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EXPONENTIAL STABILITY OF UNCERTAIN SWITCHED LINEAR SYSTEMS*

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Abstract– In this paper, sufficient conditions are proposed to investigate the robust stability of arbitrary switched linear systems with uncertain parameters belongs to the known intervals. In addition, a method is then established to determine the maximum intervals of parameters' variations which guarantee robust exponential stability of uncertain switched linear systems under arbitrary switching. In the proposed method, the known information about the parametric structure of uncertainties is considered; therefore it will result in less conservative stability margins. A generalization of the method is also provided to determine stability bounds on perturbations of entries in subsystem matrices, when subsystems are subjected to independent perturbations. Numerical examples are included to illustrate the effectiveness of the results, and compare them with the previous results. It is shown that the proposed methods provide stability intervals on the uncertain parameter for all switched linear systems which admit a common quadratic Lyapunov function for the nominal system.

Keywords– Switched linear system, exponential stability, parametric uncertainty, arbitrary switching signal

1. INTRODUCTION

Switched linear systems are an important class of dynamic systems which consist of a collection of linear time-invariant subsystems and a switching signal to arrange the switching between the subsystems. Switched linear systems have received growing attention, due to the wide range of their applications in modeling [1-3], control [4-7], and stability analysis [8-11] of complex nonlinear systems. When the switching mechanism is undetermined, or too complicated to be useful in the stability analysis, switched linear systems should be investigated under arbitrary switching [12, 13]. In this paper, we consider the continuous-time arbitrary switched linear systems of the form

$$\dot{x}(t) = A_{\sigma(t)}x(t), \tag{1}$$

where $x \in \mathbb{R}^n$ is the vector of continuous states and $\sigma(t): \mathbb{R}^+ \to \mathcal{M} = \{1,2,\dots,m\}$ is the switching signal, generated by an unknown or nondeterministic left continuous piecewise constant function.

So far, stability of arbitrary switched linear systems without perturbations has been investigated in many publications. It has been proven that an arbitrary switched linear system is globally asymptotically stable, if and only if a common Lyapunov function (CLF) exists for all subsystems. In many articles, common quadratic Lyapunov functions (CQLFs) are focused on, due to the ability of CQLFs in converting the stability analysis to a set of linear matrix inequalities (LMIs). In [14-21] and references therein, necessary and/or sufficient conditions were proposed to guarantee the existence of a CQLF for switched linear systems. In addition, some Lie algebraic condition was proposed in [22, 23], for existence of a

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CQLF. It is also shown in [19], that existence of a CQLF guarantees the exponential stability of an arbitrary switched linear system. Besides, some efforts have also been made to guarantee the existence of a CQLF for special classes of switched linear systems, such as switched positive linear systems (SPLSs) [24, 25]. Since the states of SPLSs are naturally nonnegative, the CQLF based stability conditions are generally conservative. Hence, some other forms of common positive Lyapunov functions are introduced in [26-28] for SPLSs.

In the stability analysis and control of dynamic systems, uncertainties play a critical role [29-31]. Robust stability of arbitrary switched systems has been investigated in several references, such as [32-36]. In [32], sufficient conditions are proposed based on the generalized matrix measure to guarantee the robust local stability of arbitrary switched nonlinear systems with unstructured uncertainties. The conditions are only applicable for robust stability analysis of switched linear systems, if all eigenvalues of $(A_k + A_k')$ are negative for each subsystem $k \in \mathcal{M}$. The cycle analysis method and the generalized matrix measure are also utilized in [33] to obtain some robust stability conditions for switched nonlinear systems. In [34], robust stability of a class of switched linear systems was investigated, in which the n-dimensional subsystem matrices share n-1 linearly independent common left eigenvectors. The results of [23] are also extended in [35] to offer the robust stability conditions based on the definition of closeness of the collection of subsystems to one with nice commutation relations. Some conditions, in terms of smallness of appropriate commutators of the subsystem matrices, are also formulated in [36] to guarantee the robust stability of arbitrary switched linear systems with unstructured uncertainties. In addition, robust quadratic stability of a class of switched linear systems which share a common invariant subspace is addressed in [37], utilizing the concept of invariant subspaces. It should be noted that the considered uncertainties in [32-37] are of the unstructured type, which does not possess obvious physical interpretation. Moreover, if robust stability of parametric uncertain switched linear systems is investigated by the theorems for unstructured uncertain systems, the known information about the structure of uncertainties should be ignored. Therefore, additional uncertainties which will never arise in the given system will be included, and the conservativeness will be increased [38]. So, a parametric approach is needed to decrease the conservations in stability analysis of parametric uncertain switched linear systems.

In this paper, robust exponential stability of arbitrary switched linear systems with parametric uncertainties is addressed. These kinds of uncertainties arise in practice, when state equations are derived from physical laws, based on the uncertain physical parameters. First, sufficient conditions are proposed to investigate robust exponential stability of arbitrary switched linear systems, assuming that uncertain parameters belong to known intervals. Then, upper bounds on parametric uncertainties are determined, such that robust exponential stability of switched linear systems is ensured under arbitrary switching signals. Since the estimated upper bounds depend on the choice of CQLFs, an optimization algorithm is also offered to calculate the maximum guaranteed stability intervals. Moreover, the proposed method is generalized to the unstructured uncertain switched linear systems with independent perturbations on the entries of subsystem matrices. The presented results are applicable for all switched linear systems, in which the nominal system is quadratically stable with a CQLF $V(x) = x'Px$, while the previous publications may not provide stability bounds for such cases (see, e.g., the first example in section 5, where [32, 35] do not offer any bound). It should be noted that a parametric uncertain switched linear system, with r uncertain physical parameters, may also be considered as a polytopic switched linear system with 2^r vertices. So, complexity of theorems for polytopic uncertain systems will be increased considerably when number of uncertain parameters increases, while the maximum stability intervals will not be significantly increased. It is also illustrated in the numerical examples of Section 5, that maximal guaranteed stability intervals of this paper may be larger or even infinite from one side, compared with stability results of polytopic uncertain systems such as [39]. Numerical examples illustrate the effectiveness and simplicity of the proposed results, compared with previous results.

