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"Research Note"

INSTABILITY OF SOLUTIONS OF CERTAIN NONLINEAR VECTOR DIFFERENTIAL EQUATIONS OF ORDER SEVEN[*]

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Abstract – Sufficient conditions are established under which the zero solution $X = 0$ of equation (2) is unstable.

Keywords – Nonlinear differential equations, seventh order, instability, Lyapunov's second (direct) method

1. INTRODUCTION

Since Lyapunov [1] proposed his famous theory on the stability of motion, which is now commonly known as Lyapunov's second method or direct method, many papers have been published on the instability properties for various third-, fourth-, fifth-, sixth-, seventh and eighth orders of certain nonlinear differential equations in the relevant literature. So far, the most effective method to investigate the instability of solutions of certain nonlinear differential equations of higher order is still the Lyapunov's direct (or second) method. In this connection, we refer to the papers of Bereketoglu [2], Ezeilo [3-7], Krasovskii [8], Li and Yu [9], Li and Duan [10], Reissig et al [11], Skrapek [12], [13], Tejumola [14], Tiryaki [15-17], C. Tunc and E. Tunc [18], Tunc [19-21], Tunc and Sevli [22], and the references cited therein. However, with respect to our observations in the relevant literature, in the case $n = 1$, the instability properties of nonlinear differential equations of the seventh order have been discussed only by Sadek [23] and Tejumola [14].

More recently, Sadek [23] investigated the subject for the seventh order scalar differential equation of the form:

$$x^{(7)} + a_1 x^{(6)} + a_2 x^{(5)} + a_3 x^{(4)} + a_4 \dddot{x} + f(\dot{x})\ddot{x} + g(x)\dot{x} + h(x) = 0 . \qquad (1)$$

Namely, in [23], sufficient conditions for the instability of the zero solution of equation (1) were established by the author. However, with respect to our observations in the relevant literature, no work has been found other than the papers mentioned above on the instability of solutions of certain

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seventh order nonlinear differential equations. The present work is the first attempt to obtain sufficient conditions for the instability of solutions of certain nonlinear vector differential equations of the seventh order. The motivation for the present study has come especially from the paper of Sadek [23] and the papers mentioned. Our aim is to acquire a similar result for certain nonlinear vector differential equations of the seventh order.

In the present paper, we are concerned with the instability of the trivial solution $X = 0$ of the vector differential equation of the form:

$$X^{(7)} + AX^{(6)} + BX^{(5)} + CX^{(4)} + D\dddot{X} + F(\dot{X})\ddot{X} + G(X)\dot{X} + H(X) = 0 \tag{2}$$

in the real Euclidean space $\Re^n$ (with the usual norm denoted in what follows by $\|.\|$), where A, B, C and D are constant $n \times n$-symmetric matrices; F and G are continuous $n \times n$-symmetric matrix functions depending, in each case, on the arguments shown; $H : \Re^n \to \Re^n$ and $H(0) = 0$. It will be supposed that the function H is continuous. Let $J_H(X)$, $J_G(X)$ and $J_F(\dot{X})$ denote the Jacobian matrices corresponding to the functions $H(X)$, $G(X)$ and $F(\dot{X})$, respectively, that is, $J_H(X) = \left(\frac{\partial h_i}{\partial x_j}\right)$, $J_G(X) = \left(\frac{\partial g_i}{\partial x_j}\right)$, $J_F(\dot{X}) = \left(\frac{\partial f_i}{\partial \dot{x}_j}\right)$ $(i, j = 1,2,...,n)$, where $(x_1, x_2,..., x_n)$, $(\dot{x}_1, \dot{x}_2,..., \dot{x}_n)$, $(h_1, h_2,..., h_n)$, $(g_1, g_2,..., g_n)$ and $(f_1, f_2,..., f_n)$ are the components of X, $\dot{X}$, H, G and F, respectively. Other than these, it will also be assumed that the Jacobian matrices $J_H(X)$, $J_G(X)$ and $J_F(\dot{X})$ exist and are continuous. The symbol $\langle X, Y \rangle$ corresponding to any pair X, Y in $\Re^n$ stands for the usual scalar product $\sum_{i=1}^{n} x_i y_i$, and $\lambda_i(A)$, $(i = 1,2,..., n)$ are the eigenvalues of the $n \times n$ - matrix A.

We consider though in what follows, in place of (2), the equivalent differential system:

$$\dot{X} = Y, \dot{Y} = Z, \dot{Z} = S, \dot{S} = T, \dot{T} = U, \dot{U} = W, \tag{3}$$

$$\dot{W} = -AW - BU - CT - DS - F(Y)Z - G(X)Y - H(X)$$

was obtained as usual by setting $\dot{X} = Y$, $\ddot{X} = Z$, $\dddot{X} = S$, $X^{(4)} = T$, $X^{(5)} = U$, $X^{(6)} = W$ in (2).

