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GREEN FUNCTION OF AXISYMMETRIC MAGNETOSTATICS*

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Abstract – A simple new closed form of the Green function for axisymmetric magnetostatic problems is found analytically in cylindrical coordinates. The result is verified by applying several examples.

Keywords – Magnetostatics, green functions, partial differential equations, electromagnetics

1. INTRODUCTION

The magnetic field distribution of an axisymmetric magnetostatic system with rotationally symmetric current distribution (being independent of the angle of rotation of the current in the cylindrical coordinate system) is described by an elliptic differential equation in terms of the poloidal magnetic flux [1], which is usually used in describing toroidal magnetohydrodynamic equilibrium. The numerical solution of this equation has been made possible by several methods including the Finite Difference Method, Inverse Variables, Moments Method, Method of Eigenfunctions, and the Finite Element Method [1, 2]. Also, its analytical solution has been reported by integral transform techniques [3], expansion in terms of confluent hyper-geometric functions [4], and coordinate transformations [5]. Existence of its solutions is discussed in [6].

In [1], the authors have also mentioned the possible application of Green functions for this purpose. The corresponding Green function has been reported in terms of elliptic functions [7, 8] and an infinite series of associated Legendre polynomials [9].

Here, a new and simpler form of this Green function is derived by a direct analytical approach. The final result, which is in the form of a Fourier cosine transform, is shown to be reducible to a closed form. Then to verify the resulting function, the method is applied to derive the magnetic fields of a current loop and a solenoid, in agreement with known expressions. In another example, we shall consider the poloidal flux from a magnetic quadrupole obtained from the present method and the newly developed variational axisymmetric finite element method [10, 11], to check the validity of the result.

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2. GREEN FUNCTION

The axisymmetric magnetostatics in cylindrical coordinates can be described by the equation [1]

$$\frac{1}{r}\Delta^{*}\Psi = -\mu_0 J_\phi \tag{1}$$

in which $\Psi = rA_\phi$ is the magnetic poloidal flux, and where A_ϕ is the toroidal component of the magnetic vector potential. Also, J_ϕ is the toroidal current density and Δ^* is the elliptic Grad-Shafranov operator

$$\Delta^{*} = r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{2}$$

where *r* and *z* are the radial and longitudinal coordinates in the cylindrical system of coordinates, respectively. Here we seek solutions of the form

$$\Psi(r,z) = \int_{-\infty}^{\infty}\int_{0}^{\infty} G(\mathbf{r},\mathbf{r}')J_\phi(\mathbf{r}')dr'dz' \tag{3}$$

in which $\mathbf{r} = r\hat{r} + z\hat{z}$ is the position vector, $G(\mathbf{r},\mathbf{r}')$ is referred to as the Green function, obtained through the solution of the following equation:

$$\Delta^{*}G(\mathbf{r},\mathbf{r}') = -\mu_0 r\delta(r-r')\delta(z-z') \tag{4}$$

with $\delta(\cdot)$ being the Dirac delta function.

To find the Green function $G(\mathbf{r},\mathbf{r}')$, it is instructive to first solve the homogeneous Grad-Shafranov equation, that is

$$\Delta^{*}\Psi = 0 \tag{5}$$

Using the method of separation of variables, by inserting $\Psi(r,z) = R(r)Z(z)$, the following ordinary differential equations are obtained

$$\ddot{R} - \frac{1}{r}\dot{R} - k^2 R = 0 \tag{6a}$$

$$Z'' + k^2 Z = 0 \tag{6b}$$

Here, the dot and prime stand for the differentiation with respect to *r* and *z*, respectively. Also *k* is an arbitrary non-negative real constant. The equations (6) have the general solutions:

$$R(r) = r[c_k \mathrm{K}_1(kr) + d_k \mathrm{I}_1(kr)] \tag{7a}$$

$$Z(z) = a_k \cos(kz) + b_k \sin(kz) \tag{7b}$$

in which a_k, b_k, c_k, and d_k are constants, and $\mathrm{K}_1(\cdot)$ and $\mathrm{I}_1(\cdot)$ are the first order modified Bessel functions. Since $r\,\mathrm{I}_1(kr)$ is unbounded when *r* approaches infinity, while $r\,\mathrm{K}_1(kr)$ remains finite at center and vanishes at infinity, the homogeneous Grad-Shafranov equation admits physically meaningful solutions of the form

$$\Psi(r,z)=\int_0^\infty r\,\mathrm{K}_1(kr)[a_k\cos(kz)+b_k\sin(kz)]dk \tag{8}$$

