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ON GENERAL RELATIVELY ISOTROPIC L-CURVATURE FINSLER METRICS *

B. NAJAFI[1**], A. TAYEBI[2] AND M. M. REZAII[3]

Department of Mathematics and Computer Science
of Amir-Kabir University, Tehran, I. R. of Iran
[1]b_najafi@aut.ac.ir, [2]akbar_tayebi@aut.ac.ir
[3]mmreza@aut.ac.ir

Abstract – In this paper the *general relatively isotropic* L *-curvature Finsler metrics* are studied. It is shown that on constant relatively Landsberg spaces, the concepts of weakly Landsbergian, Landsbergian and generalized Landsbergian metrics are equivalent. Some necessary conditions for a *relatively* isotropic L -curvature Finsler metric to be a Riemannian metric are also found.

Keywords – Finsler metric, (generalized) Landsberg metric, (mean) landsberg curavture, (mean) Cartan tensor, Douglas tensor, Funk metric, Randers metric

1. INTRODUCTION

There are several notions of curvature in Finsler geometry. If g_{ij} is the fundamental tensor of Finsler metric F, the vertical derivative of g_{ij} on tangent space gives rise to the *Cartan tensor* **C**, and the horizontal derivative of C along geodesics is called the *Landsberg tensor* **L**. It is natural to consider $\mathbf{L}/\mathbf{C}$ as the relative growth rate of the Cartan torsion along geodesics. This leads to a study of *general relatively isotropic Landsberg metrics* which was first considered by Izumi [1]. There are lots of Finsler metrics in this class of metrics, such as the *Funk* metric on strongly convex domains in $\mathbf{R}^\mathbf{n}$. Therefore it is natural to study this class of Finsler metrics, especially *Randers* metrics in this class, where there are many contributions to this class of Finsler metrics [2-5].

Landseberg metrics belong to this class of Finsler metrics. As a generalization of Landsberg metrics, Bejancu and Farran introduced the *generalized Landsberg metrics* [5]. Here we show that on constant relatively Landsberg spaces, this generalization does not lead to a new class.

Theorem: Let (M, F) be a constant relative isotropic Landsberg space. Then the following are equivalent:
1) F is Landsbergian.
2) F is generalized Landsbergian.

Finally, we find some necessary conditions for a relative isotropic L -curvature Finsler metric to be a Riemannian metric. More precisely we have

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**Corresponding author

Theorem: Let (M, F) be a complete Finsler manifold with bounded Cartan torsion.Suppose that F is a non-zero constant relative Landsberg manifold, then F must be Riemannian metric. In particular, non-zero constant relatively compact Landsberg manifold must be Riemannian.

Throughout this paper, we make use of the Einstein convention, that is, repeated indices with one upper index and one lower index which denote summation over their range. We also set the Chern connection on Finsler manifolds.

2. PRELIMINARIES

Let M be an n-dimensional C^{∞} manifold. The tangent space at $x \in M$ is denoted by $T_x M$, and the tangent bundle of M by $TM = \cup_{x \in M} T_x M$. Each element of TM has the form (x,y), where $x \in M$ and $y \in T_x M$. Let $TM_0 = TM \setminus \{0\}$. The natural projection $\pi : TM \to M$ is given by $\pi(x, y) = x$. The *pull-back tangent bundle* $\pi^* TM$ is a vector bundle over TM_0 whose fiber $\pi_v^* TM$ at $v \in TM_0$ is just $T_x M$, where $\pi(v) = x$. Then

$$\pi^* TM = \{(x, y, v) \mid y \in T_x M_0, v \in T_x M\}.$$

A *Finsler metric* on manifold M is a function $F : TM \to [0, \infty)$, which has the following properties:

(i) F is C^{∞} on TM_0;

(ii) $F(x, \lambda y) = \lambda F(x, y) \quad \lambda > 0$;

(iii) For any tangent vector $y \in T_x M$, the vertical Hessian of $\frac{F^2}{2}$ given by

$$(g_{ij}(x, y)) = \left(\left[\frac{1}{2} F^2 \right]_{y^i y^j} \right)$$

is positive definite.