Now, we consider the linear constant coefficient seventh order differential equation:

$$x^{(7)} + a_1 x^{(6)} + a_2 x^{(5)} + a_3 x^{(4)} + a_4 \dddot{x} + a_5 \ddot{x} + a_6 \dot{x} + a_7 x = 0. \tag{4}$$

It is well-known from the qualitative behavior of solutions of linear differential equations that the trivial solution of (4) is unstable if, and only if, the associated auxiliary equation:

$$\psi(\lambda) \equiv \lambda^7 + a_1\lambda^6 + a_2\lambda^5 + a_3\lambda^4 + a_4\lambda^3 + a_5\lambda^2 + a_6\lambda + a_7 = 0 \tag{5}$$

has at least one root with a positive real part. The existence of such a root naturally depends on (though not always all of) the coefficients a_1, a_2,…, a_7 in (5). For example, if

$$a_1 < 0 \tag{6}$$

then it is clear from a consideration of the fact that the sum of the roots of (5) equals (a_1) and that at least one root of (5) has a positive real part for arbitrary values of a_2, a_3, a_4, a_5, a_6, a_7. An analogue consideration, combined with the fact that the product of the roots of (5) is equal to ($-a_7$), will verify that at least one root of (5) has a positive real part if

$$a_1 = 0 \text{ and } a_7 \neq 0 \tag{7}$$

for arbitrary a_2, a_3, a_4, a_5 and a_6. The condition $a_1 = 0$ here in (7) is, however, superfluous when

$$a_7 < 0; \tag{8}$$

then for $\psi(0) = a_7 < 0$ and $\psi(R) > 0$ if $R > 0$ is sufficiently large, thus showing that there is a positive real root of (5) subject to (8) and for arbitrary a_1, a_2, a_3, a_4, a_5 and a_6. Moreover, a necessary and sufficient condition for (5) to have a purely imaginary root, $\lambda = i\beta$ (β real), is that the two equations

$$-a_1\beta^6 + a_3\beta^4 - a_5\beta^2 + a_7 = 0 \tag{9}$$

and

$$-\beta^6 + a_2\beta^4 - a_4\beta^2 + a_6 = 0 \tag{10}$$

are satisfied at the same time. If

$$a_1 \leq 0,\ a_3 \geq 0,\ a_5 \leq 0 \text{ and } a_7 > 0 \tag{11}$$

or

$$a_1 \geq 0,\ a_3 \leq 0,\ a_5 \geq 0 \text{ and } a_7 < 0, \tag{12}$$

then equation (5) cannot have any purely imaginary root whatever.

2. MAIN RESULT

We establish the following result:

Theorem: In addition to the basic assumptions imposed on A, B, C, D, F, G and H, we suppose that the following conditions are satisfied:

(i) There are constants a_1, a_2, a_3, a_7 such that $\lambda_i(A) \leq a_1 \leq 0$, $\lambda_i(C) \geq a_3 \geq 0$, $\lambda_i(F(Y)) \leq 0$ for all $Y \in \Re^n$ and $\lambda_i(J_H(X)) \geq a_7 > 0$ for all $X(\neq 0) \in \Re^n$ and $H(X) \neq 0$ if $X \neq 0$, $(i = 1,2,\ldots,n)$ or

$(i)'$ $\lambda_i(A) \geq a_1 \geq 0$, $\lambda_i(B) \geq a_2 > 0$, $\lambda_i(C) \leq a_3 \leq 0$, $\lambda_i(F(Y)) \geq 0$ for all $Y \in \Re^n$ and $\lambda_i(J_H(X)) \leq a_7 < 0$ for all, $X \neq 0$ $X \in \Re^n$ and $H(X) \neq 0$ if $X \neq 0$, $(i = 1,2,\ldots,n)$.

Then the zero solution $X = 0$ of (3) is unstable.

Remark 1. It should be noted that there are no restrictions on the constants a_2, a_4, a_6 in (1), as well as on the eigenvalues of the matrices B, D, G in (2) for the part (i) of the theorem.

Remark 2. In the case $n = 1$, the conditions of the theorem reduce to those of Sadek [23].

The following lemma is important for the proof of the theorem.

Lemma: Let A be a real symmetric $n \times n$- matrix and $a' \geq \lambda_i(A) \geq a > 0 \quad (i = 1,2,...,n)$, where a', a are constants. Then

$$a'\langle X, X\rangle \geq \langle AX, X\rangle \geq a\langle X, X\rangle$$

and

$$a'^2\langle X, X\rangle \geq \langle AX, AX\rangle \geq a^2\langle X, X\rangle.$$

Proof: See [24].

