be considered in further researches. In Fig. 9, the effects of gradient index on the deflection and buckling load of FGPR beam are presented.

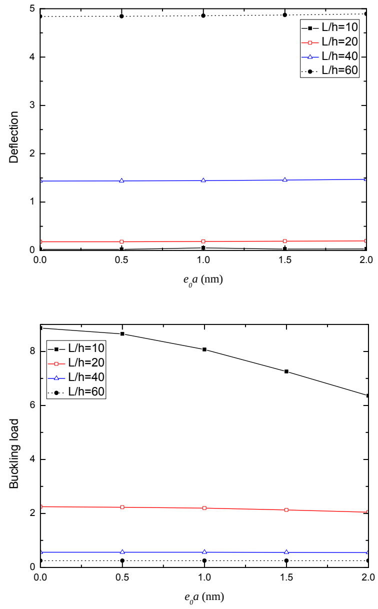

Fig. 8. Effects of nonlocal parameter on the deflection and buckling load of FGPR beam

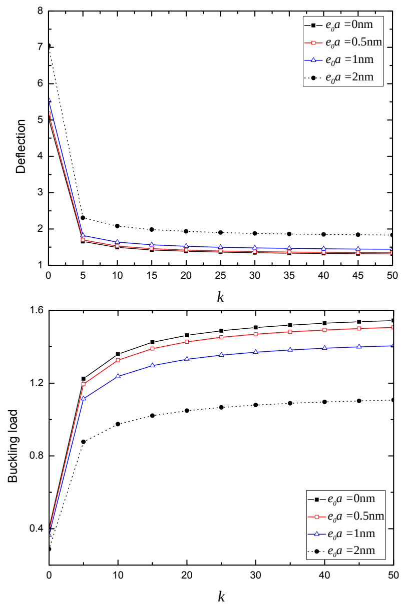

Fig. 9. Effects of gradient index on the deflection and buckling load of FGPR beam

6. Conclusions

Functionally graded Poisson's ratio (FGPR) cellular structures are one kind of innovative material. The static bending and buckling of FGPR nanoscale beam is studied in this paper based on the nonlocal Timoshenko beam model. The equilibrium equations are derived by using the principle of total potential energy. In the numerical results, three different kinds of beam models are considered, in which the effects of the nonlocal parameter, aspect ratio and the Poisson's ratio on the deflection and buckling are discussed.

(1) The expressions of the static bending and the critical buckling loads of the FGPR beam are given;

(2) Compared with FG beam with a constant Poisson's ratio in the thickness direction, the deflections of FGPR beam are smaller, but the buckling loads are larger;

(3) The deflections of the FGPR beam decreases with the increase of gradient index and tends to a constant value gradually, and the buckling loads increases with the gradient index and also tend to a constant value.

(4) The deflections of the FGPR beam increases with the nonlocal parameters, and the buckling loads decreases with it.

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