If g is a Riemannian metric on M, then $F = \sqrt{g(y, y)}$ defines a Finsler metric on M, so every Riemannian metric can be considered a Finsler metric.

Suppose (M, F) is a Finsler manifold. The global vector field G is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by

$$G = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i},$$

where $G^i(x, y)$ are local functions on TM_0 satisfying

$$G^i(x, \lambda y) = \lambda^2 G^i(x, y) \quad \lambda > 0.$$

G is called the associated *spray* to (M, F). The projection of an integral curve of G is called a *geodesic* in M. In local coordinates, a curve $c(t)$ is a geodesic if and only if its coordinates $(c^i(t))$ satisfy $\ddot{c}^i + 2G^i(\dot{c}) = 0$. F is said to be *positively complete* (resp. *negatively complete*), if any geodesic on an open interval (a, b) can be extended to a geodesic on (a, ∞) (resp. $(-\infty, b)$). F is said to be *complete* if it is positively and negatively complete.

The notion of Riemann curvature for Riemann metrics can be extended to Finsler metrics. For a vector $y \in T_x M_0$, the Riemann curvature $R_y : T_x M \to T_x M$ is defined by

$$R_y(u) = R^i_k(y)u^k \frac{\partial}{\partial x^i}$$

where

$$R^i_k(y) = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}$$

Suppose $P \subset T_x M$ (flag) is an arbitrary plane and $y \in P$ (flag pole) is a non-zero vector. The flag curvature $K(P, y)$ is defined by

$$K(P, y) = \frac{g_y(R_y(v), v)}{g_y(y, y)g_y(v, v) - g_y(v, y)g_y(v, y)}$$

where v is an arbitrary vector in P such that $P = span(y, v)$.

A Finsler metric F is said to be of scalar curvature if for any non-zero vector $y \in T_x M$ and any flag $P \subset T_x M$, $x \in M$, with $y \in P$, $K(P, y) = \lambda(y)$ is independent of P, or equivalently,

$$R_y = \lambda(y)F^2(y)\{I - g_y(y, .)y\}, \quad y \in T_x M, \quad x \in M$$

where $I : T_x M \to T_x M$ denotes the identity map and $g_y(y, .) = \frac{1}{2}[F^2]_{y^i} dx^i$. F is also said to be of constant curvature λ if the above identity holds for the constant λ.

Let $\{e_i\}_{i=1}^n$ be a local orthonormal (with respect to g) frame field for vector bundle $\pi^* TM$ and $\{\omega^i\}_{i=1}^n$ be its dual co-frame field. The Chern connection is a linear connection on $\pi^* TM$, which is defined by the following.

Theorem: ([6]) There is a unique set of local 1-forms $\{\omega^i_j\}$ on TM_0 such that

$$d\omega^i = \omega^j \wedge \omega^i_j$$

$$dg_{ij} = g_{kj}\omega^k_i + g_{ki}\omega^k_j + 2C_{ijk}\omega^{n+k}$$

$$\omega^{n+k} = dy^k + y^j \omega^k_j.$$

where $C_{ijk} = \frac{1}{4}[F^2]_{y^i y^j y^k}(y)$. We obtain a symmetric tensor **C** defined by

$$\mathbf{C}(U, V, W) := C_{ijk}(y)U^i V^j W^k,$$

where $U = U^i \frac{\partial}{\partial x^i}$, $V = V^i \frac{\partial}{\partial x^i}$ and $W = W^i \frac{\partial}{\partial x^i}$. We call **C** the *Cartan* tensor. It was *E. Cartan* who first gave a geometric interpretation of this quantity. The Cartan tensor characterizes Riemannian metrics among Finlser metrics.