Notations: Throughout this paper, the following standard notations are used. $\mathbb{R}^n$ denotes all n-dimensional real valued vectors, and $\mathbb{R}^{n\times m}$ is the space of $n \times m$ matrices with real entries. For $x \in \mathbb{R}^n$, $\|x\|_2$ denotes Euclidean norm of vector x. For a square matrix $A \in \mathbb{R}^{n\times n}$, A' indicates transpose of matrix A, and $\|A\|_2$ indicates 2-norm of matrix A which is calculated as $\|A\|_2 = \max_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2}$. $\rho(A)$ is also used to show negative of the maximum real part of the eigenvalues of A. Moreover, the largest and smallest singular values of matrix A are shown by $\sigma_{max}(A)$ and $\sigma_{min}(A)$ respectively. In addition, we mean by $A > 0$ (or $A \geq 0$) that matrix A is positive (semi) definite, while $A < 0$ (or $A \leq 0$) means that matrix A is negative (semi) definite.

2. PROBLEM FORMULATIONS AND PRELIMINARIES

In the stability analysis of switched linear systems, uncertainties of subsystem matrices play an important role. In this section, we will formulate the parametric uncertainty, which usually arises when state equations are derived based on physical considerations, but some physical parameters contain uncertainty. Parametric uncertainties may be caused by inability in precisely measuring parameters, or actual parameter variations during system operation. In the parametric uncertain switched linear systems, some entries of subsystem matrices depend on the uncertain parameters. Let us consider parametric uncertain subsystem matrices A_k as

$$A_k(Q) = A_k^o + \Delta A_k(Q^o, \Delta Q) \quad (2a)$$

$$\Delta A_k(Q^o, \Delta Q) = \sum_{i=1}^{r} \delta q_i E_i^k \quad (2b)$$

where $Q = [q_1, q_2, \ldots, q_r]$ is the vector of physical parameters, $Q^o = [q_1^o, q_2^o, \ldots, q_r^o]$ is its nominal value, and $\Delta Q = [\delta q_1, \delta q_2, \ldots, \delta q_r]$ is the perturbation around the nominal value of parameters vector ($Q = Q^o + \Delta Q$). A_k^o is also defined as $A_k^o \coloneqq A_k(Q^o)$ and E_i^k is the uncertainty structure matrix, describing how A_k depends on the uncertain parameter q_i. If physical parameter q_j does not enter in the *l*-th subsystem, E_j^l is a null matrix. Here, we present the definitions and lemmas which are used in the proceeding sections.

Definition 1 [40]: The ratio of the largest singular value to the smallest one in the singular value decomposition of a matrix A is called the condition number of matrix A and is shown by $\mathcal{K}(A)$.

Definition 2 [33]: The modulus matrix of A is shown by $|A|_m$ and is defined as $|A|_m \coloneqq \left[|a_{ij}|\right]$.

Lemma 1: For each matrix A, norm inequality $\|A\|_2 \leq \|\,|A|_m\|_2$ holds.

Proof: Let $y \in \mathbb{R}^n$ be a vector such that $\|A\|_2 = \max_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \frac{\|Ay\|_2}{\|y\|_2}$. It should be noted that $\|x\|_2 = \||x|_m\|_2$, since $\|x\|_2^2 = \sum_{i=1}^n x_i^2 = \sum_{i=1}^n |x_i|^2 = \||x|_m\|_2^2$. Therefore, $\|Ax\|_2^2 = \||Ax|_m\|_2^2 = \sum_{i=1}^n \left|\sum_{j=1}^n a_{ij}x_j\right|^2 \leq \sum_{i=1}^n \left|\sum_{j=1}^n |a_{ij}x_j|\right|^2 = \sum_{i=1}^n \left|\sum_{j=1}^n (|a_{ij}||x_j|)\right|^2 = \||A|_m|x|_m\|_2^2$ and $\|Ax\|_2 \leq \||A|_m|x|_m\|_2$. So, by setting $x = y$, it will be concluded that $\|A\|_2 = \frac{\|Ay\|_2}{\|y\|_2} \leq \frac{\||A|_m z\|_2}{\|z\|_2}$, where $z = |y|_m$. In addition, $\frac{\||A|_m z\|_2}{\|z\|_2} \leq \max_{x\neq 0} \frac{\||A|_m x\|_2}{\|x\|_2}$ holds. Hence, $\|A\|_2 \leq \|\,|A|_m\|_2$ holds for all matrices A.

Lemma 2: (Rayleigh–Ritz ratio [41]) for a given positive definite symmetric matrix $M \in \mathbb{R}^{n\times n}$, inequalities $\sigma_{min}(M)\|x\|_2^2 \leq x'Mx \leq \sigma_{max}(M)\|x\|_2^2$ hold for any nonzero vector $x \in \mathbb{R}^n$.

Assumption 1: It is assumed that a CQLF exists for the nominal switched linear system $\dot{x}(t) = A_{\sigma(t)}^o x(t)$. In other words, there exists a positive definite matrix P, such that $A_k^{o\prime} P + P A_k^o < 0$ holds for all $k \in \mathcal{M}$.

3. STABILITY OF ARBITRARY SWITCHED LINEAR SYSTEMS WITH PARAMETRIC UNCERTAINTIES

In this section, sufficient conditions are first proposed to investigate whether an arbitrary switched linear system with parameter uncertainties belongs to known intervals, $\delta q_i \in [-c_i, +c_i]$ is robustly exponentially stable. Then, another theorem is presented to estimate how wide physical parameters of a switched linear system can vary from their nominal values, such that robust exponential stability of uncertain arbitrary switched system is guaranteed.