The form of (8) together with the reciprocity property of the Green function, i.e. $G(\mathbf{r},\mathbf{r}')=G(\mathbf{r}',\mathbf{r})$ suggest the following expression:

$$G(\mathbf{r},\mathbf{r}')=\mu_0\int_0^\infty a_k(r,r')\cos[k(z-z')]dk \tag{9}$$

with $a_k(r,r')$ to be determined. Plugging (9) in (4) gives

$$\int_0^\infty \Delta^*\{a_k(r,r')\cos[k(z-z')]\}dk=-r\delta(r-r')\delta(z-z') \tag{10}$$

But the above may be rewritten as

$$\int_0^\infty \Delta_k^{\ *}\{a_k(r,r')\}\cos[k(z-z')]dk=-r\delta(r-r')\delta(z-z') \tag{11}$$

where $\Delta_k^{\ *}=\dfrac{\partial^2}{\partial r^2}-\dfrac{1}{r}\dfrac{\partial}{\partial r}-k^2$. Comparing (11) with $\int_0^\infty \cos[k(z-z')]dk=\pi\delta(z-z')$ gives

$$\Delta_k^{\ *}a_k(r,r')=-\frac{1}{\pi}r\delta(r-r') \tag{12}$$

Noting the form of solutions (7a) for the homogeneous equation (6a), and using the method described in [12], the following expression results:

$$a_k(r,r')=\frac{-1}{\pi A}r_<r_>\,\mathrm{I}_1(kr_<)\mathrm{K}_1(kr_>) \tag{13}$$

in which *A* is the Wronskian of the functions $r'\,\mathrm{I}_1(kr')$ and $r'\,\mathrm{K}_1(kr')$ divided by r', being equal to –1. Also $r_<=\min\{r,r'\}$, and $r_>=\max\{r,r'\}$.
Therefore, the Green function $G(\mathbf{r},\mathbf{r}')$ is obtained as:

$$G(\mathbf{r},\mathbf{r}')=\mu_0\frac{rr'}{\pi}\int_0^\infty \mathrm{I}_1(kr_<)\mathrm{K}_1(kr_>)\cos[k(z-z')]dk \tag{14}$$

It is observed that (14) is in the form of a Fourier cosine transformation. Fortunately, the above integral may be evaluated to the closed form [13]

$$G(\mathbf{r},\mathbf{r}')=\mu_0\frac{\sqrt{rr'}}{2\pi}\mathrm{Q}_{1/2}\left[\frac{r^2+r'^2+(z-z')^2}{2rr'}\right] \tag{15}$$

where $\mathrm{Q}_{1/2}(\cdot)$ is the Legendre function of the second kind, satisfying

$$(1-x^2)y''(x)-2xy'(x)+\nu(\nu+1)y(x)=0 \tag{16}$$

with $\nu=1/2$. It is noticeable that the latter result justifies the requirement for the reciprocity property of the Green function as stated above. This completes the assertion.

It is also possible to express $Q_{\frac{1}{2}}(x)$ for $x \geq 1$ as [12]

$$Q_\nu(x) = \frac{\sqrt{\pi}\,\Gamma(\nu+1)}{2^{\nu+1}\Gamma\left(\nu+\frac{3}{2}\right)}\,{}_2\mathrm{F}_1\left(\frac{1+\nu}{2}, \frac{2+\nu}{2}; \frac{3+2\nu}{2}; x^{-2}\right) \tag{17}$$

The other independent solution of Legendre's equation (16) is [12]

$$P_\nu(x) = {}_2\mathrm{F}_1\left(-\nu, 1+\nu; 1; \frac{1-x}{2}\right) \tag{18}$$

