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DYNAMICAL BEHAVIOR OF AN AXIALLY MOVING STRING CONSTITUTED BY A FRACTIONAL DIFFERENTIATION LAW*

D. DÖNMEZ DEMİR[1**], B. G. SİNİR[2] AND N. BİLDİK[3]

[1,3]Dept. of Math., Faculty of Art & Sciences, Celal Bayar University, 45047, Manisa, Turkey
Email: duygu.donmez@cbu.edu.tr
[2]Dept. of Civil Eng., Faculty of Engineering, Celal Bayar University, 45140 Manisa, Turkey

Abstract– In this paper, the dynamical behavior of an axially moving string modeled by fractional derivative is investigated. The governing equation represented motion is solved by the method of multiple scales. Considering principal parametric resonance, the stability boundaries for string with simple supports are obtained. Numerical results indicate the effects of fractional damping on stability.

Keywords– Axially moving string; Fractional differentiation; Method of multiple scales; Principal parametric resonance

1. INTRODUCTION

Many engineering materials such as power transmission belts, plastic films, magnetic tapes, paper sheets, thread lines, wires, chains, high speed fiber winding and textile fibers are important in terms of modeling axially moving string. The linear or non-linear vibrations of axially moving string are popular among many researchers. Sack [1] is one of the first researchers in this area. Also, Mahalingam and Archibald et al. [2, 3] investigated the transverse oscillations of travelling strings. Wickert and Mote [4] performed many studies to investigate transverse vibrations of such systems. Pakdemirli et al. [5] studied transverse vibrations of an axially accelerating string. The transverse vibrations due to tension or axial speed variation constitute a major problem. Therefore, many studies are focused on this subject.

Two most common models which are translating string or beam are used to performed the dynamical analysis of such systems. These basic models related to the class of gyroscopic systems vanish for the fundamental natural frequency [6]. Yang et al. [7] applied the Lyapunov method to a two-span axially moving string subjected to varying tension and boundary disturbance. Von Horssen and Ponomareva [8] used the Laplace transformation technique to construct the solution of the problem of moving string with constant speed. On the other hand, Chen et al. [9] considered the transverse vibration of an initially stressed moving viscoelastic string obeying fractionally. Non-linear creep vibration of the axially moving viscoelastic string constituted by fractional differentiation law is analytically studied by Yang and Fang [10].

In recent years, there has been a growing interest in the area of fractional calculus and its applications [11]. Strictly speaking, fractional derivatives are useful for describing the transverse vibrations in engineering practice. Many vibration and wave problems have been investigated for the continua

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**Corresponding author

constituted by the fractional differentiation [12]. When the literature is reviewed, it is easily seen that the number of the published papers especially regarding axially moving strings is very limited.

In this study, we investigate stability region of parametrically excited string. In this model, the damping term is modeled as a fractional derivation. The method of multiple scales is used to analyze the dynamic behavior of tensioned axially moving string.

2. EQUATION OF MOTION

For the uniform axially moving tensioned string, ρ is the density, A is the cross-sectional area, $\hat{P}(\hat{t})$ is the initial tension, $\hat{v}$ is the axial speed, L is the distance apart between two simple supports, and also $\hat{\mu}$ is the damping coefficient. The equation of motion can be obtained as

$$\rho A\left(\ddot{\hat{w}}+2\hat{v}\dot{\hat{w}}'+\hat{v}^2\hat{w}''\right)-\hat{P}\hat{w}''+\hat{\mu}D^{\alpha}\hat{w}=0 \tag{1}$$

$$\hat{w}\left(0,\hat{t}\right)=\hat{w}\left(1,\hat{t}\right)=0 \tag{2}$$

where $\hat{w}\left(\hat{x},\hat{t}\right)$ is the transverse displacement. Here, $\hat{x}$ is the spatial variable and $\hat{t}$ is the time variable in the space $\left(\hat{x},\hat{t}\right)$. The dot denotes the differentiation with respect to time $\hat{t}$. For convenience, we introduce the following non-dimensional variables and parameters:

