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Bending and Free Vibration Analyses of Rectangular Laminated Composite Plates Resting on Elastic Foundation Using a Refined Shear Deformation Theory

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Abstract: In this paper, a closed form solution for bending and free vibration analyses of simply supported rectangular laminated composite plates is presented. The static and free vibration behavior of symmetric and antisymmetric laminates is investigated using a refined first-order shear deformation theory. The Winkler–Pasternak two-parameter model is employed to express the interaction between the laminated plates and the elastic foundation. The Hamilton's principle is used to derive the governing equations of motion. The accuracy and efficiency of the theory are verified by comparing the developed results with those obtained using different laminate theories. The laminate theories including the classical plate theory, the classical first-order shear deformation theory, the higher order shear deformation theory and a three-dimensional layerwise theory are selected in order to perform a comprehensive comparison. The effects of the elastic foundation parameters, orthotropy ratio and width-to-thickness ratio on the bending deflection and fundamental frequency of laminates are investigated.

Keywords: Bending analysis, Free vibration, Refined shear deformation theory, Two-parameter elastic foundation

1. Introduction

Laminated composites are widely used in aerospace, automotive, and marine industries due to their high strength-to-weight ratio. The static, dynamic and buckling analyses of such structures have been the focus of attention for mechanical and structural engineers. Meanwhile, different theories have been developed to predict the behavior of laminated plates. The most popular theory is equivalent to single-layer (ESL) theory according to the literature. The ESL theories are usually categorized in three main categories of the classical plate theory (CPT), first-order shear deformation theory (FSDT), and higher-order shear deformation theories (HSDTs).

In this regard, Shojaee et al. [1] developed a free vibration and buckling analyses of symmetrically thin composites based on the classical plate theory. Reissner [2] developed the FSDT which takes into account the shear deformation effects. Unlike the FSDT, the HSDT satisfies the equilibrium conditions on the top and bottom surfaces without using shear correction factor. Also, Reddy [3] developed a third-order shear deformation theory (TSDT) using polynomial functions for displacement fields. However, most of the HSDTs are computationally expensive due to the additional unknowns which are introduced in the context of the theory. Recently, employing the refined form of the shear deformation theories has been the subject of many researches. Meanwhile, different forms of polynomial, trigonometric and exponential functions are implemented to investigate the mechanical behavior of one- and two-dimensional structures. For instance, Ferreira et al. [4] used sinusoidal functions for displacement fields. Soldatos [5] used hyperbolic functions to express the distribution of displacement components. Karama et al. [6] adopted exponential functions to study mechanical behavior of laminated composite beams. On the other hand, the FSDT may be inaccurate enough to predict the mechanical behavior of thick and moderately thick

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laminated composites. Unlike the classical FSDTs, Thai and Choi [7-9] presented a refined theory which contains four unknowns. They exhibited that this refined FSDT can develop accurate results for static and vibration analyses of composite plates.

The subject of plate and beam modeling resting on elastic foundations is important in analyzing structural problems. The two models of Winkler and Pasternak elastic foundations have been widely used in different studies. Although the Winkler model is simple and widely necessary, it is unable to takes into account the nonlinear behavior. Katsikadelis et al. [10] carried out the bending analysis of plates resting on elastic foundation using singular boundary integral equations. Dinev [11] obtained analytical solutions for beams on elastic foundations by singularity functions and using Pasternak foundation. An exact three-dimensional solution of simply supported rectangular plates on Pasternak foundation was developed by Dehghany and Farajpour [12]. Akavci [13] analyzed the laminated composite plates on elastic foundation employing various plate theories. Lal et al. [14] presented an investigation of the stochastic bending static response of laminated composite plates resting on elastic foundation with uncertain system parameters subjected to the static distributed loading. Akavci et al. [15] studied bending deformation of symmetrically laminated plates resting on elastic foundation based on FSDT. Also, Akavci [16] examined buckling and free vibration analyses of simply supported symmetric and antisymmetric cross-ply thick composite plates on elastic foundation by a hyperbolic displacement model. Nedri et al. [17] presented free vibration analysis of laminated composite on elastic foundation using a refined hyperbolic shear deformation theory. A comprehensive static, free vibration and buckling analyses of laminated composite plates on distributed and point elastic supports using a three-dimensional layer-wise finite element method was developed by Setoodeh and Karami [18].

In this paper, a simple refined first-order shear deformation theory is implemented to predict the bending and free vibration behavior of simply supported rectangular laminated composites on elastic foundations. To the best of the authors' knowledge, the influence of the Winkler-Pasternak two-parameter model on the bending deformation and natural frequencies of any laminate stacking sequence is investigated for the first time in the context of the refined FSDT. Furthermore, similar solutions are presented by reducing the refined theory to the CPT. The closed form solutions of cross-ply and angle-ply laminates on elastic foundation are developed and the results are successfully compared with the existing solutions.

2. Theoretical Formulation

Consider a rectangular composite plate with in-plane dimensions a and b in the x and y directions, respectively, and thickness h in the z direction as shown in Fig. 1. The reference Cartesian coordinate system (x, y, z) is located on the middle-plane of the plate. The plate is consisted of n orthotropic layers resting on elastic foundation.

