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THE INVERSE OF COVARIANCE MATRICES FOR THE ARMA (p, q) CLASS OF PROCESSES*

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Abstract – Analysis of time series data can involve the inversion of large covariance matrices. For the class of ARMA (p, q) processes there are no exact explicit expressions for these inverses, except for the MA (1) process. In practice, the sample covariance matrix can be very large and inversion can be computationally time consuming and so approximate explicit expressions for the inverse are desirable. This paper offers some of these approximations.

Keywords – ARMA processes, band matrix, covariance matrix

1. INTRODUCTION

Let $\{\varepsilon_t\}$ be a sequence of independent random variables with zero mean and variance σ^2. An ARMA (p,q) process $\{x_t\}$ is defined by

$$x_t + \sum_{j=1}^{p} \alpha_j x_{t-j} = \varepsilon_t + \sum_{j=1}^{q} \beta_j \varepsilon_{t-j}. \tag{1}$$

We assume that stationary and invertibility conditions are held. When p and q are equal to unity the parameter subscript is dropped according to usual practice. In the covariance matrix $\mathbf{\Sigma}$ of $\mathbf{x}' = (x_1, x_2, \ldots, x_T)$, $\mathbf{\Sigma}$ is a Toeplitz matrix with the additional property of symmetry. Symmetric Toeplitz matrices arise frequently in statistical work as covariance matrices of wide-sense stationary processes. The inversion of $\mathbf{\Sigma}$ is required for many features in time series such as estimation of parameters, calculation of the likelihood in normal processes, and linear and quadratic discrimination (See [1]). The motivations behind the inversion of $\mathbf{\Sigma}$ are twofold in literature. Firstly, as a computational problem, it is to find numerical procedures that operate faster than general inversion, and secondly, as a mathematical problem, to find the components of the inversion explicitly. It has been noted that the covariance matrices of the time series as was given in (1) can be very large with the dimension equal to the number of observations. As an example, consider speech data or seismology records with more than two thousand observations. In this case, the computational method for inverting $\mathbf{\Sigma}$ is both difficult and time consuming. It is desirable to find an explicit form of covariance matrices, possibly with suitable and analytic components of inversion. For convenience,

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the value of σ^2 is taken as unity. For non-unit values expressions for $\mathbf{\Sigma}^{-1}$ must be multiplied by σ^{-2}.

A necessary and sufficient condition in order for a tridiagonal symmetric matrix, (MA (1) covariance matrix structure) to have an inverse is given by [2, 3]. This theorem was invoked later by [4] to find an exact form for the inverse of the covariance matrix, $\mathbf{\Sigma}^{-1}$, for MA(1) processes. A solution for $\mathbf{\Sigma}^{-1}$ by different methods has been given by [5&6]. The method given in [5] involves obtaining the inverse of an approximate matrix and adjusting this inverse to obtain the inverse of the given matrix. This method was simplified by [7] by eliminating $(\mathbf{\Sigma})_{11}$ and $(\mathbf{\Sigma})_{TT}$ from $\mathbf{\Sigma}$, where $(\mathbf{\Theta})_{rs}$ is denoted for (*rs*) th element $\mathbf{\Theta}$ matrix. Recently [8] expressed Σ as the product of the covariance matrix of the dual autoregressive process of order one and a near identity matrix to deduce its inverse. The method of [5] was extended to MA (q) processes by [9]. It was shown that the inverse of $\mathbf{\Sigma}$ can be approximated with an AR (q) and with an inverting a matrix not larger than $\left[\frac{q}{2}\right]\times\left[\frac{q}{2}\right]$ dimension. For an AR (1) process, [10] gave a method of finding $\mathbf{\Sigma}^{-1}$ using the spectral density function. However, it leads us to consider infinite dimension vector $\boldsymbol{x}$. Another method was suggested by [11, 12] based on the symmetric property of $\mathbf{\Sigma}$ and quadratic forms. [11] also gave $\mathbf{\Sigma}^{-1}$ for AR(2) processes explicitly. For an ARMA (1, 1) process, $\mathbf{\Sigma}^{-1}$ has been investigated by [10, 13, 14]. One problem with the suggested methods is that they can involve manipulation of very large matrices. In this paper an approximation of $\mathbf{\Sigma}^{-1}$ for ARMA (p,q) processes is suggested based on symmetric band matrices. Then the inversion of $\mathbf{\Sigma}$ is approximated via $\mathbf{\Sigma}^{-1} \approx \mathbf{L}\mathbf{\Lambda}\mathbf{L}'$. The paper is organized in four sections as follows: in section 2, $\mathbf{\Sigma}$ in the MA (q) process is approximated with a polynomial of the band matrix of band width 3 and then an inverse is obtained. This leads to an exact inverse for the MA (1) process. The method is extended in section 3 to obtain an approximation to the inverse of $\mathbf{\Sigma}$ for AR (p) processes. The fourth and final section is devoted to the inverse of the covariance matrix of ARMA (p, q) processes.

