Iranian Journal of Science & Technology, Transaction A, Vol. 30, No. A3
Printed in The Islamic Republic of Iran, 2006

"Research Note"

EXISTENCE OF THREE SOLUTIONS FOR THE DIRICHLET PROBLEM INVOLVING THE P-LAPLACIAN AND MINIMAX INEQUALITY FOR RELEVANT FUNCTIONALS*

G. A. AFROUZI** AND S. HEIDARKHANI

Department of Mathematics, Faculty of Basic Sciences,
Mazandaran University, Babolsar, I. R. of Iran
alizadehafrouzi@yahoo.com

Abstract – In this paper, we establish some results on the existence of at least three weak solutions for a Dirichlet problem involving p-Laplacian using a variational approach.

Keywords – Dirichlet problem, critical point, three solutions, multiplicity results, minimax inequality

1. INTRODUCTION

We consider the boundary value problem

$$\begin{cases} -\Delta_p u + \lambda f(x,u) = m(x)u^2 & in\ \Omega, \\ u = 0 & on\ \partial\Omega, \end{cases} \tag{1}$$

where $\Delta_p u = \text{div}\ (|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $\Omega \subset R^N (N \geq 1)$ is nonempty bounded open set with a boundary $\partial\Omega$ of class C^1, $P > N$, $\lambda > 0$, $f : \Omega \times R \to R$ is a Caratheodory function and the weight $m(x) \in C(\overline{\Omega})$. In this paper, we are interested in ensuring the existence of at least three weak solutions to the problem (1). As usual, a weak solution of (1) is any $u \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega |\nabla u(x)|^{p-2}\ \nabla u(x)\nabla v(x)dx + \lambda \int_\Omega f(x,u)vdx = \int_\Omega m(x)u^2 vdx$$

for all $v \in W_0^{1,p}(\Omega)$. Multiplicity results for problem:

$$\begin{cases} \Delta_p u + \lambda f(x,u) = 0 & in\ \Omega, \\ u = 0 & on\ \partial\Omega, \end{cases} \tag{2}$$

have been broadly investigated in recent years (see [1, 2]). For instance, in [1], using variational methods, the authors ensure the existence of an arbitrarily small positive solution for the problem (2) when the function f has a suitable oscillating behavior at zero. In the present paper, the existence of multiple solutions of (1) is established. In section 2, Theorem 2.1 under novel assumptions ensures the existence of an open interval $\Lambda \subseteq [0,+\infty[$ and a positive number q such that, for each $\lambda \in \Lambda$, problem (1) admits at least three solutions in $W_0^{1,P}(\Omega)$ whose norms are less than q. In section 3, we establish an equivalent statement of minimax inequality

*Received by the editor May 9, 2005 and in final revised form January 20, 2007
**Corresponding author

$$\mathrm{Sup}_{\lambda\geq 0}\inf_{u\in X}\left(\Phi(u)+\lambda\Psi(u)+\rho\lambda\right)<\inf_{u\in X}\sup_{\lambda\geq 0}\left(\Phi(u)+\lambda\Psi(u)+\rho\lambda\right). \tag{3}$$

for a special class of functional. Our approach is based on a three critical point theorem proved in [3]. Here we recall its equivalent formulation [4, Theorem 1.1 and Remark 1.1]:

Theorem A: Let X be a separable and reflexive real Banach space; $\Phi : X \to R$ a continuously $G\hat{a}teaux$ differentiable and sequentially weakly lower semicontinuous functional whose $G\hat{a}teaux$ derivative admits a continuous inverse on X^*; $\Psi : X \to R$, a continuously $G\hat{a}teaux$ differentiable functional whose $G\hat{a}teaux$ derivative is compact. Assume that $\lim_{\|u\|\to+\infty}(\Phi(u)+\lambda\Psi(u)) = +\infty$
for all $\lambda \in [0,+\infty[$, and that there exists $\rho \in$ R such that

$$\mathrm{Sup}_{\lambda\geq 0}\inf_{u\in X}\left(\Phi(u)+\lambda\Psi(u)+\rho\lambda\right)<\inf_{u\in X}\sup_{\lambda\geq 0}\left(\Phi(u)+\lambda\Psi(u)+\rho\lambda\right).$$