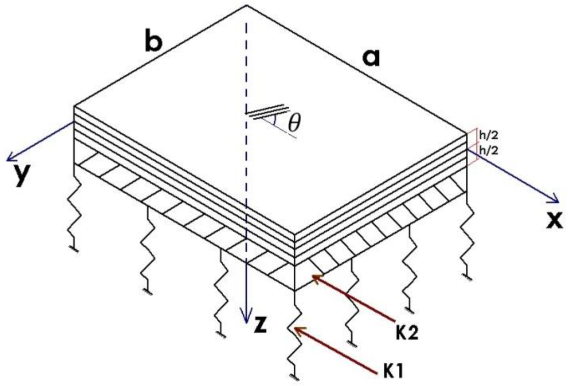

Fig. 1. Geometry of a laminated plate resting on elastic foundation.

The displacement field of the refined FSDT is given by:

$$u_1(x,y,z)=u(x,y)-z\frac{\partial w_b}{\partial x}$$
$$u_2(x,y,z)=v(x,y)-z\frac{\partial w_b}{\partial y} \tag{1}$$
$$u_3(x,y,z)=w_b(x,y)+w_s(x,y)$$

where (u_1, u_2, u_3) are respectively the components of the displacement vector and (u, v) denote the in-plane displacement components of the middle-plane in the x and y directions, respectively. Unlike the classical FSDT, the transverse displacement u_3 is divided into the bending (w_b) and shear (w_s) parts.

The nonzero in-plane strains (ε_x, ε_y, γ_{xy}) and the transverse shear strains (γ_{xz}, γ_{yz}) are associated with the displacement field in Eq. (1) as:

$$\varepsilon_x=\frac{\partial u}{\partial x}-z\frac{\partial^2 w_b}{\partial x^2},\quad \varepsilon_y=\frac{\partial v}{\partial y}-z\frac{\partial^2 w_b}{\partial y^2},\quad \gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}-2z\frac{\partial^2 w_b}{\partial x\,\partial y}$$
$$\gamma_{xz}=\frac{\partial w_s}{\partial x},\quad \gamma_{yz}=\frac{\partial w_s}{\partial y} \tag{2}$$

Under the assumption that each layer possesses a plane of elastic symmetry parallel to the x–y plane, the constitutive equations for a layer can be written as:

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_{xy}\\ \sigma_{yz}\\ \sigma_{xz}\end{Bmatrix}=\begin{bmatrix}Q_{11}&Q_{12}&0&0&0\\ Q_{12}&Q_{22}&0&0&0\\ 0&0&Q_{66}&0&0\\ 0&0&0&Q_{44}&0\\ 0&0&0&0&Q_{55}\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{xz}\end{Bmatrix} \tag{3}$$

Again, (σ_x, σ_y, σ_{xy}) and (σ_{xz}, σ_{yz}) denote the in-plane and the transverse shear stresses, respectively. Here, for each layer Q_{ij} is given in terms of the Young's modulus (E), the Poisson's ratio (ν) and the shear modulus (G) in different orthotropic directions as below:

$$Q_{11}=\frac{E_1}{1-\nu_{12}\nu_{21}},\ Q_{22}=\frac{E_2}{1-\nu_{12}\nu_{21}},\ Q_{12}=\nu_{12}Q_{22},\ Q_{66}=G_{12},\ Q_{44}=G_{23},\ Q_{55}=G_{13} \tag{4}$$

The stress–strain relations in the laminate coordinates of the kth layer are given as:

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_{xy}\\ \sigma_{yz}\\ \sigma_{xz}\end{Bmatrix}^{(k)}=\begin{bmatrix}\bar{Q}_{11}&\bar{Q}_{12}&\bar{Q}_{16}&0&0\\ \bar{Q}_{12}&\bar{Q}_{22}&\bar{Q}_{26}&0&0\\ \bar{Q}_{16}&\bar{Q}_{26}&\bar{Q}_{66}&0&0\\ 0&0&0&\bar{Q}_{44}&\bar{Q}_{45}\\ 0&0&0&\bar{Q}_{45}&\bar{Q}_{55}\end{bmatrix}^{(k)}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{xz}\end{Bmatrix}^{(k)} \tag{5}$$

where $\bar{Q}_{ij}$ are the transformed elastic stiffness coefficients, and are defined according to the following equations.

$$\begin{aligned}
\bar{Q}_{11} &= Q_{11}cos^4\theta + 2\left(Q_{12}+2Q_{66}\right)sin^2\theta cos^2\theta + Q_{22}sin^4\theta \\
\bar{Q}_{12} &= \left(Q_{11}+Q_{22}-4Q_{66}\right)sin^2\theta cos^2\theta + Q_{12}\left(sin^4\theta + cos^4\theta\right) \\
\bar{Q}_{22} &= Q_{11}sin^4\theta + 2\left(Q_{12}+2Q_{66}\right)sin^2\theta cos^2\theta + Q_{22}cos^4\theta \\
\bar{Q}_{16} &= \left(Q_{11}-Q_{12}-2Q_{66}\right)sin\theta cos^3\theta + \left(Q_{12}-Q_{22}+2Q_{66}\right)sin^3\theta cos\theta \\
\bar{Q}_{26} &= \left(Q_{11}-Q_{12}-2Q_{66}\right)sin^3\theta cos\theta + \left(Q_{12}-Q_{22}+2Q_{66}\right)sin\theta cos^3\theta \\
\bar{Q}_{66} &= \left(Q_{11}+Q_{22}-2Q_{12}-2Q_{66}\right)sin^2\theta cos^2\theta + Q_{66}\left(sin^4\theta + cos^4\theta\right) \\
\bar{Q}_{44} &= Q_{44}cos^2\theta + Q_{55}sin^2\theta \\
\bar{Q}_{45} &= \left(Q_{55}-Q_{44}\right)sin\theta cos\theta \\
\bar{Q}_{55} &= Q_{55}cos^2\theta + Q_{44}sin^2\theta
\end{aligned} \tag{6}$$