For the special case of $\nu = \frac{1}{2}$ these functions turn out to be [14]

$$P_{\frac{1}{2}}(x) = \frac{4}{\pi}\mathrm{E}\left(\frac{1-x}{2}\right) - \frac{2}{\pi}\mathrm{K}\left(\frac{1-x}{2}\right) \tag{19a}$$

$$Q_{\frac{1}{2}}(x) = \sqrt{\frac{2}{x+1}}\left\{x\,\mathrm{K}\left(\frac{2}{x+1}\right) - (x+1)\mathrm{E}\left(\frac{2}{x+1}\right)\right\} \tag{19b}$$

where $\mathrm{E}(\cdot)$ and $\mathrm{K}(\cdot)$ are full elliptic integrals of the first and second kind. These relations may be verified by direct substitution in (16). It is an easy task to observe the consistency of (15) with [7] through (19b).

While it is possible to compute $Q_{\frac{1}{2}}(x)$ numerically by (19b), it is considerably advantageous to use (17) as shown in the next section. Furthermore, in a more recent article [15], a new method for efficient evaluation of the Legendre functions of the second kind has been described for this purpose. The Green function $G(\mathbf{r}, \hat{r}+\hat{z})$ is plotted in Fig. 1.

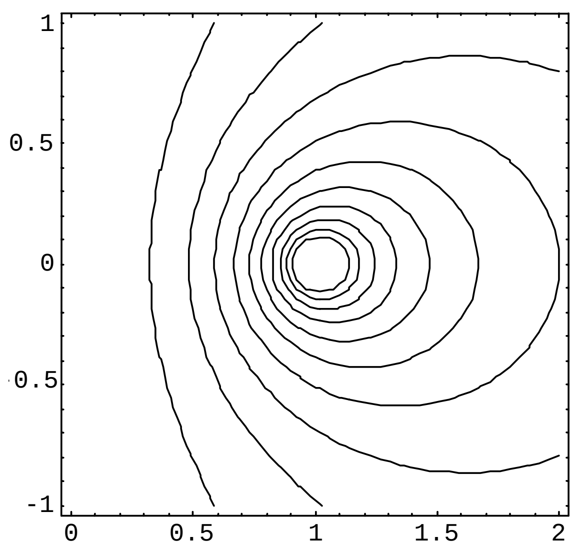

Fig 1. Contour plot of the Green function as given by (15)

As a final remark, (15) may be rewritten as

$$G(\mathbf{r}, \mathbf{r}') = \mu_0 \frac{\sqrt{rr'}}{2\pi} \mathrm{Q}_{\frac{1}{2}}\left[1 + \frac{|\mathbf{r}-\mathbf{r}'|^2}{2rr'}\right] \tag{20}$$

The Legendre function $\mathrm{Q}_{1/2}(x)$ tends to infinity at $x=1$. However, it may be easily seen that this is possible if and only if $\mathbf{r}=\mathbf{r}'$ in (15). Moreover, the above form meets the requirement for the reciprocity feature of the Green function as postulated above.

The behavior of (20) either at infinity with $r \to \infty$ or near the symmetry axis with $r \to 0^+$ may be studied using the asymptotic forms of the second kind of Legendre function $\mathrm{Q}_{1/2}(x)$ with large *x*. From [12], we have

$$\mathrm{Q}_\nu(x)=\int_0^\infty \frac{d\theta}{\left(x+\sqrt{x^2-1}\cosh\theta\right)^{1+\nu}}, \qquad \nu>-1, x\geq 1 \tag{21}$$

which may be approximated for large *x* as

$$\mathrm{Q}_\nu(x)\approx x^{-\nu-1}\int_0^\infty \frac{d\theta}{(1+\cosh\theta)^{1+\nu}}=\frac{\sqrt{\pi}\,\Gamma(\nu+1)}{2^{\nu+1}\Gamma\left(\nu+\frac{3}{2}\right)}x^{-\nu-1}, \qquad \nu>-1, x\geq 1 \tag{22}$$

For $\nu=\frac{1}{2}$, the asymptotic expansion therefore would be $\mathrm{Q}_{\frac{1}{2}}(x)\approx\frac{\pi}{\sqrt{32}}x^{-\frac{3}{2}}$. Thus, $G(\mathbf{r},\mathbf{r}')$ may be approximated by the following expression near the symmetry axis as

$$G(\mathbf{r},\mathbf{r}')\approx\frac{\mu_0 r^2 r'^2}{4\left[r^2+r'^2+(z-z')^2\right]^{3/2}} \tag{23}$$