Theorem 1: Consider an uncertain switched linear system with A_k described as (2) and $\delta q_i \in [-c_i, +c_i]$. The switched linear system is robustly exponentially stable under arbitrary switching signal, if a positive definite matrix $P \geq I$ and positive scalars α_k exist such that $A_k^{o\prime} P + P A_k^o \leq -\alpha_k P$ and inequality (3) holds.

$$\left\| \sum_{i=1}^{r} c_i \left| E_i^k \right|_m \right\|_2 < \frac{\alpha_k}{2\|P\|_2} \;\; ; \;\; \forall k \in \mathcal{M} \tag{3}$$

Proof: According to [19], existence of a CQLF is a sufficient condition for exponential stability of a switched linear system. Therefore, uncertain arbitrary switched linear system (1), (2) is robustly exponentially stable, if CQLF $V(x) = x'Px$ stays decreasing ($\dot{V}(x) < 0$) along with the trajectories of each subsystem $k \in \mathcal{M}$, for all uncertainties of parameters, satisfying (3). In other words, the uncertain switched linear system (1), (2) will be robustly exponentially stable, if

$$x'\left(A_k^{o\prime} P + P A_k^o\right)x + x'\left(\sum_{i=1}^{r} \delta q_i \left[E_i^{k\prime} P + P\, E_i^k\right]\right)x < 0 \tag{4}$$

for all $k \in \mathcal{M}$ and $x \in \mathbb{R}^n$.

Since $V(x) = x'Px$ is a CQLF for nominal switched linear system, $A_k^{o\prime} P + P A_k^o = -R_k < 0$ holds for all $k \in \mathcal{M}$ where R_k is a positive definite matrix. So, (4) will be assured if $\sum_{i=1}^{r} \delta q_i \left[E_i^{k\prime} P + P\, E_i^k\right] < R_k$ for each subsystem $k \in \mathcal{M}$. In addition, according to Lemma 2, inequality $\sum_{i=1}^{r} \delta q_i \left[E_i^{k\prime} P + P\, E_i^k\right] < R_k$ holds, if

$$\left\| \sum_{i=1}^{r} \delta q_i \left[E_i^{k\prime} P + P\, E_i^k\right] \right\|_2 < \sigma_{min}(R_k) \tag{5}$$

for all $k \in \mathcal{M}$, where $\left\| \sum_{i=1}^{r} \delta q_i \left[E_i^{k\prime} P + P\, E_i^k\right] \right\|_2 \leq 2\|P\|_2 \left\| \sum_{i=1}^{r} \left(\delta q_i E_i^k\right) \right\|_2$, since $\|AB\|_2 \leq \|A\|_2 \|B\|_2$ holds for any pair of matrices A and B, with appropriate dimensions [42]. Moreover, Lemma 1 shows that $\left\| \sum_{i=1}^{r} \left(\delta q_i E_i^k\right) \right\|_2 \leq \left\| \sum_{i=1}^{r} \left|\delta q_i E_i^k\right|_m \right\|_2 = \left\| \sum_{i=1}^{r} |\delta q_i| \left|E_i^k\right|_m \right\|_2 \leq \left\| \sum_{i=1}^{r} c_i \left|E_i^k\right|_m \right\|_2$ for all $k \in \mathcal{M}$. Therefore, (5) will be satisfied if

$$\left\| \sum_{i=1}^{r} c_i \left| E_i^k \right|_m \right\|_2 < \frac{\sigma_{min}(R_k)}{2\|P\|_2} \tag{6}$$

for all $k \in \mathcal{M}$.

Besides, since positive scalar α_k exist such that $A_k^{o\prime} P + P A_k^o \leq -\alpha_k P$, it will be concluded that $\sigma_{min}\left(A_k^{o\prime} P + P A_k^o\right) \geq \alpha_k \sigma_{min}(P)$. So, (6) will be satisfied if inequality

$$\left\| \sum_{i=1}^{r} c_i \left| E_i^k \right|_m \right\|_2 < \frac{\alpha_k \sigma_{min}(P)}{2\|P\|_2} \tag{7}$$

holds. Moreover, $\sigma_{min}(P) \geq 1$ holds, since $P \geq I$. Therefore, inequality (7) will be satisfied, if (3) holds, and the proof is completed.

Now, upper bounds on intervals of the uncertain parameters will be determined, such that robust stability is ensured. To estimate the parametric stability margins, parametric uncertainties are considered as $\delta q_i = \gamma w_i$, where $w_i > 0\,; i \in \{1,2,\dots,r\}$ indicates the weight of uncertainty on the i-th parameter and

$\gamma > 0$ is also the weighted bound of uncertainties. So, the problem will be changed to finding maximum value of γ, such that uncertain switched linear system is robustly exponentially stable for $\delta q_i \in [-\gamma w_i, +\gamma w_i]$. Theorem 2 gives an upper bound on γ, considering that Assumption 1 is satisfied.