where θ is the angle between the global *x*-axis and the longitudinal direction of fibers of each lamina. The Hamilton's principle is used to derive the governing equations of motion as:

$$\delta\int_{t_1}^{t_2}\left(U+U_F-V-T\right)dt=0 \tag{7}$$

where U and T are respectively the strain energy and kinetic energy of the plate, U_F denotes the strain energy of the elastic foundation, and V is the work done by external forces. The variation form of the strain energy is expressed as:

$$\begin{aligned}
\delta U &= \int_A\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(\sigma_x\delta\varepsilon_x + \sigma_y\delta\varepsilon_y + \sigma_{xy}\delta\gamma_{xy} + \sigma_{xz}\delta\gamma_{xz} + \sigma_{yz}\delta\gamma_{yz}\right)dzdA \\
&= \int_A [N_x\frac{\partial\delta u}{\partial x} - M_x\frac{\partial^2\delta w_b}{\partial x^2} + N_y\frac{\partial\delta v}{\partial y} - M_y\frac{\partial^2\delta w_b}{\partial y^2} + N_{xy}\left(\frac{\partial\delta u}{\partial y}+\frac{\partial\delta v}{\partial x}\right) - 2M_{xy}\frac{\partial^2\delta w_b}{\partial x\,\partial y} \\
&\quad + Q_x\frac{\partial\delta w_s}{\partial x} + Q_y\frac{\partial\delta w_s}{\partial y}]dA
\end{aligned} \tag{8}$$

where the stress resultants are defined by:

$$\begin{aligned}
&\left(N_x,N_y,N_{xy}\right) = \int_{-\frac{h}{2}}^{\frac{h}{2}}\left(\sigma_x,\sigma_y,\sigma_{xy}\right)dz\,, \quad \left(M_x,M_y,M_{xy}\right) = \int_{-\frac{h}{2}}^{\frac{h}{2}}\left(\sigma_x,\sigma_y,\sigma_{xy}\right)zdz \\
&\left(Q_x,Q_y\right) = \int_{-\frac{h}{2}}^{\frac{h}{2}}\left(\sigma_{xz},\sigma_{yz}\right)dz
\end{aligned} \tag{9}$$

By using Eqs. (2), (5) and (9), the stress resultants in terms of the displacements can be obtained:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} -\dfrac{\partial^2 w_b}{\partial x^2} \\ -\dfrac{\partial^2 w_b}{\partial y^2} \\ -2\dfrac{\partial^2 w_b}{\partial x \partial y} \end{Bmatrix}$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} -\dfrac{\partial^2 w_b}{\partial x^2} \\ -\dfrac{\partial^2 w_b}{\partial y^2} \\ -2\dfrac{\partial^2 w_b}{\partial x \partial y} \end{Bmatrix}$$

$$\begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = k \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial w_s}{\partial x} \\ \dfrac{\partial w_s}{\partial y} \end{Bmatrix} \tag{10}$$

where k is the shear correction factor. Also, the elements of the stiffness matrices A, B, D are defined as follows:

$$\left(A_{ij}, B_{ij}, D_{ij}\right) = \int_{-h/2}^{h/2} \bar{Q}_{ij}\left(1, z, z^2\right) dz \tag{11}$$

The variation of the work done by the transverse load q and the kinetic energy are calculated respectively by Eqs. (12) and (13).

$$\delta V = -\int_A q\delta\left(w_b + w_s\right) dA \tag{12}$$

$$\begin{aligned} \delta K &= \int_V \left(\dot{u}_1 \delta \dot{u}_1 + \dot{u}_2 \delta \dot{u}_2 + \dot{u}_3 \delta \dot{u}_3\right) \rho dAdz \\ &= \int_A \left\{ I_0 \left[\dot{u}\delta\dot{u} + \dot{v}\delta\dot{v} + \left(\dot{w}_b + \dot{w}_s\right)\delta\left(\dot{w}_b + \dot{w}_s\right) \right] + I_2 \left(\frac{\partial \dot{w}_b}{\partial x}\frac{\partial \delta \dot{w}_b}{\partial x} + \frac{\partial \dot{w}_b}{\partial y}\frac{\partial \delta \dot{w}_b}{\partial y} \right) \right\} \end{aligned} \tag{13}$$

The point above a variable denotes, as usual, a time derivative. The various inertias I_0 and I_2 are defined as:

$$\left(I_0, I_2\right) = \int_{-h/2}^{h/2} \left(1, z^2\right) \rho dz \tag{14}$$

The variation of U_F can be written as:

$$\delta U_F = \int_A f_e \delta\left(w_b + w_s\right) dA \tag{15}$$

where f_e is the density of reaction force of the foundation and is expressed as follows.

$$f_e = k_0 w + k_1 \nabla^2 w \tag{16}$$

The coefficients k_0 and k_1 show the elastic stiffness of the Winkler and Pasternak foundations, respectively.