2. INVERSE OF THE COVARIANCE MATRIX FOR MA (Q) PROCESSES

A symmetric band matrix of dimension T and of band width (2q+1), q integer, is defined by

$$\left(B_{2q+1}\right)_{rs} = \begin{cases} x_{|r-s|} & |r-s| \le q \\ 0 & otherwise, \end{cases} \tag{2}$$

where $x_{|r-s|}$ is real, $r, s = 1, \ldots, T$. It is clearly a finitely correlated equation of order q.

Some of the spectral properties of symmetric band matrices have been investigated by [15] (see also [16]. It can be shown that B_3^i is an approximate symmetric band matrix of band width (*2i+1*), but with some of the elements in the upper left corner and the lower right corner slightly different. Then Σ can be approximated by a polynomial in $\mathbf{B}_3$, i.e.

$$\mathbf{\Sigma} \approx c_0\,\mathbf{I} + c_1\,\mathbf{B}_3 + \cdots + c_q\,\mathbf{B}_3^q \tag{3}$$

where I is the identity matrix and $c_0, \ldots, c_q$ are constants that can be obtained by equating Σ to (3), (see also [17]). In a numerical study, [18] has shown that the best matrix for use in this polynomial approximation is $\mathbf{B}_3$ given in (2) with $x_0 = 0$, and $x_1 = x_{-1} = -1$. For example, the covariance matrix of an MA(2) process is approximated by

$$\mathbf{\Sigma} \approx \{\beta_1^2 + (1+\beta_2)^2\,\mathbf{I} - (\beta_1 + \beta_1\beta_2)\mathbf{B}_3 + \beta_2\,\mathbf{B}_3^2\,. \tag{4}$$

The only error in the approximation is that $(\mathbf{\Sigma})_{11}$ and $(\mathbf{\Sigma})_{TT}$ have the approximate value $1+\beta_1^2+\beta_2^2+\beta_2$. All the other elements take their true values.
The *r*th eigenvalue of band matrix $\mathbf{B}_3$ is given by $\lambda_r = -2\cos(r\omega)$, where $\omega = \pi/(T+1)$. The *r*th normalized eigenvector associated with λ_r, denoted by $\boldsymbol{\xi}_r$, is given by

$$\boldsymbol{\xi}_r' = \{2/(T+1)\}^{\frac{1}{2}}\,(\sin(r\omega)\,, \sin(2r\omega), \ldots, \sin(Tr\omega)).$$

The $T \times T$ symmetric matrix of eigenvectors is $\mathbf{L} = (\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_T)$ and $\mathbf{L\Sigma L}' = \mathbf{\Lambda}$ where $\mathbf{\Lambda}$ is a diagonal matrix given by $(\mathbf{\Lambda})_{rr} = -2\cos(r\omega)$.