Theorem 2: Consider a switched linear system with uncertain matrices A_k described as (2). Assume that a positive definite matrix P and positive scalars α_k exist such that $A_k^{o\prime} P + P A_k^o \leq -\alpha_k P$. The arbitrary switched linear system will be robustly exponentially stable, for all $\delta q_i \in [-\gamma w_i, +\gamma w_i]$, if γ holds in

$$\gamma < \frac{\alpha_k}{2\mathcal{K}(P)\left\|\sum_{i=1}^{r}\left(w_i |E_i^k|_m\right)\right\|_2} \tag{8}$$

for all $k \in \mathcal{M}$. In addition, for $j \in \{1,2,\dots,r\}$, if matrices $\left[E_j^{k\prime} P + P E_j^k\right]$; $\forall\, k \in \mathcal{M}$ are negative semi-definite (or positive semi-definite), the uncertain switched system will also be exponentially stable for δq_j which belongs to the following interval:

$$\delta q_j \in [-\gamma w_j,\ +\infty)\text{, if } \left[E_j^{k\prime} P + P E_j^k\right] \leq 0\,;\ \forall k \in \mathcal{M} \tag{9a}$$

$$\delta q_j \in (-\infty,\ +\gamma w_j]\text{, if } \left[E_j^{k\prime} P + P E_j^k\right] \geq 0\,;\ \forall k \in \mathcal{M} \tag{9b}$$

Proof: Similar to the proof of Theorem 1, the uncertain switched linear system will be robustly exponentially stable, if inequality (4) holds for all $k \in \mathcal{M}$ and $x \in \mathbb{R}^n$. Inequality (4) is rewritten as

$$\left(\sum_{i=1}^{r} \delta q_i \left[E_i^{k\prime} P + P E_i^k\right]\right) < -\left(A_k^{o\prime} P + P A_k^o\right)\,;\ \forall k \in \mathcal{M} \tag{10}$$

Since $V(x) = x^T P x$ is a CQLF for the nominal switched linear system, and $A_k^{o\prime} P + P A_k^o \leq -\alpha_k P$ holds, then (10) will be satisfied, if $\left(\sum_{i=1}^{r} \delta q_i \left[E_i^{k\prime} P + P E_i^k\right]\right) < \alpha_k P$ for each subsystem $k \in \mathcal{M}$. Moreover, according to Rayleigh–Ritz ratio (Lemma 2), $\left(\sum_{i=1}^{r} \delta q_i \left[E_i^{k\prime} P + P E_i^k\right]\right) < \alpha_k P$ will be satisfied if

$$\left\|\sum_{i=1}^{r} \delta q_i \left[E_i^{k\prime} P + P E_i^k\right]\right\|_2 < \alpha_k \sigma_{min}(P) \tag{11}$$

for all $k \in \mathcal{M}$. Moreover, Lemma 1 ensures that $\left\|\sum_{i=1}^{r} \delta q_i \left[E_i^{k\prime} P + P\, E_i^k\right]\right\|_2 \leq 2\gamma \|P\|_2 \left\|\sum_{i=1}^{r}\left(w_i |E_i^k|_m\right)\right\|_2$ for each $k \in \mathcal{M}$. So, (11) will be satisfied if $\left\|\sum_{i=1}^{r}\left(w_i |E_i^k|_m\right)\right\|_2 \gamma < \frac{\alpha_k}{2\gamma\mathcal{K}(P)}$, or equally (8) holds. Thus, the proposed uncertain switched linear system with $\delta q_i \in [-\gamma w_i, +\gamma w_i]$ will be exponentially stable if γ holds in (8).

In addition, if there exists a $j \in \{1,2,\dots,r\}$, such that matrices $\left[E_j^{k\prime} P + P E_j^k\right]$; $\forall k \in \mathcal{M}$ are all negative semi-definite, it is easy to verify that (10) holds for all $\delta q_j \geq 0$ and $\delta q_i \in [-\gamma w_i, +\gamma w_i]\,; i \neq j$, where γ holds in (8). Hence, if $\delta q_j \in [-\gamma w_j, +\gamma w_j] \cup [0, +\infty) = [-\gamma w_j, +\infty)$, the proposed switched linear system will be robustly exponentially stable. Correspondingly, if matrices $\left[E_j^{k\prime} P + P E_j^k\right]$ are positive semi-definite for a $j \in \{1,2,\dots,r\}$ and all subsystems $k \in \mathcal{M}$, the considered switched system will be robustly exponentially stable for $\delta q_j \in (-\infty,\ +\gamma w_j]$. So, the proof is completed.

Remark 1: If weights of parameter uncertainties are chosen equal to the nominal values of parameters, i.e., $w_i = q_i^o$, the uncertain parameters will belong to $q_i \in q_i^o[1-\gamma, 1+\gamma]$. In other words, the resulted γ from Theorem 2 shows the percent tolerance bound of parameters ($q_i^o \pm (100\gamma)\%$).

Remark 2: As it was mentioned in the Introduction, if switched linear systems (2) are considered as polytopic switched linear systems, stability bounds on the physical parameters will not significantly be increased, while the complexity will be increased so much for the large number of unknown parameters. Moreover, in some examples, especially when matrices $\left[E_j^{k\prime} P + P E_j^k\right]$ are negative semi-definite (or positive semi-definite) for all $k \in \mathcal{M}$, applying Theorem 2 will result in larger stability margins than those theorems for polytopic uncertain switched linear systems.

Remark 3: The proposed theorems for parametric switched linear systems can be easily extended to unstructured uncertain switched linear systems, in which subsystem matrices are exposed to independent perturbations of entries. For this purpose, we need to consider each uncertain entry as an uncertain parameter. Therefore, an unstructured uncertain switched linear system can be considered as a parametric switched linear system with at most n^2 uncertain parameters.