By substituting Eqs. (8), (12), (13) and (15) into Eq. (7), doing some manipulations and collecting the coefficients of $\delta u, \delta v, \delta w_b$ and δw_s, the governing equations are obtained as below:

$$\begin{aligned}
&\delta u : \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_0 \ddot{u} \\
&\delta v : \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = I_0 \ddot{v} \\
&\delta w_b : \frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x\, \partial y} + \frac{\partial^2 M_y}{\partial y^2} + q + k_0 (w_b + w_s) + k_1 \nabla^2 (w_b + w_s) = I_0 (\ddot{w}_b + \ddot{w}_s) - I_2 \nabla^2 \ddot{w}_b \\
&\delta w_s : \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q + k_0 (w_b + w_s) + k_1 \nabla^2 (w_b + w_s) = I_0 (\ddot{w}_b + \ddot{w}_s)
\end{aligned} \tag{17}$$

By using the stress resultants from Eq. (10), the governing equations can be expressed in terms of displacement components (u, v, w_b, w_s) as:

$$\begin{aligned}
&A_{11}\frac{\partial^2 u}{\partial x^2} + 2A_{16}\frac{\partial^2 u}{\partial x \partial y} + A_{66}\frac{\partial^2 u}{\partial y^2} + A_{16}\frac{\partial^2 v}{\partial x^2} + (A_{12} + A_{66})\frac{\partial^2 v}{\partial x \partial y} + A_{26}\frac{\partial^2 v}{\partial y^2} \\
&- B_{11}\frac{\partial^3 w_b}{\partial x^3} - 3B_{16}\frac{\partial^3 w_b}{\partial x^2 \partial y} - (B_{12} + 2B_{66})\frac{\partial^3 w_b}{\partial x \partial y^2} - B_{26}\frac{\partial^3 w_b}{\partial y^3} = I_0 \ddot{u}
\end{aligned} \tag{18a}$$

$$\begin{aligned}
&A_{16}\frac{\partial^2 u}{\partial x^2} + (A_{12} + A_{66})\frac{\partial^2 u}{\partial x \partial y} + A_{26}\frac{\partial^2 u}{\partial y^2} + A_{66}\frac{\partial^2 v}{\partial x^2} + 2A_{26}\frac{\partial^2 v}{\partial x \partial y} + A_{22}\frac{\partial^2 v}{\partial y^2} \\
&- B_{16}\frac{\partial^3 w_b}{\partial x^3} - (B_{12} + 2B_{66})\frac{\partial^3 w_b}{\partial x^2 \partial y} - 3B_{26}\frac{\partial^3 w_b}{\partial x \partial y^2} - B_{22}\frac{\partial^3 w_b}{\partial y^3} = I_0 \ddot{v}
\end{aligned} \tag{18b}$$

$$\begin{aligned}
&B_{11}\frac{\partial^3 u}{\partial x^3} + 3B_{16}\frac{\partial^3 u}{\partial x^2 \partial y} + (B_{12} + 2B_{66})\frac{\partial^3 u}{\partial x \partial y^2} + B_{26}\frac{\partial^3 u}{\partial y^3} + B_{16}\frac{\partial^3 v}{\partial x^3} + (B_{12} + 2B_{66})\frac{\partial^3 v}{\partial y \partial x^2} \\
&+ 3B_{26}\frac{\partial^3 v}{\partial x \partial y^2} + B_{22}\frac{\partial^3 v}{\partial y^3} - [D_{11}\frac{\partial^4 w_b}{\partial x^4} + 2(D_{12} + 2D_{66})\frac{\partial^4 w_b}{\partial x^2 \partial y^2} + 4D_{16}\frac{\partial^4 w_b}{\partial y \partial x^3} \\
&+ 4D_{26}\frac{\partial^4 w_b}{\partial x \partial y^3} + D_{22}\frac{\partial^4 w_b}{\partial y^4}] + q + k_0 (w_b + w_s) + k_1 \nabla^2 (w_b + w_s) = I_0 (\ddot{w}_b + \ddot{w}_s) - I_2 \nabla^2 \ddot{w}_b
\end{aligned} \tag{18c}$$

$$kA_{55}\frac{\partial^2 w_s}{\partial x^2} + 2kA_{45}\frac{\partial^2 w_s}{\partial x \partial y} + kA_{44}\frac{\partial^2 w_s}{\partial y^2} + q + k_0 (w_b + w_s) + k_1 \nabla^2 (w_b + w_s) = I_0 (\ddot{w}_b + \ddot{w}_s) \tag{18d}$$

3. Analytical Solution for Cross-Ply and Angle-Ply Laminates

We can assume the solutions based on Navier approach, as below:

$$w_s(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}W_{smn}e^{i\omega t}\sin\alpha x\,\sin\beta y$$
$$w_b(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}W_{bmn}e^{i\omega t}\sin\alpha x\,\sin\beta y$$
$$\left.\begin{array}{l} u(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}U_{mn}e^{i\omega t}\cos\alpha x\,\sin\beta y \\ v(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}V_{mn}e^{i\omega t}\sin\alpha x\,\cos\beta y \end{array}\right. \quad (antisymmetric\,cross-ply) \tag{19}$$
$$\left.\begin{array}{l} u(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}U_{mn}e^{i\omega t}\sin\alpha x\,\cos\beta y \\ v(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}V_{mn}e^{i\omega t}\cos\alpha x\,\sin\beta y \end{array}\right. \quad (antisymmetric\,angle-ply)$$

where $i=\sqrt{-1},\alpha=m\pi/a,\beta=n\pi/b$. Also $(U_{mn},V_{mn},W_{bmn},W_{smn})$ are the coefficients, and ω is the frequency of the free vibration. The transverse load q is defined using the double Fourier sinusoidal series expansions as:

$$q(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}Q_{mn}\sin\alpha x\sin\beta y \tag{20}$$