3. EXAMPLES

I. ***Current Loop***

The magnetic field intensity **B** on the symmetry axis of a loop with the radius *b*, placed on the $z=0$ plane, and carrying the current *I* may be expressed as [16]

$$\mathbf{B}=\hat{z}\mu_0 I\frac{b^2}{2\left(z^2+b^2\right)^{3/2}} \tag{24}$$

Using the relations (3) and (15) with $J_\phi(\mathbf{r}')=I\delta(r'-b)\delta(z')$, the poloidal flux Ψ is obtained as

$$\Psi=\mu_0 I\frac{\sqrt{br}}{2\pi}\mathrm{Q}_{1/2}\left(\frac{r^2+z^2+b^2}{2br}\right) \tag{25}$$

The magnetic field intensity vector **B** is related to the poloidal flux Ψ as [1]

$$\mathbf{B}=-\frac{1}{r}\frac{\partial\Psi}{\partial z}\hat{r}+\frac{1}{r}\frac{\partial\Psi}{\partial r}\hat{z} \tag{26}$$

The radial component of the magnetic field intensity vector **B** on the *z*-axis is thus obtained by using the asymptotic expression (23) near the *z*-axis as

$$B_r=-\mu_0 I\lim_{r\to 0^+}\frac{-3b^2rz}{4\left(r^2+z^2+b^2\right)^{5/2}}=0 \tag{27}$$

Hence, the magnetic field has no radial component on the z-axis as expected due to symmetry considerations. Similarly, the longitudinal component is obtained as:

$$B_z = \mu_0 I \lim_{r\to 0^+} \frac{b^2\left(2z^2 + 2b^2 - r^2\right)}{4\left(r^2 + z^2 + b^2\right)^{5/2}} = \mu_0 I \frac{b^2}{2\left(z^2 + b^2\right)^{3/2}} \tag{28}$$

The latter result coincides with the expression in (24).

II. *Solenoid*

Putting (15) and $J_\phi(\mathbf{r}') = J_s \delta(r' - b)$ in (3) for an infinitely long solenoid centered on the z-axis with radius b, the poloidal flux Ψ is obtained as

$$\Psi = \frac{\mu_0 J_s}{2\pi} \sqrt{br} \int_{-\infty}^{\infty} Q_{1/2}\left[\frac{r^2 + b^2 + (z - z')^2}{2br}\right] dz' \tag{29}$$

Putting the asymptotic expression (23) for the symmetry axis, we have

$$\Psi = \frac{\mu_0 J_s b^2}{4} r^2 \int_{-\infty}^{\infty} \frac{dz'}{\left[r^2 + b^2 + (z - z')^2\right]^{3/2}} \tag{30}$$

which may be simplified to

$$\Psi = \frac{\mu_0 J_s b^2}{2} \frac{r^2}{r^2 + b^2} \tag{31}$$

This produces the magnetic field

$$\mathbf{B} = \hat{z} \frac{\mu_0 J_s b^2}{2} \lim_{r\to 0^+} \frac{2b^2}{\left(r^2 + b^2\right)^2} = \hat{z} \mu_0 J_s \tag{32}$$

in agreement with the well-known value for an infinitely long solenoid.

III. *Magnetic Quadrupole*

In this section, the flux resulting from a magnetic quadrupole consisting of four poloidal coils located at (r,z)=(1,2), (2,1), (1,-2), (2,-1) with toroidal currents +1, -1, +1, and -1, respectively, is considered. In Fig. 2, the computation is done by the Variational Axisymmetric Finite Element Method (VAFEM) [10, 11] using an optimized integration scheme [17]. Please notice that since the system is symmetric with respect to the z=0 plane, only the upper half is shown. The resulting poloidal flux from the calculation of the Green function by either (17) or (19b) is shown in Fig. 3, in justification of Fig. 2. The computation time for the optimized VAFEM is approximately 44 seconds on a Pentium PC, 10 seconds for the Green function method using Elliptic integrals (19b), and just 6 seconds for the case of our suggested closed form using the Legendre function (17). The calculations have been performed on a 54x54 grid.