$$x=\frac{\hat{x}}{L},\ w=\frac{\hat{w}}{r},\ r=\sqrt{\frac{I}{A}},\ t=\frac{\hat{t}}{L^2}\sqrt{\frac{EI}{\rho A}},\ v=\hat{v}L\sqrt{\frac{\rho A}{EI}},\ P=\frac{\hat{P}L^2}{EI},\ \varepsilon\mu=\frac{\hat{\mu}}{L^{2\alpha-4}}\sqrt{\frac{(EI)^{\alpha-2}}{(\rho A)^{\alpha}}} \tag{3}$$

where ε is a small dimensionless parameter, I is the moment of inertia and E is modulus of elasticity, and r is radius of gyration. Thus, non-dimensional linear equation of motion for transverse vibration

$$\ddot{w}+2v\dot{w}'+v^2w''-\left(P_0+\varepsilon P_1\cos\Omega t\right)w''+\varepsilon\mu D^{\alpha}w=0 \tag{4}$$

subject to the boundary conditions in non-dimensional form

$$w\left(0,t\right)=w\left(1,t\right)=0 \tag{5}$$

is obtained. Also, we assume that the initial tension $P(t)$ is characterized as a small periodic perturbation $\varepsilon P_1\cos\Omega t$ superimposed on the steady state tension P_0, i.e. $P=P_0+\varepsilon P_1\cos\Omega t$; which is the same as in previous studies [9].

3. METHOD OF SOLUTION

Let us consider the Eq. (4) which is a linear partial-differential equation with fractional derivatives. To solve this equation, the method of multiple scales is applied. A first-order approximation is considered in the form

$$w(x,t;\varepsilon)=w_0\left(x,T_0,T_1\right)+\varepsilon\, w_1\left(x,T_0,T_1\right)+\ldots \tag{6}$$

where $t=T_0$ is the usual fast-time scales, and $\varepsilon t=T_1$ is the slow-time scales. Thus, the time derivatives are expressed in terms of fast and slow time scales as follows:

$$d/dt=D_0+\varepsilon D_1+\ldots,\ d^2/dt^2=D_0^2+2\varepsilon D_0D_1+\ldots \tag{7}$$

where $D_n=\partial/\partial T_n$ [13]. On the other hand, the fractionally time derivative [14] is given by

$$\left(\frac{d}{dt}\right)^{\alpha} = \left(D_0 + \varepsilon D_1 + \ldots\right)^{\alpha} = D_{+}^{\alpha} + \varepsilon\alpha D_{+}^{\alpha-1} D_1 + \ldots \tag{8}$$

Besides the definition of Riemann-Liouville fractional derivative [15] is introduced as

$$D_{+}^{\alpha} w(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_{-\infty}^{t} \frac{w(\tau)\,d\tau}{(t-\tau)^{\alpha}}. \tag{9}$$

Using Eq. (9), the fractional derivative of the exponential function is calculated as [14]

$$D_{0+}^{\alpha} e^{i\omega t} = (i\omega)^{\alpha} e^{i\omega t} + \frac{\sin \pi\alpha}{\pi} \int_{0}^{\infty} \frac{u^{\alpha}}{u + i\omega} e^{ut} du. \tag{10}$$

Since the integral in the second term of Eq. (10) decays rapidly with respect to time t, it can be neglected as compared with the first term in Eq. (9) in the following cases [15-18]: (1) α is slightly different from unity, (2) the magnitude of fractional parameter α is small, (3) for large magnitudes of ω. The second term in Eq. (9) can be also ignored from some instants of time after beginning of vibratory motion. Then, the fractional derivative of the exponential function can be obtained as

$$D_{+}^{\alpha} e^{i\omega t} = (i\omega)^{\alpha} e^{i\omega t} \tag{11}$$

where D_{+}^{α}, $D_{+}^{\alpha-1}$, $D_{+}^{\alpha-2}$, $\ldots$ are the Riemann-Liouville fractional derivative [14]. Substituting Eqs. (6), (7) and (8) into Eq. (4) and separating each order of approximation, one obtains

$$O(1): D_0^2 w_0 + 2vD_0 w_0' + \left(v^2 - P_0\right) w_0'' = 0 \tag{12}$$

$$O(\varepsilon): D_0^2 w_1 + 2vD_0 w_1' + \left(v^2 - P_0\right) w_1'' = -2D_1\left(D_0 w_0 + v w_0'\right) + P_1 w_0'' \cos \Omega T_0 - \mu D_0^{\alpha} w_0. \tag{13}$$