The coefficient Q_{mn} for some typical loads is obtained as follows:

$$Q_{mn}=\frac{4}{ab}\int_0^a\int_0^b q(x,y)\sin\alpha x\sin\beta y\,dxdx=\begin{cases} q_0 & for\,sinusoidal\,load \\ \dfrac{16q_0}{mn\pi^2} & for\,uniform\,load \end{cases} \tag{21}$$

Finally, the matrices related to the analytical solutions are obtained by substituting Eqs. (19) and (20) into Eq. (18):

$$\left(\begin{pmatrix} s_{11} & s_{12} & s_{13} & 0 \\ s_{12} & s_{22} & s_{23} & 0 \\ s_{13} & s_{23} & s_{33} & s_{34} \\ 0 & 0 & s_{34} & s_{44} \end{pmatrix}-\omega^2\begin{pmatrix} I_0 & 0 & 0 & 0 \\ 0 & I_0 & 0 & 0 \\ 0 & 0 & \bar{I}_0 & I_0 \\ 0 & 0 & I_0 & I_0 \end{pmatrix}\right)\begin{Bmatrix} U_{mn} \\ V_{mn} \\ W_{bmn} \\ W_{smn} \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ Q_{mn} \\ Q_{mn} \end{Bmatrix} \tag{22}$$

where,

$$s_{11}=A_{11}\alpha^2+A_{66}\beta^2,\ s_{12}=(A_{12}+A_{66})\alpha\beta,\ s_{22}=A_{66}\alpha^2+A_{22}\beta^2$$
$$s_{33}=D_{11}\alpha^4+2(D_{12}+2D_{66})\alpha^2\beta^2+D_{22}\beta^4+k_0+k_1(\alpha^2+\beta^2)$$
$$s_{44}=k\,(A_{55}\alpha^2+A_{44}\beta^2)+k_0+k_1(\alpha^2+\beta^2),\ s_{34}=k_0+k_1(\alpha^2+\beta^2)$$
$$\left.\begin{array}{l} s_{13}=-B_{11}\alpha^3-(B_{12}+2B_{66})\alpha\beta^2 \\ s_{23}=-B_{22}\beta^3-(B_{12}+2B_{66})\alpha^2\beta \end{array}\right. \quad for\,antisymmetric\,cross-ply \tag{23}$$
$$\left.\begin{array}{l} s_{13}=-B_{26}\beta^3-3B_{16}\alpha^2\beta \\ s_{23}=-B_{16}\alpha^3-3B_{26}\alpha\beta^2 \end{array}\right. \quad for\,antisymmetric\,angle-ply$$
$$\bar{I}_0=I_0+I_2(\alpha^2+\beta^2)$$

It is worth nothing that Eq. (22) is in general form and in the case of the static analysis, it is reduced as follows:

$$\begin{pmatrix} s_{11} & s_{12} & s_{13} & 0 \\ s_{12} & s_{22} & s_{23} & 0 \\ s_{13} & s_{23} & s_{33} & s_{34} \\ 0 & 0 & s_{34} & s_{44} \end{pmatrix} \begin{Bmatrix} U_{mn} \\ V_{mn} \\ W_{bmn} \\ W_{smn} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ Q_{mn} \\ Q_{mn} \end{Bmatrix} \tag{24}$$

4. Numerical Results

In this section, different examples are demonstrated to indicate the accuracy and efficiency of the present formulation. The results are successfully compared with available solutions and those predicted by various laminate theories. To perform a comprehensive comparison, a separate computer code is prepared based on FSDT theory of Reddy [19] and an analytical solution is developed to generate comparable results. Similarly, the results based on the CPT are provided by setting $w_s = 0$ in the present theory. Afterwards, the influences of elastic foundation parameters and laminate stacking sequence on the bending behavior and natural frequencies are illustrated. The symmetric and antisymmetric arrangements of laminated composites are considered which exhibit the efficacy of the model. In static analysis, the plate is subjected to sinusoidal loading. The shear correction factor is considered to be $k = 5/6$. Also, the following material properties are assumed in the solutions.

Material 1: $E_1 = 25E_2,\ G_{12} = G_{13} = 0.5E_2,\ G_{23} = 0.2E_2,\ \nu_{12} = 0.25$ (25)

Material 2: $E_1 / E_2 = \text{variable},\ G_{12} = G_{13} = 0.6E_2,\ G_{23} = 0.5E_2,\ \nu_{12} = 0.25$ (26)

The following non-dimensional parameters are defined throughout the paper:

$$K_0 = \frac{k_0 b^4}{E_2 h^3},\ K_1 = \frac{k_1 b^2}{E_2 h^3},\quad \bar{w} = 100\frac{E_2 h^3}{q_0 a^4} w\left(\frac{a}{2},\frac{b}{2}\right),\quad \bar{\omega} = \frac{\omega a^2}{h}\sqrt{\frac{\rho}{E_2}}$$

$$\bar{\sigma}_x(z) = \frac{h^2}{q_0 a^2}\sigma_x\left(\frac{a}{2},\frac{b}{2},z\right),\ \bar{\sigma}_{xz}(z) = \frac{h^2}{q_0 a^2}\sigma_{xz}\left(0,\frac{b}{2},z\right)$$