Theorem: C=0 if and only if F is Riemannian.
Put

$$L_{ijk} := \frac{\partial C_{ijk}}{\partial x^l} y^l - 2C_{ijkl}G^l - C_{ljk}N^l_i - C_{ilk}N^l_j - C_{ijl}N^l_k,$$

where $C_{ijkl} := \frac{1}{4}[F^2]_{y^i y^j y^k y^l}(y)$. It is easy to show that,

$$L_{ijk}y^i = L_{ijk}y^j = L_{ijk}y^k = 0$$

From the above argument we obtain the symmetric tensor $\mathbf{L}$ on π^*TM defined by

$$\mathbf{L}(U,V,W) := L_{ijk}(y)U^iV^jW^k,$$

where $U = U^i \frac{\partial}{\partial x^i}$, $V = V^i \frac{\partial}{\partial x^i}$ and $W = W^i \frac{\partial}{\partial x^i}$. We call $\mathbf{L}$ the *Landsberg tensor*. The Landsberg tensor plays an important role in Finsler geometry. A Finsler metric is called a *Landsberg metric* if $L = 0$.

Let $c(t)$ be an arbitrary geodesic in (M, F). Take arbitrary parallel vector fields $U(t)$, $V(t)$, and $W(t)$ along c. Then by definition of the Landsberg tensor

$$L_{\dot c(t)}(U(t),V(t),W(t)) = \frac{d}{dt}[C_{\dot c(t)}(U(t),V(t),W(t))].$$

Thus the Landsberg curvature measures the rate of changes of the Cartan torsion along geodesics.

Now we introduce two important non-Riemannian curvatures for Finsler spaces. Let $\{b_i\}_{i=1}^n$ be an arbitrary basis for T_xM. We define the mean of L (resp. C) by

$$J_y(u) := \sum_{i,j=1}^n g^{ij}(y)L_y(u,b_i,b_j),$$

$$\text{(resp. } I_y(u) := \sum_{i,j=1}^n g^{ij}(y)C_y(u,b_i,b_j))$$

where $g_{ij}(y) = g_y(b_i, b_j)$. The family $J = \{J_y\}_{y\in TM_0}$ (resp. $I = \{I_y\}_{y\in TM_0}$) is called the *mean Landsberg curvature* (resp. *mean Cartan curvature*). A Finsler metric is called a *weak Landsberg metric* if $J = 0$. In dimension two, J completely determines L.

Theorem: ([6]) $I = 0$ if and only if F is Riemannian.

Definition: (**C-reducible**)
A Finsler metric is said to be C-reducible if the Cartan tensor of F is in the following form

$$C_{ijk} = \frac{1}{1+n}\{h_{ij}I_k + h_{jk}I_i + h_{ki}I_j\},$$

where $h_{ij} = g_{ij} - \ell_i \ell_j$ is the angular metric tensor [7, 8].

Definition: (Douglas tensor)
Let F be a Finsler metric, and G be its associated spray. In a standard coordinate (x^i, y^i) for TM_0, we define

$$B^i_{jkl} = \frac{\partial G^i}{\partial y^j \partial y^k \partial y^l}, \quad E_{jk} = B^i_{jki}.$$

Now the Douglas tensor is given by

$$D^i_{jkl} = B^i_{jkl} - \frac{2}{n+1}\{E_{jk}\delta^i_l + E_{jl}\delta^i_k + E_{kl}\delta^i_j + \frac{\partial E_{jk}}{\partial y^l} y^i\}.$$

Definition: (Isotropic mean Berwald metric)
A Finsler manifold (M, F) is said to be *isotropic mean Berwald metric* if

$$E_{ij} = \frac{n+1}{2} c F_{y^i y^j}.$$

where $c = c(x)$ is a scalar function on M.

Definition: (Generalized Landsberg metrics)
We say that a Finsler metric F is a *generalized Landsberg metric* if the h-curvature of the Berwald and Chern connections coincide [5].

Remark 1. Every Landsberg manifold is a generalized Landsberg manifold, but the opposite is not true.

Definition: (Funk metrics)
The Funk metric on a strongly convex domain $\Omega \subset R^n$ is a nonnegative function on $T\Omega = \Omega \times R^n$, which satisfies the following

$$F_{x^i} = F F_{y^i}.$$

Then for every Funk metric we have

$$F_{x^k y^i} y^k = F_{x^i}$$

and the geodesic coefficients G^i of F are given by

$$G^i(y) = \frac{1}{2} F(y) y^i.$$

Definition: (Randers metrics)
Let $\alpha = \sqrt{a_{ij}(x) y^i y^j}$ be a Riemannian metric, and $\beta = b_i(x) y^i$ be a 1-form on M with $b := \sqrt{a^{ij} b_i b_j} < 1$. The Finsler metric $F = \alpha + \beta$ is called a Randers metric. This class of Finsler metrics is very important in Finlser geometry, and can be seen in many areas such as mathematics, physics and biology [9]. Randers metrics were first studied by physicist G. Randers in 1941, from the standard point of general relativity [9, 10]. Since then, many Finslerists have made efforts to investigate the geometric properties of Randers metrics.