The solution of Eq. (12) can be written as follows [19]:

$$w_0\left(x, T_0, T_1\right) = A_n\left(T_1\right) X_n(x) e^{i\omega_n T_0} + \bar{A}_n\left(T_1\right) \bar{X}_n(x) e^{-i\omega_n T_0} \tag{14}$$

where A_n and $\bar{A}_n$ are complex amplitudes and their conjugates, respectively. Then, the mode function is found as

$$X_n(x) = c e^{iv\omega_n x / P_0 - v^2} \sin \frac{\omega_n \sqrt{P_0}}{P_0 - v^2} x \tag{15}$$

where

$$\omega_n = \frac{n\pi\left(P_0 - v^2\right)}{\sqrt{P_0}};\ n = 1, 2, \ldots \tag{16}$$

Here, n is the mode number and ω_n is the natural frequency. Substituting Eq. (14) into Eq. (13) for the solution of $\varepsilon-$order, one obtains

$$D_0^2 w_1 + 2vD_0 w_1' + \left(v^2 - P_0\right) w_1'' = \left[-2D_1 A_n\left(i\omega_n X_n + vX_n'\right)\right] e^{i\omega_n T_0} + \frac{1}{2} P_1 A_n X_n''\left[e^{i(\Omega+\omega_n)T_0} + e^{-i(\Omega-\omega_n)T_0}\right]$$
$$+ \frac{1}{2} P_1 \bar{A}_n \bar{X}_n''\left[e^{i(\Omega-\omega_n)T_0} + e^{-i(\Omega+\omega_n)T_0}\right] - \mu\left(i\omega_n\right)^{\alpha} A_n X_n e^{i\omega_n T_0} + cc \tag{17}$$

where cc is the complex conjugates. In the next section, three different cases will be discussed.

4. CASE STUDIES

In this section, we assume that one dominant mode of vibrations exists. Then, the direct-perturbation method gives more accurate results for finite mode truncations. This is because the spatial functions appearing at higher orders of approximation represent the best to real system in the case of the direct-perturbation methods [20].

Case 1: Ω Away from 2ω_n and 0

In this case, no resonance exists. Hence, Eq. (17) becomes

$$D_0^2 w_1 + 2vD_0 w_1' + \left(v^2 - P_0\right) w_1'' = \left[-2D_1 A_n \left(i\omega_n X_n + vX_n'\right) - \mu\left(i\omega_n\right)^\alpha X_n A_n\right] e^{i\omega_n T_0} + cc + NST \quad (18)$$

where NST represents non-secular terms. Thus, the solution of Eq. (18) is as follows [19]:

$$w_1\left(x, T_0, T_1\right) = \varphi_n\left(x, T_1\right) e^{i\omega_n T_0} + W\left(x, T_0, T_1\right) + cc\,. \quad (19)$$

The first term represents secular terms and the latter term is related to non-secular terms. Substituting Eq. (19) into Eq. (18), φ_n provides

$$\left(P_0 - v^2\right)\varphi_n'' - 2iv\omega_n\varphi_n' + \omega_n^2\varphi_n = 2D_1 A_n\left(i\omega_n X_n + vX_n'\right) + \mu\left(i\omega_n\right)^\alpha X_n A_n \quad (20)$$

$$\varphi_n(0) = \varphi_n(1) = 0\,. \quad (20\text{-a})$$

The solvability condition requires [21]

$$D_1 A_n + k_0 A_n = 0 \quad (21)$$

where

$$k_0 = \frac{\mu\left(i\omega_n\right)^\alpha \int_0^1 X_n \bar{X}_n dx}{2\left(i\omega_n \int_0^1 X_n \bar{X}_n dx + v\int_0^1 X_n' \bar{X}_n dx\right)}\,. \quad (22)$$