Proof of the theorem: The proof of the theorem depends on a scalar differentiable Lyapunov function $V_0 = V_0(X, Y, Z, S, T, U, W)$. This function and its total time derivative satisfy some fundamental inequalities. We define V_0 as follows:

$$\begin{aligned} V_0 = &- \int_0^1 \langle F(\sigma Y)Y, X\rangle d\sigma - \int_0^1 \langle \sigma G(\sigma X)X, X\rangle d\sigma - \langle X, W\rangle - \langle X, AU\rangle - \langle X, BT\rangle \\ &- \langle X, CS\rangle - \langle X, DZ\rangle + \langle Y, U\rangle + \langle Y, AT\rangle + \langle Y, BS\rangle + \langle Y, CZ\rangle - \langle Z, AS\rangle \\ &- \langle Z, T\rangle + \frac{1}{2}\langle S, S\rangle + \frac{1}{2}\langle DY, Y\rangle - \frac{1}{2}\langle BZ, Z\rangle. \end{aligned} \tag{13}$$

It is clear from (13) that $V_0(0,0,0,0,0,0,0) = 0$. Obviously, it also follows from the assumptions of the theorem, the above lemma and (13) that:

$$V_0(0,0,0,\varepsilon,0,0,0) = \frac{1}{2}\langle \varepsilon, \varepsilon\rangle = \frac{1}{2}\|\varepsilon\|^2 > 0,$$

for all arbitrary, $\varepsilon \neq 0$, $\varepsilon \in \Re^n$. So, in every neighborhood of $(0,0,0,0,0,0,0)$ there exists a point $(\xi, \eta, \zeta, \mu, \tau, \omega, \rho)$ such that $V_0(\xi, \eta, \zeta, \mu, \tau, \omega, \rho) > 0$ for all $\xi, \eta, \zeta, \mu, \tau, \omega, \rho$ in $\Re^n$. Next, let $(X, Y, Z, S, T, U, W) = (X(t), Y(t), Z(t), S(t), T(t), U(t), W(t))$ be an arbitrary solution of (3). A straightforward calculation from (13) and (3) yields

$$\begin{aligned} \dot{V}_0 = \frac{d}{dt}V_0(X, Y, Z, S, T, U, W) = &- \langle AS, S\rangle + \langle CZ, Z\rangle + \langle H(X), X\rangle + \langle F(Y)X, Z\rangle + \langle G(X)Y, X\rangle \\ &- \frac{d}{dt}\int_0^1 \langle F(\sigma Y)Y, X\rangle d\sigma - \frac{d}{dt}\int_0^1 \langle \sigma G(\sigma X)X, X\rangle d\sigma. \end{aligned} \tag{14}$$

Remember that

$$\begin{aligned} \frac{d}{dt}\int_0^1 \sigma\langle G(\sigma X)X, X\rangle d\sigma &= \int_0^1 \langle \sigma G(\sigma X)Y, X\rangle d\sigma + \int_0^1 \sigma^2\langle J_G(\sigma X)XY, X\rangle d\sigma + \int_0^1 \sigma\langle G(\sigma X)X, Y\rangle d\sigma \\ &= \int_0^1 \langle \sigma G(\sigma X)Y, X\rangle d\sigma + \int_0^1 \sigma\frac{\partial}{\partial\sigma}\langle \sigma G(\sigma X)Y, X\rangle d\sigma = \sigma^2\langle G(\sigma X)Y, X\rangle\Big|_0^1 = \langle G(X)Y, X\rangle. \end{aligned} \tag{15}$$

and

$$\frac{d}{dt}\int_0^1 \langle F(\sigma Y)Y, X\rangle d\sigma = \frac{d}{dt}\int_0^1 \langle F(\sigma Y)X, Y\rangle d\sigma = \int_0^1 \langle F(\sigma Y)X, Z\rangle d\sigma + \int_0^1 \sigma\langle J_F(\sigma Y)XZ, Y\rangle d\sigma$$

$$+\int_0^1 \langle F(\sigma Y)Y, Y\rangle d\sigma = \int_0^1 \langle F(\sigma Y)X, Z\rangle d\sigma + \int_0^1 \sigma\frac{\partial}{\partial\sigma}\langle F(\sigma Y)X, Z\rangle d\sigma + \int_0^1 \langle F(\sigma Y)Y, Y\rangle d\sigma$$

$$= \sigma\langle F(\sigma Y)X, Z\rangle\Big|_0^1 + \int_0^1 \langle F(\sigma Y)Y, Y\rangle d\sigma = \langle F(Y)X, Z\rangle + \int_0^1 \langle F(\sigma Y)Y, Y\rangle d\sigma . \qquad (16)$$