Theorem A: The Finlser metric F^n, $(n \geq 3)$ is C-reducible if and only if the metric is a Randers metric [7, 8].

3. GENERAL RELATIVE ISOTROPIC L -CURVATURE METRICS

Definition A: A Finlser metric F is said to be *General Relative Isotropic Landsberg Metric* if $L=\lambda C$, for some scalar function λ on TM, and to be brief we call it *G.R.I Landsberg metric*. In the case $\lambda = cF$ for some scalar function c on M, it is also said to be Relatively Isotropic Landsberg metric or briefly R.I Landsberg metric.

Remark 2. It is proved that in this case the scalar $\lambda(x,y)$ is given by [1]

$$\lambda=\frac{J_i I^i}{I_j I^j}.$$

Definition B: A Finlser metric F is said to be *General Relative Isotropic J-curvature Metric* if $J=\lambda I$, for some scalar function λ on TM, and to brief we call it *G.R.I J-curvature metric*. In the case $\lambda = cF$ for some scalar function c on M, it is also said to be *Relatively Isotropic J-curvature Metric* or briefly R.I J-curvature metric.

Remark 3. Every R.I Landsberg metric is a G.R.I Landsberg metric, every R.I J-curvature metric is a G.R.I J-curvature metric, every G.R.I Landsberg metric is a G.R.I J-curvature metric and every R.I Landsberg metric is a R.I J-curvature metric.

Theorem B: Let F^n, $(n>3)$ be a C-reducible G.R.I Landsberg metric and $n>3$. Then F is a R.I Landsberg metric [1, 3].

Lemma 1. Every C-reducible G.R.I J-curvature metric F is a G.R.I Landsberg metric.

Proof: Since F is C-reducible, then

$$C_{ijk}=\frac{1}{n+1}\{h_{ij}I_k+h_{ik}I_j+h_{jk}I_i\}.$$

By taking the horizontal covariant derivation of the above identity with respect to the Chern connection we get

$$C_{ijk|s}=\frac{1}{n+1}\{h_{ij}I_{k|s}+h_{ik}I_{j|s}+h_{jk}I_{i|s}\}.$$

By contracting the last identity with (y^s) and using $I_{k|s}y^s=J_k$ we have

$$L_{ijk}=-\lambda C_{ijk}.$$

This means that F is G.R.I Landsberg metric.

Corollary 1. Let F^n be a C-reducible G.R.I J-curvature metric and $n>3$. Then F is a R.I J-curvature metric.

Proof: By Theorem B and remark 3 the result is obtained.

Theorem C: Let F^n, $(n>3)$ be a C-reducible G.R.I Landsberg metric and $n>3$. Then F is a Douglas metric [3].

Corollary 2. Let F^n, $(n>3)$ be a C-reducible G.R.I J-curvature metric. Then F is a Douglas metric.

Proof: By lemma 1 and Theorem C, we have the above corollary.

Corollary 3. Let F^n, $(n>3)$ be a C-reducible R.I J-curvature Finsler metric. Then F is a Douglas metric.

Proof: It is a consequence of Remark 3 and corollary 3.

Corollary 4. Let F^n, $(n>3)$ be a Randers metric which is R.I J-curvature Finsler metric. Then F is a Douglas metric.

Proof: Theorem A and corollary 4 lead to the result.

Definition E: A R. I Landsberg metric is said to be a *constant relatively Landsberg metric* if $c=c_0$, where c_0 is a constant. In this case we write $\mathbf{L}=c_0$.

Example 1. Let F be the Funk metric on a strongly convex domain Ω in $\mathbf{R^n}$. Then we have

$$L_{ijk}=\frac{-1}{2}FC_{ijk},$$

so Funk metric is a constant relatively isotropic Landsberg metric with $c_0=\frac{-1}{2}$.