Then, there exists an open interval $\Lambda \subseteq [0,+\infty[$ and a positive number q such that, for each $\lambda \in \Lambda$, the equation $\Phi'(u)+\lambda\Psi'(u)=0$, has at least three solutions in X whose norms are less than q.
In Theorem 2.3 of [4], if we choose $-J=\Psi,\ u_0=0$. Then we have:

Theorem B: Let X be a separable and reflexive real Banach space; $\Phi,\Psi : X \to R$ two sequentially weakly lower semicontinuous functional. The following assertions are equivalent:
(a) there are $\rho \in R,\ u_1 \in X$ such that: (i) $0<\rho<\Psi(u_1)$, (ii) $\inf_{u\in\Psi^{-1}(]-\infty,\rho])}\Phi(u) > \rho\dfrac{\Phi(u_1)}{\Psi(u_1)}$;
(b) there are $r \in R,\ u_1 \in X$ such that: (j) $0<r<\Phi(u_1)$, (jj) $\inf_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u) > \rho\dfrac{\Psi(u_1)}{\Phi(u_1)}$;
(c) there exists $\rho \in R$ such that:

$$\mathrm{Sup}_{\lambda\geq 0}\inf_{u\in X}\left(\Phi(u)+\lambda\Psi(u)+\rho\lambda\right)<\inf_{u\in X}\sup_{\lambda\geq 0}\left(\Phi(u)+\lambda\Psi(u)+\rho\lambda\right).$$

2. EXISTENCE OF THREE WEAK SOLUTIONS

In the sequel, X will denote the Sobolev space $W_0^{1,P}(\Omega)$ with the norm

$$\|u\| = \left(\int_\Omega |\nabla u(x)|^p dx\right)^{1/p}.$$

Put $k=\sup_{u\in X\setminus\{0\}}\dfrac{\max_{x\in\bar\Omega}|u(x)|}{\|u\|}$ and put $g(x,t)=\int_0^t f(x,\xi)d\xi$ for each $(x,t)\in\Omega\times R$.
Since $p>N$, one has $k<+\infty$.
Now, we define $\|u\|_I = \left(\int_\Omega \left(\left|\nabla u\right|^p - \dfrac{m(x)}{3}u^3\right)dx\right)^{1/p}$. We have $\|u\|_I$ and $\|u\|$ equivalent such that for positive suitable constants c_1 and c_2:

$$\|u\|_I \leq c_1\|u\| \leq c_2\|u\|_I.$$

Theorem 2. 1. Assume that there exist positive constant c and $w \in X$ such that:
(i) $\|w\| > c/k$; (ii) $\int_\Omega \inf_{t\in[-c,c]} g(x,t)dx > (c/k)^p \int_\Omega g(x,w)dx \,/\, \|w\|^p$.
Then, there exists an open interval $\Lambda \subseteq [0,+\infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$, problem (1) admits at least three solutions in X whose norms are less than q.

Proof: For each $u \in X$, we put $\Phi(u) = \frac{\|u\|_I^p}{p}$, $\Psi(u) = \int_\Omega g(x,u(x))dx$ and $J(u) = \Phi(u) + \lambda\Psi(u)$. In particular, for each $u, v \in X$ one has $\Phi'(u)(v) = \int_\Omega (|\nabla u|^{p-2}\nabla u \nabla u - mu^{p-1}v)dx$, $\Psi'(u)(v) = \int_\Omega f(x,u)vdx.$. It is well known that the critical points of J are the weak solutions of (1), our goal is to prove that Φ and Ψ satisfy the assumptions Theorem A. Clearly, Φ is a continuously $G\hat{a}te$aux differentiable and sequentially weakly lower semi continuous functional with a differentiable whose $G\hat{a}te$aux derivative admits a continuous inverse on X^*, and Ψ is a continuously $G\hat{a}te$aux differentiable functional whose $G\hat{a}te$aux derivative is compact. For each $\lambda > 0$, one has that $\lim_{\|u\| \to +\infty} (\Phi(u) + \lambda\Psi(u)) = +\infty$ and so assumption (i) Theorem 1.1 and Remark 1.1 of [4] holds. Now put $r = \frac{1}{p}(cc_1 / c_2 k)^p$. By assumption (i), one has $0 < r < \Phi(w)$. Moreover, since sup $|u(x)| \le k\|u\|$ for every $u \in X$ and $\|u\| \le \frac{c}{k}$ for every $u \in X$, such that $\Phi(u) \le r$, thus, and from (ii) one has

$$\inf_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u) = \inf_{t \in [-c,c]} \int_\Omega g(x,u)dx \ge \int_\Omega \inf_{t \in [-c,c]} g(x,u)dx$$

$$> (c/k)^p \int_\Omega g(x,w)dx / \|w\|^p \ge (c/c_2 k)^p \int_\Omega g(x,w)dx / \|w\|_I^p = r\frac{\Psi(w)}{\Phi(w)}.$$