By collecting estimates (15) and (16) into (14) we obtain

$$\dot{V}_0 = -\langle AS, S\rangle + \langle CZ, Z\rangle + \langle H(X), X\rangle - \int_0^1 \langle F(\sigma Y)Y, Y\rangle d\sigma$$

Since $\frac{\partial}{\partial\sigma}H(\sigma X) = J_H(\sigma X)X$ and $H(0) = 0$, then

$$H(X) = \int_0^1 J_H(\sigma X)X d\sigma .$$

Therefore, it follows from (*i*) and $(i)'$, respectively, that

$$\langle H(X), X\rangle = \int_0^1 \langle J_H(\sigma X)X, X\rangle d\sigma \geq a_7\int_0^1 \langle X, X\rangle d\sigma = a_7\langle X, X\rangle = a_7\|X\|^2 \qquad (17)$$

and

$$\langle H(X), X\rangle = \int_0^1 \langle J_H(\sigma X)X, X\rangle d\sigma \leq a_7\int_0^1 \langle X, X\rangle d\sigma = a_7\langle X, X\rangle = a_7\|X\|^2 . \qquad (18)$$

Now, if we assume the assumption (*i*) of the theorem and (17) hold, then we have

$$\dot{V}_0 \geq -a_1\|S\|^2 + a_3\|Z\|^2 + a_7\|X\|^2 .$$

Thus, the assumption (*i*) shows that $\dot{V}_0(t) \geq 0$ for all $t \geq 0$, that is, $\dot{V}_0$ is positive semi-definite. Furthermore, $\dot{V}_0 = 0$ $(t \geq 0)$ necessarily implies that $Y = 0$ for all $t \geq 0$, and also that $X = \xi$ (a constant vector), $Z = \dot{Y} = 0$, $S = \ddot{Y} = 0$, $T = \dddot{Y} = 0$, $U = Y^{(4)} = 0$, $W = Y^{(5)} = 0$, $\dot{W} = Y^{(6)} = 0$ for all $t \geq 0$. Substituting the estimates

$$X = \xi ,\ Y = Z = S = T = U = W = 0$$

in (3) it follows that $H(\xi) = 0$, which necessarily implies that $\xi = 0$ because of $H(0) = 0$ and $H(X) \neq 0$ if $X \neq 0$. So

$$X = Y = Z = S = T = U = W = 0 \text{ for all } t \geq 0 .$$

Therefore, the function V_0 has the entire requisite Krasovskii criterion [8] if the condition (*i*) in the theorem holds. This proves the proof of part (*i*) of the theorem.

Similarly, for the proof of part $(i)'$ of the theorem, we consider the Lyapunov function $V_1 = V_1(X,Y,Z,S,T,U,W)$ defined by the function $V_1 = - V_0$, where V_0 is defined as the same the function in (13). Trivially,

$$V_1(0,0,0,0,0,0,0) = 0$$

and

$$V_1(0,0,\varepsilon,0,\varepsilon,0,0) = \langle \varepsilon,\varepsilon \rangle + \frac{1}{2}\langle B\varepsilon,\varepsilon \rangle \geq \|\varepsilon\|^2 + \frac{1}{2}a_2\|\varepsilon\|^2 > 0$$

for all arbitrary, $\varepsilon \neq 0$, $\varepsilon \in \Re^n$. Hence, in every neighborhood of $(0,0,0,0,0,0,0)$ there exists a point $(\bar{\xi},\bar{\eta},\bar{\zeta},\bar{\mu},\bar{\tau},\bar{\omega},\bar{\rho})$, such that $V_1(\bar{\xi},\bar{\eta},\bar{\zeta},\bar{\mu},\bar{\tau},\bar{\omega},\bar{\rho}) > 0$ for all $\bar{\xi},\bar{\eta},\bar{\zeta},\bar{\mu},\bar{\tau},\bar{\omega},\bar{\rho}$ in $\Re^n$. For the remaining of the proof, if we follow the line indicated just above, then we can easily conclude that the function V_1 has the entire requisite Krasovskii criterion [8] if the condition $(i)'$ in the theorem holds. This completes the proof of part $(i)'$ of the theorem.

Thus, the basic properties of the functions $V_0(X,Y,Z,S,T,U,W)$ and $V_1(X,Y,Z,S,T,U,W)$, which are proved just above, verify that the zero solution of (3) is unstable. (See, Theorem 1.15 in Reissig [11] and Krasovskii [8]) The system of equation (3) is equivalent to the differential equation (2). Consequently, it follows the original statement of the theorem.

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