Let $F^n=\alpha+\beta$ be a Randers metric with $n>3$, which is also a G.R.I Landsberg metric. *S. Bacso* and *I. Papp* say that if F is Douglas space, then F must be a Riemannain metric [2]. But example 1 shows this claim is not correct. Here, in some way, by theorem 1.2 of [11], remark 3 and corollary 2, we state the correct version of the above claim.

Example 2. Let $F^n=\alpha+\beta$ be a Randers metric with $n>3$ which is also a G. R. I Landsberg metric. Suppose F has constant flag curvature $\mathbf{K}=\lambda$, then $\lambda=-c^2\leq 0$. F is either locally Minkowskian $(\lambda=-c^2=0)$ or in the following form after a scaling $(\lambda<0)$:

$$F_a=\frac{\sqrt{|y|^2-(|x|^2|y|^2-<x,y>^2)}}{1-|x|^2}\pm\frac{<x,y>}{1-|x|^2}\pm\frac{<a,y>}{1+<a,x>},\quad y\in T_xR^n$$

where $a\in\mathbf{R^n}$ is a constant vector with $|a|<1$, and $<>$ is the Euclidean inner product on $\mathbf{R^n}$.

Remark: Let $F=\alpha+\beta$ be a Randers metric on a manifold M. There is a pair (h,W) corresponding to F by Zermelo's navigation problem [12], where h is a Riemannian metric and W is a vector field on M with $h(W,W)<1$. Suppose F is Douglas, i.e. β is closed, then it is proved that F is G.R.I Landsberg curvature metric if and only if W is a homotetic vector filed with respect to h. Moreover, $L=cFC$ if and only if $Ł_W(h)=-4ch$, where $Ł_W$ is the Lie derivation with respect to W. On the other hand, it has recently been proven that on Douglas spaces, relatively isotropic Landsberg metrics and isotropic mean Berwald metrics are the same [13].

Theorem 5. Let (M,F) be a constant relative isotropic Landsberg space. Then the following are

equivalent:
1) F is Landsbergian.
2) F is generalized Landsbergian.

Proof: Let F be a Finsler metric. By remark 1, it is sufficient to prove that every generalized Landsberg metric is Landsberg metric. The relation between h-curvatures of Berwald and Chern connections is given by

$$\mathbf{R}^{i}_{jkl} = R^{i}_{jkl} + [L^{i}_{jl|k} - L^{i}_{jk|l} + L^{i}_{sk}L^{s}_{jl} - L^{i}_{sl}L^{s}_{jk}]$$

where $\mathbf{R}$ and R are h-curvatures of Berwald and Chern connections, respectively [14]. By definition of generalized Landsberg metric we have

$$L^{i}_{jl|k} - L^{i}_{jk|l} + L^{i}_{sk}L^{s}_{jl} - L^{i}_{sl}L^{s}_{jk} = 0$$

Since $L_{ijk} = cFC_{ijk}$, then the above result becomes the following

$$cF\{C^{i}_{jl|k} - C^{i}_{jk|l}\} + c^2F^2\{C^{i}_{sk}C^{s}_{jl} - C^{i}_{sl}C^{s}_{jk}\} = 0$$

By contracting with y^k, we have

$$L_{ijl} = 0$$

This means that F is a Landsberg metric.

By using the above theorem and remark 2, we have the following

Corollary 6. Let (M, F) be a constant relative isotropic Landsberg space. Then the following are equivalent:
1) F is weakly Landsbergian.
2) F is Landsbergian.
3) F is generalized Landsbergian.

Proposition ([4]): Let F be a Finsler metric of scalar curvature on an n-dimensional manifold with $\mathbf{L} = \mathbf{c}$, then

$$K = -c^2 + \sigma(x)e^{\frac{-3\tau}{n+1}},$$

where $\sigma(x)$ is a scalar function on M, and τ is a scalar function on TM_0.

With the above notations and conditions, we have the following

Corollary 7. If $(n > 3)$ and $\sigma \neq 0$ then F is a Randers metric.