The following corollary is a generalization of Theorem 2, which gives upper stability bounds on entry perturbations in arbitrary switched linear systems. For this purpose, the unstructured uncertainties of subsystem matrices $\Delta A_k = \left[\delta a_{i,j}^k\right]_{n\times n}$ are considered as

$$\Delta A_k = \sum_{j=1}^{n} \sum_{i=1}^{n} \delta a_{i,j}^k E_{i,j}, \tag{12}$$

where, $\delta a_{i,j}^k$ shows the perturbation of (i,j)-th entry in the k-th subsystem matrix, and uncertainty structure matrix $E_{i,j}$ is defined as the sparse matrix $E_{i,j} \coloneqq [e_{s,t}]_{n\times n}$, where

$$e_{s,t} = \begin{cases} 1 & ; s = i\ ,\ t = j \\ 0 & ; else\ where \end{cases}. \tag{13}$$

Similarly, the entry perturbations are considered as $\delta q_{i,j}^k = \gamma_k w_{i,j}^k$, where $w_{i,j}^k > 0\ ;\ i,j \in \{1,2,\dots,n\},\ k \in \mathcal{M}$, indicates the given weight of perturbation of the (i,j)-th entry, and $\gamma_k > 0$ indicates the weighted perturbation of the k-th subsystem. So, the problem will be changed to finding the maximum values of γ_k, such that uncertain switched linear system (1), (12) is robustly exponentially stable for $\delta a_{i,j}^k \in \left[-w_{i,j}^k\gamma_k\ ,+w_{i,j}^k\gamma_k\right]$.

Corollary 1: Consider an uncertain switched linear system (1), (12) with $\delta a_{i,j}^k \in \left[-w_{i,j}^k\gamma_k\ ,+w_{i,j}^k\gamma_k\right]$. The arbitrary switched linear system is robustly exponentially stable if a positive definite matrix P and positive scalars $\alpha_k;\ \forall k \in \mathcal{M}$ exist, such that $A_k^{o\prime}P + PA_k^o \leq -\alpha_k P$ and γ_k holds in

$$\gamma_k < \frac{\alpha_k}{2\mathcal{K}(P)\left\|W^k\right\|_2}, \tag{14}$$

for each $k \in \mathcal{M}$, where W^k is defined as $W^k \coloneqq [w_{i,j}^k]_{n\times n}$.

Proof: From Theorem 2, it is known that uncertain switched system (1), (12), and (13) is robustly exponentially stable under arbitrary switching signal, if $\gamma_k < \frac{\alpha_k}{2\mathcal{K}(P)\left(\left\|\sum_{i,j=1}^{n} w_{i,j}^k \left|E_{i,j}\right|_m\right\|_2\right)}$. Since $\left|E_{i,j}\right|_m = E_{i,j}$ for all $i,j \in \{1,2,\dots,n\}$, then $\sum_{i,j=1}^{n} w_{i,j}^k \left|E_{i,j}\right|_m = W^k \coloneqq [w_{i,j}^k]_{n\times n}$, and $\frac{\alpha_k}{2\mathcal{K}(P)\left(\left\|\sum_{i,j=1}^{n} w_{i,j}^k \left|E_{i,j}\right|_m\right\|_2\right)} = \frac{\alpha_k}{2\mathcal{K}(P)\left\|W^k\right\|_2}$. Thus, the proposed uncertain switched linear system is robustly exponentially stable, if γ_k holds in (14).

4. MAXIMUM GUARANTEED STABILITY INTERVALS

In this section, a computational algorithm is proposed to maximize the guaranteed stability bounds on γ, given in the previous section. The maximum upper bound of γ, given from Theorem 2 (or Corollary 1), will be obtained corresponding to the positive definite symmetric matrix P which maximizes cost functions $J_k \coloneqq \frac{\alpha_k}{\mathcal{K}(P)}\ ;\ \forall k \in \mathcal{M}$, where α_k is the maximum positive scalar which satisfies the constraint $A_k^{o\prime}P + PA_k^o \leq -\alpha_k P$. Obviously, α_k depends on the choice of matrix P. The following algorithm is proposed to maximize J_k on positive definite matrices P and positive scalars α_k. Hence, for each $k \in \mathcal{M}$, the proposed algorithm should carry out the min-max problem (15a) over positive definite matrices P and supremum problem (15b) over positive scalars $\alpha \in (0\,,\ 2\rho_{min}]$, subject to $A_k^{o\prime}P + PA_k^o < -\alpha P$, where $\rho_{min} = \min_{k\in\mathcal{M}} (\rho(A_k^o))$.

$$\bar{J_\alpha} \coloneqq \min_{k\in\mathcal{M}}\left\{\max_{P>0} J_k\Big|_{\alpha_k=\alpha}\right\} \tag{15a}$$

$$J^* \coloneqq \sup_{\alpha\in(0\,,2\rho_{min}]} \{\bar{J_\alpha}\} \tag{15b}$$

For a fixed value of $\alpha_k = \alpha$, $\min_{k\in\mathcal{M}}\left\{\max_{P>0} J_k\right\}$ will be obtained corresponding to the positive definite matrix P with minimum condition number, such that constraint $A_k^{o\prime}P + PA_k^o < -\alpha P$ is satisfied for each $k \in \mathcal{M}$. The following lemma which is given from [40], formulates the problem of finding the positive definite matrix P with minimum condition number, such that constraints $A_k^{o\prime}P + PA_k^o < -\alpha P\ ;\ \forall k \in \mathcal{M}$ are satisfied.

Lemma 3: The problem of finding the positive definite symmetric matrix P with minimum condition number, which holds in constraints $A_k^{o\prime}P + PA_k^o < -\alpha P$, can be formulated as LMIs shown in (16).

$$\begin{aligned}&\mathit{minimize}\ \mu\\ &\mathit{subject\ to}\,,\quad I < P < \mu I\ \ \text{and}\ \ A_k^{o\prime}P + PA_k^o < -\alpha P\quad;\ \forall k \in \mathcal{M}\end{aligned} \tag{16}$$

So, the maximum value of $\max_{P>0} J_k\Big|_{\alpha_k=\alpha}$, in (15a), will be obtained by solving LMIs (16). Maximizing problem (15b) can also be solved numerically, searching for the maximizing value of α in the linearly spaced sequence $\mathrm{E} = \left\{2\rho_{min} \times \left(\frac{i}{N}\right)\ ;\ i = 1,2,\dots,N-1\right\}$, and $J^* \cong \max_{\alpha\in E}\{\bar{J_\alpha}\}$ for sufficiently large values of N. For the choice of N, there is a tradeoff between complexity of the algorithm and optimality of the result.