4. 1. Bending analysis

Problem 1. Tables 1 and 2 demonstrate respectively the effects of the elastic foundation parameters on the dimensionless central deflections $\bar{w}$ for (0°,90°,0°,90°) and (0°,90°,90°,0°) laminated plates. The results are presented for thin and thick plates with different width-to-thickness ratios (*a/h*). The present solutions are compared with those obtained using the HSDT [14], FSDT and CPT. It is found that the calculated analytical results agree well with the results of HSDT. It is also seen that the CPT provides acceptable results for the laminated plates with *a/h*≥ 20. As expected, the value of $\bar{w}$ decreases with increasing the stiffness of the elastic foundation. By comparing the results reported in Tables 1 and 2, it can be concluded that the dimensionless transverse central deflection of symmetric laminates is smaller than those of antisymmetric laminates. This trend is clarified in Fig. 2. It is seen that the difference in bending deformation of the aforementioned laminates is decreased when the elastic foundation parameters are significantly increased.

Problem 2. In this problem, a two-layer cross-ply square laminated composite plate is considered. The effects of Winkler and Pasternak elastic parameters on the dimensionless transverse deflection of laminates with different width-to-thickness ratios are investigated in Table 3. The developed results are compared with the solution presented in Ref. [18]. This comparison demonstrates clearly the effectiveness and accuracy of the present model as Setoodeh and Karami [18] used a three-dimensional layerwise theory (LW3D).

Table 1. Dimensionless transverse central deflection $\bar{w}$ of (0°,90°,0°,90°) laminated square plates (material 1).

a/h	Theory	K_0=0,K_1=0	K_1=100,K_1=0	K_0=100,K_1=10
5	CPT	0.5065	0.2301	0.1108
	FSDT	1.2013	-	-
	Present	1.2013	0.5457	0.2627
	HSDT [14]	1.2455	0.4920	0.2561
10	CPT	0.5065	0.3015	0.1676
	FSDT	0.6802	-	-
	Present	0.6802	0.4048	0.2250
	HSDT [14]	0.7551	0.4001	0.2559
20	CPT	0.5065	0.3268	0.1922
	FSDT	0.5500	-	-
	Present	0.5500	0.3548	0.2087
100	CPT	0.5065	0.3358	0.2017
	FSDT	0.5083	-	-
	Present	0.5083	0.3370	0.2023

Problem 3. The static analysis of a square two-layer angle-ply $(\theta,-\theta)$ laminate is firstly studied and then the resulted bending deformations are compared with those of similar cross-ply laminate as depicted in Fig. 3. Three different ply-orientation angles of (15°,-15°), (30°,-30°), and (45°,-45°) are considered for angle-ply laminates. The variations of the dimensionless central transverse deflection with respect to (*a/h*) ratio are presented. It is observed that the values of $\bar{w}$ for all of the angle-ply laminates under consideration are smaller than the corresponding deflections predicted for the cross-ply arrangement.

Table 2. Dimensionless transverse central deflection $\bar{w}$ of (0°,90°,90°,0°) symmetric square laminates (material 1).

a/h	Theory	K_0=0,K_1=0	K_1=100,K_1=0	K_0=100,K_1=10
5	CPT	0.4312	0.2028	0.0992
	FSDT	1.2801	-	-
	Present	1.1260	0.5296	0.2589
10	CPT	0.4312	0.2687	0.1541
	FSDT	0.6627	-	-
	Present	0.6049	0.3769	0.2161
20	CPT	0.4312	0.2924	0.1788
	FSDT	0.4912	-	-
	Present	0.4747	0.3219	0.1968
100	CPT	0.4312	0.3009	0.1885
	FSDT	04337	-	-
	Present	0.4330	0.3021	0.1893

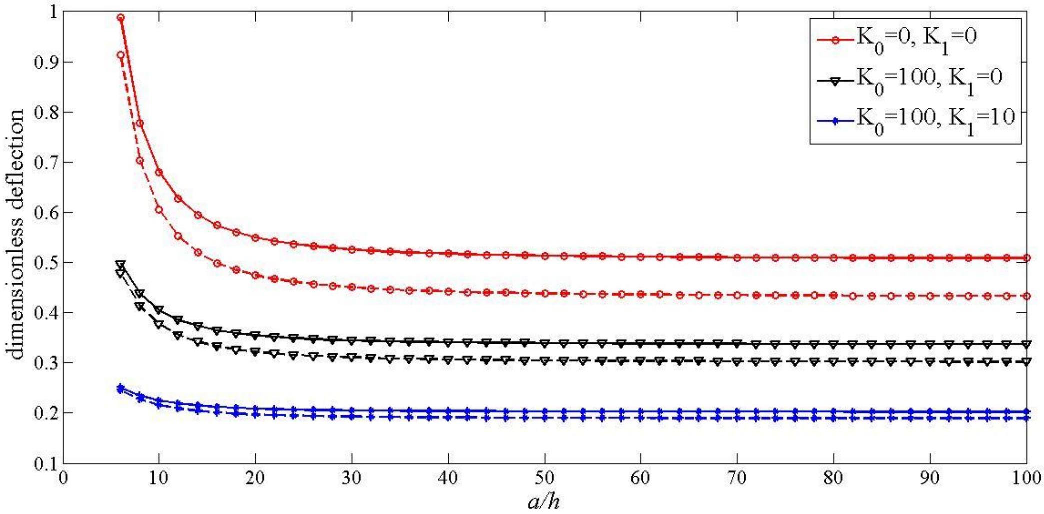

Fig. 2. Variation of dimensionless central deflection $\overline{w}$ of (0°,90°,90°,0°) symmetric (dash lines) and (0°,90°,0°,90°) antisymmetric (solid lines) laminates versus (*a/h*) ratio for different elastic foundation parameters (material 1).