Then, the amplitude solution is

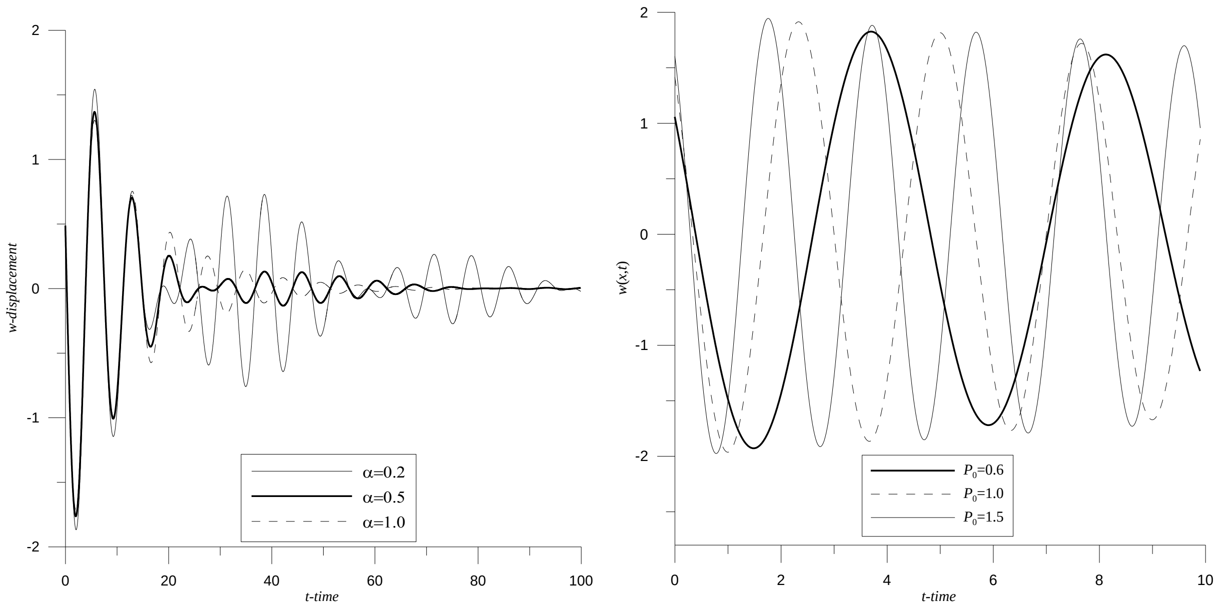
$$A_n\left(T_1\right) = A_0 e^{-k_0 T_1} \quad (23)$$

where A_0 is constant. k_0 is a complex number where $k_0 = k_0^R + i\,k_0^I$. The real part of k_0 is always positive. Substituting the solution (23) into Eq. (14), then the approximate solution is obtained as

$$w \cong w_0 \cong A_0 e^{\left[-k_0^R\varepsilon + i\left(\omega_n - k_0^I\varepsilon\right)\right]t} X_n(x) + cc \quad (24)$$

The relation (24) shows that the system is always stable in this case and the natural frequency may be changed slowly due to fractional order.

In Fig. 1, the effects of fractional order on the displacement-time curves are clearly shown. It is seen that the damping accelerates as the value α increases. The displacement-time curves for the variation of P_0 are observed afterwards. It is concluded that the frequency increases by growth of the value P_0.