5. NUMERICAL EXAMPLES

In this section, two examples are presented to show the effectiveness of the proposed theorems and corollary in calculating upper bounds on both parametric uncertainties and independent entry uncertainties. Therefore, a three-dimensional two-form uncertain switched linear system and a planar two-form uncertain switched linear system are investigated using the YALMIP solver. The results are also compared with the results of [32, 35], and [39].

Example 1: Consider a parameterized arbitrary switched linear system with subsystem matrices

$$\begin{aligned}A_1 &= \begin{bmatrix} -3-0.2a & 5 & -2-b \\ -1 & 0.5-0.1a-c & 3 \\ 2b-c & -5 & -0.5b-0.3a \end{bmatrix},\\ A_2 &= \begin{bmatrix} -0.2a-b & 2-c & 2b-5 \\ 1 & 0.5-0.5a-c & 0 \\ c-2 & 0 & -b-0.4a \end{bmatrix},\end{aligned} \tag{17}$$

where parameter vector Q is defined as $Q = [q_1, q_2, q_3] \coloneqq [a, b, c]$, and $Q^o = [5,1,3]$. The nominal switched linear system, with subsystem matrices

$$A_1^o = \begin{bmatrix} -4 & 5 & -3 \\ -1 & -3 & 3 \\ -1 & -5 & -2 \end{bmatrix},\ A_2^o = \begin{bmatrix} -2 & -1 & -3 \\ 1 & -5 & 0 \\ 1 & 0 & -3 \end{bmatrix}, \tag{18}$$

is globally exponentially stable, since there exists a CQLF for the nominal subsystems. In addition, uncertainty structure matrices are $E_1^1 = \begin{bmatrix} -0.2 & 0 & 0 \\ 0 & -0.1 & 0 \\ 0 & 0 & -0.3 \end{bmatrix}$, $E_2^1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 2 & 0 & -0.5 \end{bmatrix}$,

$E_3^1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$, $E_1^2 = \begin{bmatrix} -0.2 & 0 & 0 \\ 0 & -0.5 & 0 \\ 0 & 0 & -0.4 \end{bmatrix}$, $E_1^2 = \begin{bmatrix} -0.2 & 0 & 0 \\ 0 & -0.5 & 0 \\ 0 & 0 & -0.4 \end{bmatrix}$, $E_2^2 = \begin{bmatrix} -1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, and $E_3^2 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

It should be noted that, Theorem 2 of [32] does not bring out any suitable answer for the stability margin, since $\lambda_{max}\left([A_1^o + A_1^{o\prime}]/2\right) > 0$. Actually Theorem 2 of [32] provides a satisfactory stability margin, when $V(x) = x'x$ is a CQLF for the nominal subsystems. In addition, results of [35] will not give a stability bound, since proposition 4 of [35] states that if $\bar{\lambda}_R + \hat{\lambda}_P = \max_{k\in\mathcal{M}} \frac{1}{n} trace(A_k) + \max_{k\in\mathcal{M}} \sigma_{max}\left(\frac{1}{2}(A_k + A_k') - \frac{1}{n} trace(A_k)I\right) < 0$, the arbitrary switched linear system is stable, while for the nominal switched linear system (18), $\bar{\lambda}_R + \hat{\lambda}_P = 0.3876$.

a) Assume that unknown parameters belong to intervals $a \in [4.5\,, 5.5]$, $b \in [0.9\,, 1.1]$, and $c \in [2.85\,, 3.15]$. Theorem 1 will be used to investigate robust exponential stability of the considered switched linear system. Since there exist positive definite matrix $P = \begin{bmatrix} 1.1 & 0 & 0.1 \\ * & 2.1 & -0.25 \\ * & * & 1.25 \end{bmatrix}$ $(P \geq I, \|P\|_2 = 2.1687)$ such that $A_k^{o\prime} P + P A_k^o \leq -2.5P$ and $\left\|\sum_{i=1}^{r} c_i \left|E_i^k\right|_m\right\|_2 \leq \frac{2.5}{2\times\|P\|_2}$. Therefore, the uncertain switched linear system (17) is robustly exponentially stable for uncertain parameters belong to the considered intervals.

b) Now, the maximum percentage tolerance of uncertain parameters will be calculated, so that the robust stability is guaranteed by Theorem 2. For this purpose, weights of uncertainties will be chosen as $w_i = q_i^o$. As it is shown in Fig. 1, the maximum value $\frac{\alpha_k}{\mathcal{K}(P)}$, using the proposed algorithm, is 1.2628, which corresponds to $P = \begin{bmatrix} 1.0008 & 0.0228 & -0.0055 \\ * & 2.0078 & -0.2753 \\ * & * & 1.0772 \end{bmatrix}$ and $\alpha_k = 2.6316$. Hence, uncertain switched linear system (17) is robustly exponentially stable for $\delta q_i \in q_i^o[-0.2231\,, +0.2231]$. In other words, if each uncertain parameter lies in an interval with $\pm 22.3\%$ tolerance from its nominal value, the switched linear system (17) is robustly exponentially stable. Moreover, since $\left[E_1^{k\prime} P + P\, E_1^k\right] < 0\,;\ k = 1{,}2\,,$ switched linear system (17) will be exponentially stable for all $a \in [3.8845\,,\ +\infty), b \in [\,0.7769\,,\ 1.2231]$, and $c \in [\,2.3307\,,\ 3.6693]$. The states trajectory of switched linear system (17) is plotted in Fig. 2, for $a = 3.8845, b = 1.2231$, and $c = 2.3307$, under an arbitrarily fast switching signal.