Table 3. Dimensionless transverse central deflection $\overline{w}$ of a two-layer cross-ply square laminate (material 1).

a/h	Theory	K_0=0,K_1=0	K_0=100,K_1=0	K_0=100,K_1=10
5	CPT	1.0636	0.3855	0.1707
	FSDT	1.7583	-	-
	Present	1.7583	0.6374	0.2822
	LW3D[18]	1.6671	0.6639	0.3374
10	CPT	1.0636	0.4754	0.2273
	FSDT	1.2373	-	-
	Present	1.2373	0.5530	0.2644
	LW3D[18]	1.2162	0.5525	0.2679

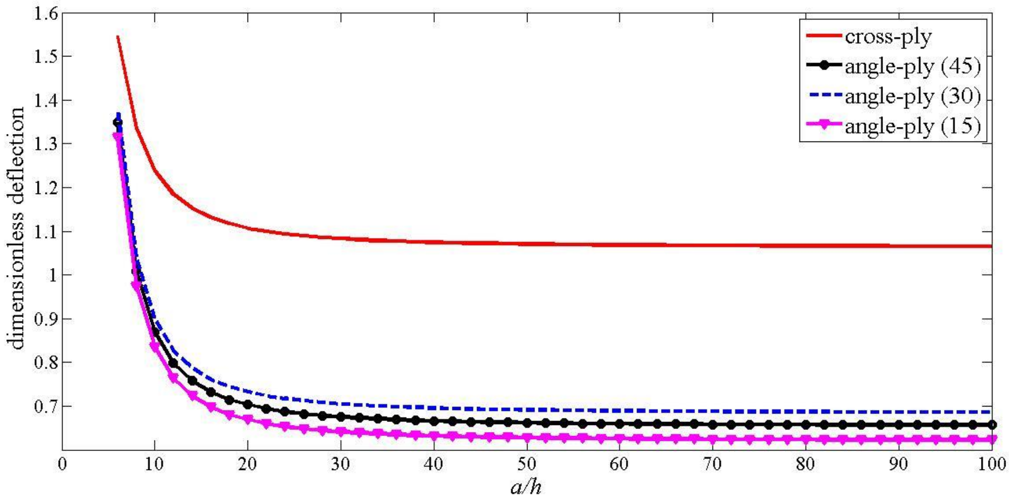

Fig. 3. Variations of dimensionless central deflection of two-layer cross-ply (0°,90°) and angle-ply $(\theta,-\theta)$ laminates with three different ply-orientation angles versus (*a/h*) ratio (material 1).

Problem 4. This problem is performed for bending analysis of four-layer angle-ply (45°,-45°,-45°,45°) symmetric and (45°,-45°,45°,-45°) antisymmetric square laminates with different (*a/h*) and $\left(E_1/E_2\right)$ ratios. The non-dimensional transverse central deflections are determined for Winkler and Pasternak elastic parameters as listed in Table 4. As mentioned before, similar solutions are developed employing the CPT and classical first-order shear deformation theory. The good agreement between the results of the present theory and those obtained by the classical FSDT is obvious. Furthermore, The CPT exhibits acceptable results only for thin laminated composite plates due to neglect of the shear deformation effect. Also, it is seen that the value of $\bar{w}$ increases with decreasing the degree of orthotropy.

Problem 5. In the next problem, a stress analysis is carried out for a square four-layer angle-ply laminate arranged as $(0°,90°)_2$ with various $\left(E_1/E_2\right)$ ratios. The variations of the dimensionless normal and transverse stress components in the *x*-direction with respect to different values of the elastic foundation parameters are illustrated in Table 5. It is observed that $\bar{\sigma}_x$ decreases with increasing the elastic foundation stiffness or with decreasing the $\left(E_1/E_2\right)$ ratio.

4.2. Free vibration analysis

Problem 6. In this problem, free vibration analysis of square antisymmetric cross-ply laminated plates is investigated using Eq. (22) in the absence of external load. In Table 6, the dimensionless natural frequencies of laminates obtained by using different theories are shown for various values of $\left(E_1/E_2\right)$ ratios. The present results are compared with those reported in Ref. [17] using the HSDT, and the three-dimensional elasticity (3D) solutions given in Ref. [20]. It is observed that the present approach can provide accurate results in comparison with the three-dimensional elasticity solutions. It is found that the dimensionless fundamental frequency increases with increasing the orthotropy $\left(E_1/E_2\right)$ ratio.