Since $\mathbf{L}\,\mathbf{B}_3^i\,\mathbf{L}' = \mathbf{\Lambda}^i$, then for the MA (q) process $\mathbf{L\Sigma L}'$ is an approximate diagonal matrix with elements

$$(\mathbf{L\Sigma L}')_{r,r} \approx \left(c_0\mathbf{I} + \sum_{i=1}^{q} c_i\mathbf{\Lambda}^i\right)_{r,r} = \sum_{i=0}^{q} c_i\{-2\cos(r\omega T)\}^i$$

and hence $(\mathbf{L\Sigma L}')^{-1} = \mathbf{L\Sigma}^{-1}\mathbf{L}'$ is also an approximate diagonal matrix with elements

$$(\mathbf{L\Sigma L}')_{r,r} = \left[\sum_{i=0}^{q} c_i\{-2\cos(r\omega T)\}^i\right]^{-1}. \tag{5}$$

Thus the approximate inverse of the covariance matrix, $\mathbf{\Sigma}$, for an MA(q) process is given by $\mathbf{L}'\mathbf{\Lambda}_{\boldsymbol{\beta}}^{-1}\,\mathbf{L}$ with the (*rs*)th element of $\mathbf{\Sigma}^{-1}$ given by

$$(\mathbf{\Sigma}^{-1})_{rs} \approx \frac{2}{T+1}\sum_{j=1}^{T}\{\sum_{i=0}^{q} c_i(-2\cos(j\omega T))^i\}^{-1}\sin(rj\omega T)\sin(sj\omega T).$$

As an example for *q=1*, $c_0 = 1+\beta^2, c_1 = -\beta$, $\mathbf{\Sigma} = (1+\beta^2)\mathbf{I} + \beta\mathbf{B}_3$ and the exact inverse matrix $\mathbf{\Sigma}$ for MA(1) process is given by

$$(\mathbf{\Sigma}^{-1})_{rs} = \frac{2}{T+1}\sum_{j=1}^{T}\frac{\sin(rj\omega T)\sin(sj\omega T)}{1+\beta^2+2\beta\cos(j\omega T)}. \tag{6}$$

It should be noted that the $(\mathbf{\Sigma}^{-1})_{rs}$ given in [9] are more complicated than that given in (6). For an MA(2) process using (4),

$$\left(\boldsymbol{\Sigma}^{-1}\right)_{rs} \approx \frac{2}{T+1}\sum_{j=1}^{T}\frac{\sin(rj\omega T)\sin(sj\omega T)}{\beta_1{}^2 + (1-\beta_2)^2 + 2(\beta_1 + \beta_1\beta_2)\cos(j\omega T) + 4\beta_2\cos^2(j\omega T)}.$$

It should be pointed out that the most common used result for inversion of matrix of covariance of ARMA process is given by [19] as

$$\left(\boldsymbol{\Sigma}^{-1}\right)_{rs} = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\cos((r-s)\lambda) f^{-1}(\lambda)d\lambda$$

where $f(\lambda)$ is a spectral density function. It is assumed that $\boldsymbol{\Sigma}$ is circulated and differs from the true $\boldsymbol{\Sigma}$ in the MA (q) process by blocks in the corners. The method given in this section suggests an exact inverse of the $\boldsymbol{\Sigma}$ matrix which differs from the required covariance matrix by at most $(q-1)\times(q-1)$ blocks at the upper left and lower right corners. This corner effect can be easily obtained for small *q*. The method also leads to an exact form of $\boldsymbol{\Sigma}^{-1}$ with *q=1* and with more appropriate elements given in [11] for MA (2).
The next section is devoted to covariance matrix of autoregressive processes.