Proof: By [3], since $K + c^2 = \sigma e^{\frac{-3\tau}{n+1}} \neq 0$ then F is C-reducible, so by theorem A the result is obtained.

4. REDUCTION TO A RIEMANNIAN METRIC

Theorem: Let F be a Finsler space of scalar curvature $\mathbf{K} = k(x, y)$ [15] then

$$L_{ijk|s}y^s = \frac{-1}{3}\{k_{,i}h_{jk} + k_{,j}h_{ik} + k_{,k}h_{ij} + 3kC_{ijk}\}.$$

Corollary 8. Every $\mathbf{R}$-flat and non-zero constant relative Landsberg manifold is a Riemannian manifold.

Theorem 9. Let (M, F) be a complete non-zero constant relative Landsberg manifold with bounded Cartan torsion. Then (M, F) is a Riemannian manifold.

Proof: Let p be an arbitrary point of M, and $y, u, v, w \in T_pM$. Let $c:(-\infty,\infty)\to M$ is the unit speed geodesic passing from p and $\frac{dc}{dt}(0) = y$. If $U(t), V(t)$ and $W(t)$ are the parallel vector fields along c with $U(0)=u, V(0)=v$ and $W(0)=w$, we put $C(t)=C(U(t),V(t),W(t))$ and $\dot{C}(t)=\dot{C}(U(t),V(t),W(t))$. By definition, we have the following ODE,

$$\dot{C}(t) = \lambda_0 C(t).$$

With a general solution of

$$C(t) = C(0)e^{\lambda_0 t}.$$

Using $\|C\| < \infty$, and letting $t \to +\infty$ or $t \to -\infty$, we have $C(0) = C(u,v,w) = 0$, so $C = 0$ i.e. (M, F) is a Riemannian manifold.

Corollary 1. Every non-zero constant relative isotropic compact Landsberg metric is a Riemannian metric.

REFERENCES

1. Izumi, H. (1977). On $^*P$-Finsler spaces I., II. Memoris of the Defense Academy. *Japan*, *16*, 133-138, *17*, 1-9.
2. Bacso, S. & Papp, I. (1998). On $^*P$-Finsler spaces with vanishing Douglas tensor. *Acta Acad. Paedagog. Agriensis sect. Mat. (N. S) 25*, 91-95.
3. Izumi, H. (1982). On $^*P$-Finsler spaces of scalar curvature. *Tensor, N. S. 38*, 220-222.
4. Shen, Z. (2004). *Landsberg Curvature, S-Curvature and Riemann Curvature.* MSRI Publication Series, Cambridge University Press.
5. Bejancu, A. & Farran, H. (2000). Generalized Landsberg manifolds of scalar curvature. *Bull. Korean Math. Soc. 37(3)*, 543–550.
6. Shen, Z. (2001). *Lectures on Finsler Geometry.* Singapore World Scientific.
7. Matsumoto, M. (1986). *Foundation of Finsler Geometry and Special Finsler spaces.* Saikawa, Otsu, Japan, Kaiseisha press.
8. Matsumoto, M. (1972). On C-reducible Finsler spaces. *Tensor, N. S.*, 29-37.
9. Antonelli, P. L., Ingarden, R. S. & Matsumoto, M. (1993). The theory of sprays and Finsler spaces with applications in physics and biology. *FTPH 58, Kluwer Academic Publishers*.
10. Randers, G. (1941). On an asymetric metric in the four-space of general relativity. *Phys. Rev., 59*, 195-199.
11. Chen, X. & Shen, Z. (2003). Randers metric with special curvature properties. *Osaka J. of Math*, 87-101.
12. Bao, D., Robles, C. & Shen, Z. Zermelo navigation on Riemanifolds. *Journal of Differential Geometry, accepted*.
13. Chen, X. & Shen, Z. (2004). On Douglas metrics. *Preprint*.

14. Bao, D., Chern, S. S. & Shen, Z. (2000). An Introduction to Riemann-Finsler Geometry. *Springer-Verlage*.
15. Chen, X., Mo, X. & Shen, Z. (2003). On the flag curvature of Finsler metrics of scalar curvature. *Journal of the London Mathematical society, 68(2)*, 762-780.