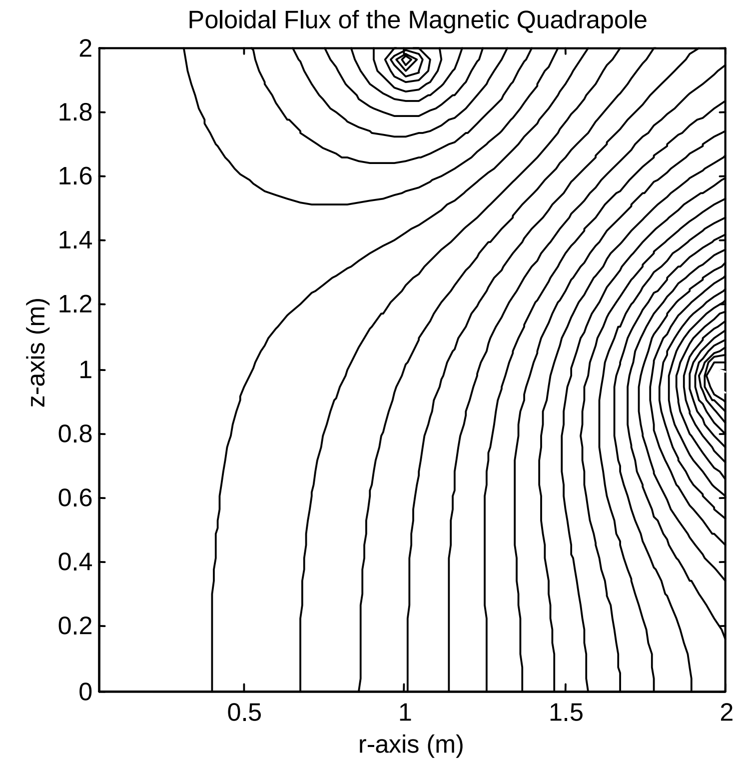

Fig. 2. Constant contours of the poloidal flux of the magnetic quadrupole computed by VAFEM [11]

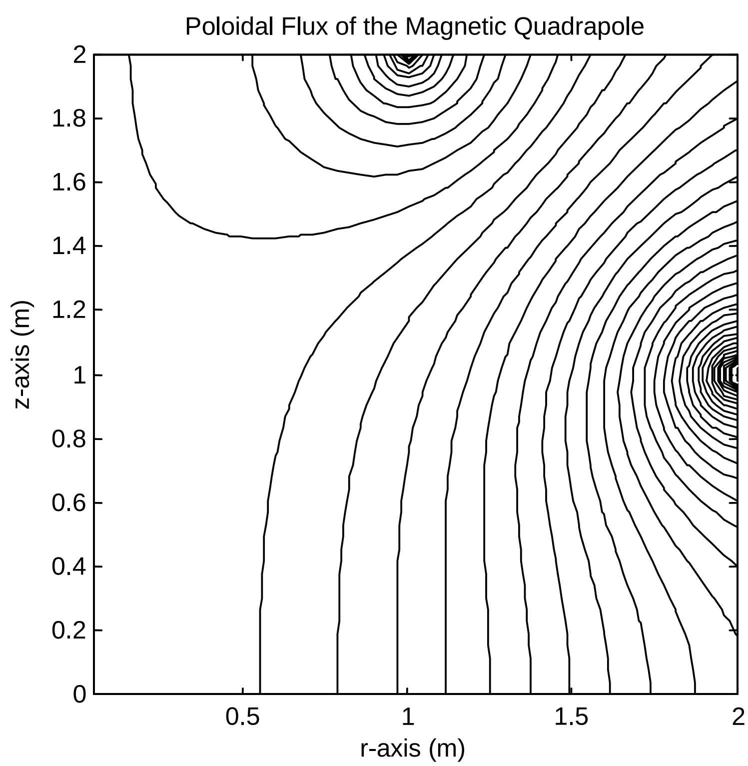

Fig. 3. Constant contours of the poloidal flux of the magnetic quadrupole computed by Green function method

4. CONCLUSIONS

A simple closed form of the Green function of axisymmetric magnetostatics was found analytically. Several examples were considered as verification of this function, and an agreement between this approach and other results or numerical methods were observed.

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