3. INVERSE OF THE COVARIANCE MATRIX FOR AR (P) PROCESSES

The inverse of the covariance matrix for an AR (1) process can be approximated by the covariance matrix of an MA (1) process [8, 20]. For AR (p) processes the Yule-Walker equations allow the covariance matrix to be expressed as

$$\boldsymbol{\Sigma} = a_1\boldsymbol{\Theta}_1 + \cdots + a_p\boldsymbol{\Theta}_p \tag{7}$$

$(\boldsymbol{\Theta}_i)_{rs} = \theta_i^{|r-s|}$, the θ_j's are the roots of $\theta^p + a_1\theta^{p-1} + \cdots + a_p = 0$ and the a_j's are constants determined from initial conditions (for example see [21, 22]). $\boldsymbol{\Sigma}$ can be approximately diagonalised by employing an **L** matrix. In [18] it is shown that

$$\left(\mathbf{L}\boldsymbol{\Theta}_{\mathbf{i}}\mathbf{L}'\right)_{rs} = \frac{2}{T+1}\sum_{j=1}^{T}\sum_{k=1}^{T}\theta_i^{|j-k|}\sin(rj\omega T)\sin(sk\omega T)$$

$$= \begin{cases} \dfrac{1-\theta_i^2}{1+\theta_i^2 - 2\theta_i\cos(r\omega T)} + o\left(\dfrac{1}{T}\right) & r = s \\ o\left(\dfrac{1}{T}\right) & r \neq s, (r+s) \text{ even} \\ 0 & r \neq s, (r+s) \text{ odd.} \end{cases} \tag{8}$$

where

$$o\left(\frac{1}{T}\right) = \frac{4}{(T+1)(1-\theta_i^2)}\frac{\theta_i^2[1+(-1)^{r+1}\theta_i^{T+1}]\sin(r\omega T)\sin(s\omega T)}{(1+\theta_i^2 - 2\theta_i\cos(r\omega T)(1+\theta_i^2 - 2\theta_i\cos(s\omega T)}.$$

Now

$$\mathbf{L}\boldsymbol{\Sigma}\mathbf{L}' \approx \mathbf{L}\left(\sum_{i=1}^{p} a_i\boldsymbol{\Theta}_i\right)\mathbf{L}' = \sum_{i=1}^{p} a_i\boldsymbol{\Lambda}_i = \boldsymbol{\Lambda}_\alpha$$

where $\left(\mathbf{\Lambda}_i\right)_{rs}$ is given in (8) and hence $\mathbf{L\Sigma L'}$ is an approximate diagonal matrix. This gives the

$$\left(\mathbf{\Sigma}^{-1}\right)_{rs} \approx \left(\mathbf{L'\Lambda}_\alpha^{-1}\mathbf{L}\right)_{rs} = \frac{2}{T+1}\sum_{j=1}^{T}\frac{\sin(rj\omega T)\sin(sj\omega T)}{\sum_{i=1}^{p}\frac{a_i(1-\theta_i^2)}{1+\theta_i^2-2\theta_i\cos(j\omega)}}. \tag{9}$$

Now, the last two sections can be combined to generalize the method to autoregressive moving average processes.

4. INVERSE OF THE COVARIANCE MATRIX FOR ARMA (P, Q) PROCESSES

Following [21, 23], express the ARMA (p,q) process (1) as

$$x_t = \left(1+\sum_{i=1}^{p}a_i B^i\right)^{-1}\left(\sum_{j=0}^{q}\beta_j\,\varepsilon_{t-j}\right) = \sum_{j=0}^{q}\beta_j\left\{\left(1-\sum_{i=1}^{p}a_i\,B^i\right)^{-1}\varepsilon_{t-j}\right\},$$

where B is the backward shift operator and $\beta_{\mathbf{0}}=1$. Hence $x_t = \sum_{j=0}^{q}\beta_j z_{t-j}$ where $\{z_t\}$ is the AR (p) process defined by $(1+\sum_{i=1}^{p}\alpha_i B^i)^{-1}\varepsilon_{t-j}$. By considering $\gamma_X(\tau)=E(x_t x_{t-\tau})$

$$\begin{aligned}\gamma_X(\tau) &= \sum_{i=0}^{q}\sum_{j=0}^{q}\beta_i\beta_j\gamma_Z(\tau-i-j)\\ &= \left(\sum_{i=0}^{q}\beta_i^2\right)\gamma_Z(\tau)+\sum_{k=1}^{q}\left(\sum_{k=1}^{q-k}\beta_i\beta_{i+k}\right)\left\{\gamma_Z(\tau-k)+\gamma_Z(\tau+k)\right\}.\end{aligned} \tag{10}$$