(a) (b)

Fig. 1. Displacement-time curves for $\varepsilon = 0.1$ and $x = 0.5$ (a) $P_0 = 0.9, v = 0.8, \mu = 1.5$ (b) $\alpha = 0.5, v = 0.5, \mu = 0.8$

Case 2: Ω Close to 0

For this case, the nearness of fluctuation frequency to zero can be expressed as

$$\Omega \cong \varepsilon \sigma_n \tag{25}$$

where σ_n is a detuning parameter. Then, Eq. (16) becomes

$$D_0^2 w_1 + 2vD_0 w_1' + \left(v^2 - P_0\right) w_1'' = \left[-2D_1 A_n \left(i\omega_n X_n + vX_n'\right) + P_1 A_n X_n'' \cos \sigma_n T_1 - \mu \left(i\omega_n\right)^{\alpha} A_n X_n\right] e^{i\omega_n T_0} + cc + NST \tag{26}$$

From the solvability condition, we obtain

$$D_1 A_n - k_1 \cos\left(\sigma_n T_1\right) A_n + k_0 A_n = 0 \tag{27}$$

where

$$k_1 = \frac{P_1 \int_0^1 X_n'' \bar{X}_n dx}{2\left(i\omega_n \int_0^1 X_n \bar{X}_n dx + v\int_0^1 X_n' \bar{X}_n dx\right)}. \tag{28}$$

The solution of Eq. (27) is

$$A_n\left(T_1\right) = A_0 e^{\frac{k_1}{\sigma_n}\sin\left(\sigma_n T_1\right) - k_0 T_1}. \tag{29}$$

Since $-1 \le \sin \sigma_n T_1 \le 1$, it is concluded that there is no instability.

Case 3: Ω Close to $2\omega_n$

Let us consider the principal parametric resonance as

$$\Omega \cong 2\omega_n + \varepsilon\sigma_n \,. \tag{30}$$

Thus, Eq. (17) turns into

$$\begin{aligned} D_0^2 w_1 + 2vD_0 w_1' + \left(v^2 - P_0\right) w_1'' = [-2D_1 A_n \left(i\omega_n X_n + vX_n'\right) + \frac{P_1}{2}\bar{A}_n \bar{X}_n'' e^{i\sigma_n T_1} \\ - \mu\left(i\omega_n\right)^{\alpha} A_n X_n] e^{i\omega_n T_0} + cc + NST \end{aligned} \tag{31}$$

The solvability condition for this case is

$$D_1 A_n - \frac{1}{2}k_1 \bar{A}_n e^{i\sigma_n T_1} + k_0 A_n = 0 \,. \tag{32}$$

Expressing the solution of Eq. (32) in the polar form

$$A_n\left(T_1\right) = B_n\left(T_1\right) e^{i\sigma_n T_1/2} \,,\ \bar{A}_n\left(T_1\right) = \bar{B}_n\left(T_1\right) e^{-i\sigma_n T_1/2} \,. \tag{33}$$

Substituting Eq. (33) into Eq. (32) yields

$$D_1 B_n + i\frac{\sigma_n}{2} B_n - \frac{k_1}{2}\bar{B}_n + k_0 B_n = 0 \tag{34}$$

where complex amplitudes are

$$B_n\left(T_1\right) = b_n e^{\lambda T_1} \,,\ \bar{B}_n\left(T_1\right) = \bar{b}_n e^{\lambda T_1} \tag{35}$$

and $b_n = b_n^R + ib_n^I$, $k_0 = k_0^R + ik_0^I$ and $k_1 = k_1^R + ik_1^I$ such that R and I denote real and imaginary parts, respectively. If we substitute Eq. (35) into Eq. (34) and separate real and imaginary parts, then the matrix equation may be represented as

$$\begin{bmatrix} \lambda + k_0^R - \frac{k_1^R}{2} & -\left(\frac{\sigma_n}{2} + k_0^I + \frac{k_1^I}{2}\right) \\ \frac{\sigma_n}{2} + k_0^I - \frac{k_1^I}{2} & \lambda + k_0^R + \frac{k_1^R}{2} \end{bmatrix} \begin{bmatrix} b_n^R \\ b_n^I \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} . \tag{36}$$