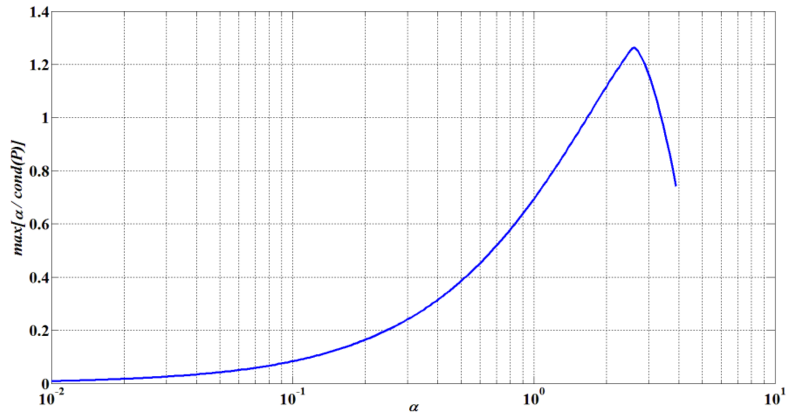

Fig. 1: Graph of $\max_{P>0} \{\frac{\alpha}{\mathcal{K}(P)}$; such that $A_k^{o\prime} P + P A_k^o < -\alpha P\}$ with respect to different values of α, for the nominal linear subsystems (18). There is no CQLF for the nominal switched linear system such that $A_k^{o\prime} P + P A_k^o < -\alpha P$ for $\alpha > 3.881$

If the switched linear system (17) is considered as a switched linear inclusion system and the results of [39] are used, the stability bound will be 26.4 percent of the nominal values. It shows that although the complexity has been increased rather than the result of Theorem 2, stability bound has not been notably improved. In addition, the obtained stability interval from Theorem 2 is infinite from one side, for parameter a.

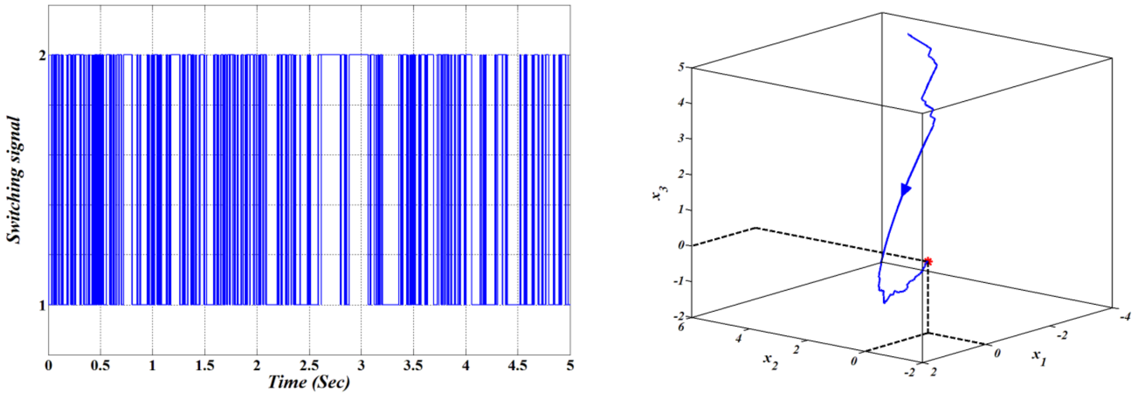

Fig. 2. Switching signal and state trajectory of the switched linear system for a = 3.8845 , b = 1.2231, and c = 2.3307

c) For the case that entries of subsystem matrices are subject to weighted perturbations, with the given weights $w_{i,j}^1 = w_{i,j}^2 = \frac{1}{1+|i-j|}$ for $i,j = 1,2,3$, i.e., $W^k = \begin{bmatrix} 1 & 0.5 & 0.333 \\ * & 1 & 0.5 \\ * & * & 1 \end{bmatrix}$, Corollary 1 gives a bound on perturbation of entries of each subsystem matrix. Arbitrary switched linear system (17) is robustly exponentially stable, if perturbation of each entry ($\delta a_{i,j}^k$) belongs to $\delta a_{i,j}^k \in \frac{1}{1+|i-j|}[-0.3335\,,+0.3335]$, e.g., $\delta a_{1,2}^k \in [-0.1668\,,+0.1668]$ or equally, $a_{1,2}^1 \in [4.8332 \quad 5.1668]$ and $a_{1,2}^2 \in [-1.1668 \quad -0.8332]$.

Example 2: Consider a two-dimensional switched linear system $\dot{x}(t) = A_{\sigma(t)}x(t)$, where subsystem matrices are

$$A_1(a,b) = \begin{bmatrix} -2 & b-a-6 \\ 8+a-b & \frac{3}{2}-\frac{1}{2}b \end{bmatrix} \;,\; A_2(a,b) = \begin{bmatrix} 4-b & 1+a \\ a+b-9 & 2-b \end{bmatrix}. \tag{19}$$

a, b are unknown parameters with the nominal values $a^o = 2$ and $b^o = 5$, and the nominal switched linear system, corresponded to the nominal subsystems

$$A_1^o = \begin{bmatrix} -2 & -3 \\ 5 & -1 \end{bmatrix} \;,\; A_2^o = \begin{bmatrix} -1 & 3 \\ -2 & -3 \end{bmatrix}, \tag{20}$$

is globally exponentially stable with the CQLF $V(x) = 2x_1^2 + x_2^2$. So, the proposed algorithm will be used to determine the maximum percentage tolerance of unknown parameters (from Theorem 2) and maximum perturbation of entries (from Corollary 1) such that robust stability is ensured. As it is shown in Fig. 3, the maximum value of $\frac{\alpha_k}{\mathcal{K}(P)}$, such that $A_k^{o\,'}P + PA_k^o < -\alpha_k P$ for $k = 1,2$, is 1.3697, which corresponds to $P = \begin{bmatrix} 1.3047 & 0.1200 \\ * & 1.0472 \end{bmatrix}$ ($\mathcal{K}(P) = 1.3519$ and $\alpha_k = 1.8518$). Moreover, uncertainty structure matrices E_i^k are written as $E_1^1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $E_2^1 = \begin{bmatrix} 0 & 1 \\ -1 & -0.5 \end{bmatrix}$, $E_1^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and $E_2^2 = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$.