Hence, it can then be shown that the covariance matrix, $\mathbf{\Sigma}$, for the ARMA (p,q) process can be approximately expressed as the product of the covariance matrix, $\mathbf{\Sigma}_\alpha$, of the AR(p) process, $x_t+\sum_{j=1}^{p}\alpha_j x_{t-j}=\varepsilon_t$, and the covariance matrix, $\mathbf{\Sigma}_\beta$, of the MA(q) process, $x_t=\sum_{j=0}^{q}\beta_j\varepsilon_{t-j}$, since if $\tau=i-k$, noting that γ_Z and γ_X are even and $\gamma_Z(u)=0$ for $u>q$,

$$\begin{aligned}\left(\mathbf{\Sigma}_\alpha\mathbf{\Sigma}_\beta\right)_{jk} &= \sum_{j=1}^{T}\left(\mathbf{\Sigma}_\alpha\right)_{ij}\left(\mathbf{\Sigma}_\beta\right)_{jk}\\ &= \sum_{j=1}^{T}\gamma_X(i-j)\gamma_Z(j-k) = \sum_{j=1}^{q}\gamma_X(i-j)\gamma_Z(j+\tau-i)\\ &\approx \sum_{j=1}^{q}\gamma_X(i-j-\tau)\gamma_Z(j-i) = \sum_{i=1}^{q}\sum_{j=1}^{q}\beta_i\beta_j\,\gamma_X(i-j-\tau)\\ &= \sum_{i=1}^{q}\sum_{j=1}^{q}\beta_i\beta_j\,\gamma_X(\tau-i+j)\end{aligned}$$

which is given in (10). Now $\mathbf{\Sigma}^{-1}\approx\mathbf{\Sigma}_\beta^{-1}\,\mathbf{\Sigma}_\alpha^{-1}\approx\mathbf{L'\Lambda}_\beta^{-1}\mathbf{LL'\Lambda}_\alpha^{-1}\mathbf{L}=\mathbf{L'\Lambda}_\beta^{-1}\,\mathbf{\Lambda}_\alpha^{-1}\mathbf{L}$. Hence from (6) and (9) it is easily seen that $\left(\mathbf{\Sigma}^{-1}\right)_{rs}$ is approximately equal to

$$\frac{2}{T+1}\sum_{m=1}^{T}\left[c_j\{-2\cos(m\omega T)\}^j\sum_{i=1}^{p}\frac{a_i(1-\theta_i^2)}{1-\theta_i^2-2\theta_i\cos(m\omega T)}\right]^{-1}\sin(rm\omega T)\sin(sm\omega T).$$

In particular for the ARMA (1, 1) process

$$\left(\mathbf{\Sigma}\right)_{rs} = \frac{1}{1-\alpha^2}\left\{(1+\beta^2)(-\alpha)^{|\tau|} + \beta\left((-\alpha)^{|\tau-1|} + (-\alpha)^{|\tau+1|}\right)\right\}$$

where $\tau = |r-s|$, $\tau = 0, 1, \ldots, T-1$. This matrix can be approximately expressed as the product of the covariance matrix of an AR (1) process and that of an MA (1) process, i.e.

$$\mathbf{\Sigma} = \mathbf{\Sigma}_\alpha \, \mathbf{\Sigma}_\beta$$

except for slight differences in the first row and the first column. For this case $c_0 = 1+\beta^2, c_1 = -\beta$, $a_1 = \frac{1}{1-\alpha^2}$ and $\theta_1 = -\alpha$. Hence

$$\left(\mathbf{\Sigma}^{-1}\right)_{rs} \approx \frac{2}{T+1}\sum_{j=1}^{T}\frac{1+\alpha^2+2\alpha\cos(j\omega T)}{1+\beta^2+2\beta\cos(j\omega T)}\sin(rj\omega T)\sin(sj\omega T).$$

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