Problem 7. In the next problem, a square three-layer cross-ply laminate is considered. The obtained dimensionless fundamental frequencies for different elastic foundation parameters and width-to-thickness ratios (*a/h*) are demonstrated in Table 7. The present results are compared with those predicted using the HSDT [14], FSDT and CPT. It is observed that the present solution is in close agreement with the results reported in Ref. [16]. It can be also concluded that the frequencies of laminates increase with increasing the stiffness of the elastic foundation.

Table 4. Dimensionless transverse central deflection $\bar{w}$ of square four-layer angle-ply laminates with various orientations (material 2).

(E_1/E_2) ratio	***a/h***	**Theory**	$\left(45^\circ,-45^\circ,45^\circ,-45^\circ\right)$			$\left(45^\circ,-45^\circ,-45^\circ,45^\circ\right)$		
			K_0=0,K_1=0	**K_0=100,K_1=0**	**K_0=100,K_1=10**	**K_0=0, K_1=0**	**K_0=100,K_1=0**	**K_0=100,K_1=10**
40	10	CPT	0.1806	0.1399	0.0968	0.1500	0.1190	0.0845
		FSDT	0.2912	-	-	0.2605	-	-
		Present	0.2912	0.2255	0.1560	0.2605	0.2067	0.1468
	100	CPT	0.1806	0.1529	0.1173	0.1500	0.1303	0.1035
		FSDT	0.1817	-	-	0.1511	-	-
		Present	0.1817	0.1538	0.1180	0.1511	0.1313	0.1042
30	10	CPT	0.2372	0.1760	0.1166	0.1983	0.1515	0.1033
		FSDT	0.3477	-	-	0.3088	-	-
		Present	0.3477	0.2580	0.1709	0.3088	0.2359	0.1610
	100	CPT	0.2372	0.1915	0.1388	0.1983	0.1653	0.1245
		FSDT	0.2383	-	-	0.1994	-	-
		Present	0.2383	0.1924	0.1395	0.1994	0.1662	0.1252
10	10	CPT	0.6363	0.3643	0.1975	0.5565	0.3338	0.1865
		FSDT	0.7469	-	-	0.6670	-	-
		Present	0.7469	0.4275	0.2319	0.6670	0.4001	0.2235
	100	CPT	0.6363	0.3886	0.2197	0.5565	0.3573	0.2093
		FSDT	0.6374	-	-	0.5576	-	-
		Present	0.6374	0.3893	0.2201	0.5576	0.3580	0.2098

Table 5. Dimensionless stress components of a square four-layer angle-ply laminate arranged as $(0°,90°)_2$ with various (E_1/E_2) ratios (material 2).

(E_1/E_2) ratio	$\bar{\sigma}_x$			$\bar{\sigma}_{xz}$		
	K_0=0, K_1=0	K_1=100,K_1=0	K_0=100,K_1=10	K_0=0, K_1=0	K_0=100,K_1=0	K_0=100,K_1=10
40	-0.5061	-0.3504	-0.2180	0.1736	0.1202	0.0748
30	-0.4905	-0.3188	-0.1886	0.1736	0.1128	0.0667
10	-0.3988	-0.1904	-0.0937	0.1736	0.0829	0.0408

Table 6. Dimensionless fundamental frequency $\bar{\omega}$ of square antisymmetric cross-ply laminates (material 2, *a/h*=5).

lamination	**Theory**	E_1/E_2=10	E_1/E_2=20	E_1/E_2=40
(0/90)1	3D [20]	6.9845	7.6745	8.5625
	HSDT [17]	6.9839	7.8095	9.0610
	Present	6.9392	7.7060	8.8333
(0/90)2	3D [20]	8.1445	9.4055	10.6790
	HSDT [17]	8.1999	9.6353	11.1853
	Present	8.2246	9.6885	11.2708
(0/90)3	3D [20]	8.4143	9.8398	11.2720
	HSDT [17]	8.4069	9.9205	11.5019
	Present	8.4183	9.9427	11.5264
(0/90)5	3D [20]	8.5625	10.0843	11.6245
	HSDT [17]	8.5131	10.0670	11.6682
	Present	8.5132	10.0638	11.6444

Table 7. Dimensionless fundamental frequency $\bar{\omega}$ of a square three-layer cross-ply laminate resting on elastic foundation (material 2, E_1/E_2=40).

a/h	**Theory**	K_0=0,K_1=0	K_1=100,K_1=0	K_0=100,K_1=10
5	CPT	18.299	20.704	24.777
	HSDT [16]	10.265	14.246	19.880
	Present	11.707	15.364	20.769
10	CPT	18.738	21.201	25.371
	HSDT [16]	14.700	17.751	22.595
	Present	15.930	18.786	23.423
20	CPT	18.853	21.331	25.526
	HSDT [16]	17.481	20.131	24.535
	Present	17.993	20.578	24.902
50	CPT	18.885	21.368	25.570
	HSDT [16]	18.640	21.152	25.390
	Present	18.738	21.238	25.462

5. Conclusion

In this article, a refined shear deformation theory is employed for bending and free vibration analyses of simply supported rectangular laminated composite plates resting on elastic foundation. The Winkler–Pasternak two-parameter foundation model is considered in the analysis and a closed form solution is developed. The effects of laminate stacking sequence, width-to-thickness ratio, elastic foundation parameters and orthotropy ratio on the bending deformation and natural frequencies of plates are investigated. Comparison studies are performed to verify the validity and efficacy of the present model. It is exhibited that the theory is capable of predicting reliable results for thick and moderately thick laminated composites with reducing the computational cost.

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