a) To determine the maximum percentage tolerance, $w_1 = a^o$ and $w_2 = b^o$ will be chosen and the largest weighted bound γ will be obtained as 0.1931, which is equal to 19.3% of the nominal values of parameters. Moreover, $\left[E_2^{k\,'}P + P\,E_2^k\right] \leq 0$ for $k = 1,2$. So, switched linear system (19) will be exponentially stable for $a \in [1.6140\ ,\ 2.3860], b \in [4.0350\ , +\infty)$. In addition, for the nominal switched linear system (20), $\bar{\lambda}_R + \hat{\lambda}_P = -0.3820 < 0$. So, it will be concluded from [35] that the nominal switched linear system is asymptotically stable, and the maximum robust stability bounds on uncertain parameters will be obtained as 15.8% of their nominal values.

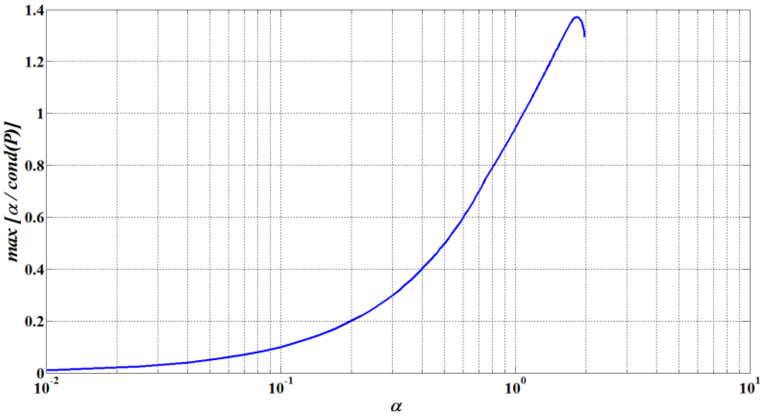

Fig. 3. Graph of $\max_{P>0} \{\frac{\alpha}{\mathcal{K}(P)}$; such that $A_k^{o\prime} P + P A_k^o < -\alpha P\}$ with respect to different values of α, for the nominal linear subsystems (20). There is no CQLF for the nominal switched linear system such that $A_k^{o\prime} P + P A_k^o < -\alpha P$ for $\alpha > 1.978$

To illustrate the effectiveness of Theorem 2, state trajectory of switched linear system (19) is plotted in Fig. 4, for $a = 2.3860$ and $b = 10^3$, under an arbitrarily fast switching signal.

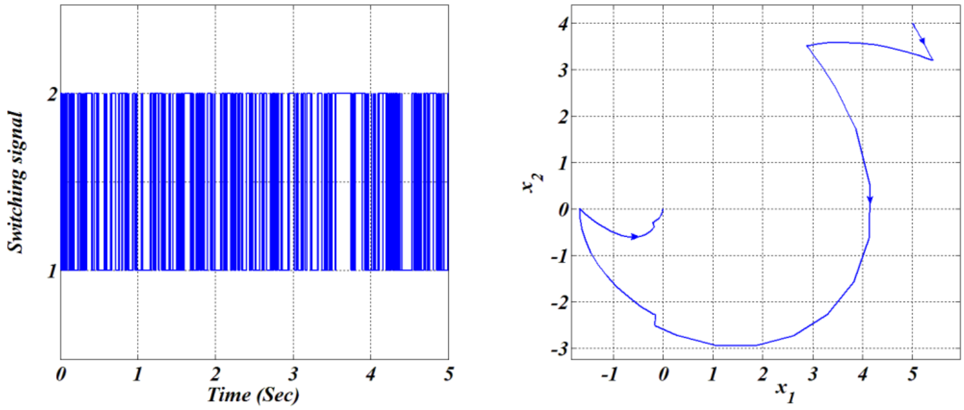

Fig. 4. Switching signal and state trajectory of the switched linear system for $\mathrm{a} = 2.3860$ and $\mathrm{b} = 10^3$

b) In the case that all entries are subject to independent uncertainties, $W^k = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, k = 1{,}2$, an upper bound on entry perturbations, which guarantee the exponential stability is calculated as $\gamma = 0.3424$. Moreover, if Theorem 2 of [32] is used to find a stability margin for the entries, it will be concluded that for the entry uncertainties which hold in $\begin{bmatrix} |\delta a_{11}^k| & |\delta a_{12}^k| \\ |\delta a_{21}^k| & |\delta a_{22}^k| \end{bmatrix} \leq 0.1910 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, the switched system is locally asymptotically stable. It shows that although stability notion in this paper is stronger than [32], the calculated stability radius from Corollary 1 is much larger.

6. CONCLUSION

Parametric uncertainties constitute a very common type of uncertainties in real-world applications. In this paper, robust exponential stability of arbitrary switched linear systems with respect to parametric uncertainties was analyzed. Sufficient conditions were proposed to investigate the robust exponential stability of arbitrary switched linear systems, when the uncertain parameters belong to the known intervals. Another theorem was also given and proved to provide bounds on uncertain parameters such that robust exponential stability of the arbitrary switched linear systems is ensured. In addition, a corollary is proposed to generalize the results of the second theorem for the cases with independent perturbations on entries of subsystem matrices. Numerical examples illustrate the effectiveness and simplicity of the proposed results, compared with the previous published works. Numerical examples show that, although the previous results are more complicated, their maximum stability intervals are not significantly increased, or are even less than our results.

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