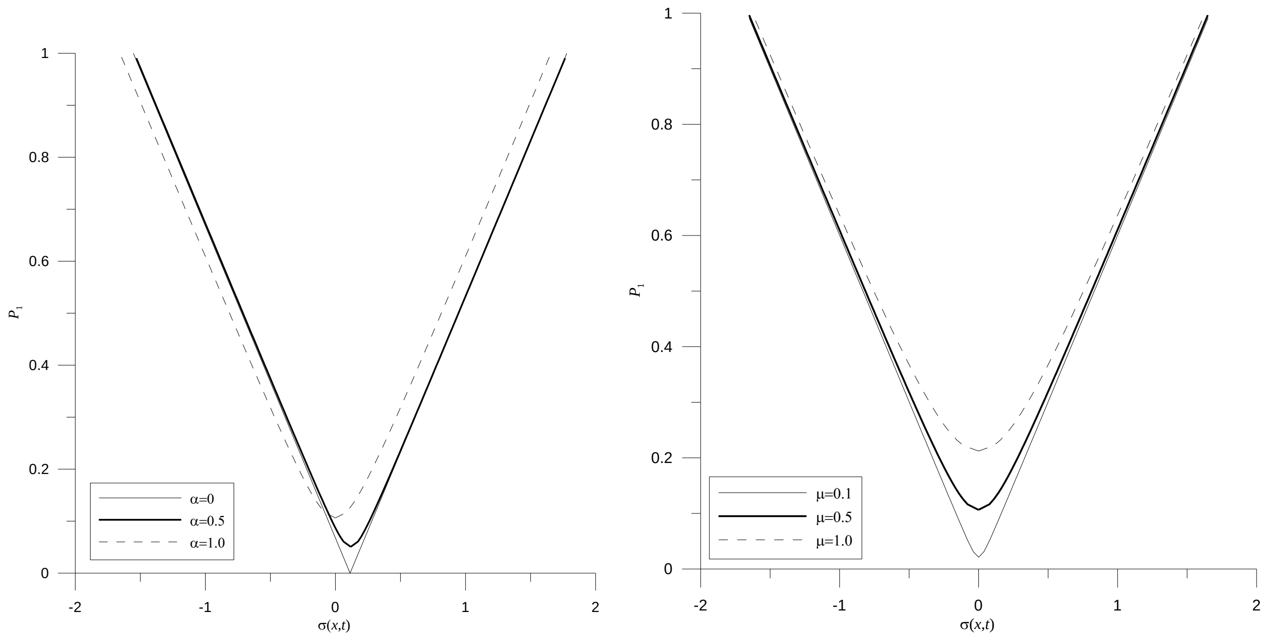
For non-trivial solution, the determinant of the coefficient matrix must be zero. Then,

$$\lambda = -k_0^R \pm \frac{1}{2}\sqrt{\left(k_1^R\right)^2 + \left(k_1^I\right)^2 - \left(2k_0^I + \sigma_n\right)^2} \,. \tag{37}$$

On the other hand, the steady-state solution requires $\lambda = 0$. Thus, the stability boundaries are written as

$$\sigma_n = -2k_0^I \pm \sqrt{\left(k_1^R\right)^2 + \left(k_1^I\right)^2 - \left(2k_0^R\right)^2} \,. \tag{38}$$

The variation of unstable region is depicted for some value α and μ in Fig. 2. The stability region moves to the left side of the graph in Fig. 2a. as the value of α is increasing. This feature comes from the fractional derivation. It is observed that the stability boundary moves to upward as the fractional order α increases.



(a) (b)

Fig. 2. Stability boundaries (a) for various fractional order-α $\left(P_0 = 2.0, v = 1.0, \mu = 0.5\right)$
(b) for different values μ $\left(P_0 = 2.0, v = 1.0, \alpha = 1.0\right)$

5. CONCLUSION

In this paper, the dynamic response of an axially moving tensioned string is investigated. It is obviously known that the string has fractional damping. In addition, the method of multiple scales is applied directly to solve the equation of motion. The case of the parametric resonance is investigated in detail. Besides, the stability boundaries are analytically determined and also natural frequencies are calculated for different fractional order. Finally, it is found that instabilities occur when the variation frequency is close to two times the natural frequencies and observed that the order fractional derivative has a meaningful effect on stability boundaries and natural frequencies.

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