Now, our conclusion follow $(b) \to (c)$ Theorem B by choosing $-J = \psi, u_0 = 0$ and Theorem A if we choose $u_1 = w$.

Example 2. 2. Consider the problem

$$\begin{cases} -u''(x) + \lambda(3u^2) = e^x u^2(x), \\ u(0) = u(1) = 0. \end{cases} \tag{4}$$

Taking into account $k = \frac{1}{2}$, choosing $c = \frac{1}{2}$, $w(x) = \begin{cases} -3x & x \in (0,1), \\ 0 & e.w, \end{cases}$ $p = 2$, $f(x,u) = 3u^2$ and $m(x) = e^x$, all assumptions of Theorem 2.1 are satisfied. So there exists an open interval $\Lambda \subseteq [0, +\infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$, problem (4) admits at least three solutions in $W_0^{1,2}([0,1])$ whose norms are less than q.

3. MINIMAX INEQUALITY FOR A SPECIAL CLASS OF FUNCTIONALS

Let r, $\rho \in R$, $w \in X$ be such that $0 < r < \Phi(w)$ *and* $0 < \rho < \Psi(w)$. We put $A(r,w) = r\frac{\Psi(w)}{\Phi(w)}$, $B(r,w) = \frac{1}{p}(A(r,w))^{1/p}$ and $C(\rho,w) = \rho\frac{\Phi(w)}{\Psi(w)}$ and, taking into account that $A(r,w)$>0, C(ρ, w)>0. Put Φ and Ψ, defined as before (in proof of the Theorem 2.1). Assume that there exists continuous function $a(x) \ge 1$ on $[a_1, a_2]$, $(a_1, a_2 \in R)$ such that $g(x,u) \ge a(x)|\nabla u|^p$ for each $u \in X$. So, we have

$$\int_\Omega g(x,u)dx \ge \|u\|^p. \tag{5}$$

Now, we put

$$\beta_1 = \inf\{\|u\|^p \in R^+; \Phi(u) \le r\},$$

$$\beta_2 = \inf\left\{\|u\|^p \in R^+; \frac{1}{P}\inf_{|u(x)|\le\frac{1}{p}\|u\|}(|\nabla u(x)|^p - \frac{m}{3}u^3(x)) \le r\right\} \text{ and } \beta_r = \beta_1 - \beta_2.$$

Clearly, $\beta_1 \ge \beta_2$. In fact $|u(x)| \le \frac{1}{p}\|u\|$ for every $x \in \Omega$ and for every $u \in X$, so that

$$\frac{1}{p}\inf_{|u(x)|\le\frac{1}{p}||u||}(|\nabla u(x)|^p - \frac{m}{3}u^3(x)) \le \frac{1}{p}\int_\Omega(|\nabla u(x)|^p - \frac{m}{3}u^3(x))dx$$

for every $u \in X$, namely

$$\frac{1}{p}\inf_{|u(x)|\le\frac{1}{p}||u||}(|\nabla u(x)|^p - \frac{m}{3}u^3(x)) \le \Phi(u)$$

for every $u \in X$; therefore

$$\left\{\|u\|^p \in R^+; \Phi(u) \le r\right\} \subseteq \left\{\|u\|^p \in R^+; \frac{1}{P}\inf_{|u(x)|\le\frac{1}{p}||u||}(|\nabla u(x)|^p - \frac{m}{3}u^3(x)) \le r\right\}. \text{ Hence, } \beta_r \ge 0.$$

Theorem 3. 1. Assume that there exist $r \in R$, $w \in X$ such that:

(i) $0 < r < \Phi(w)$, (ii) $\frac{1}{p}\inf_{|u(x)|\le B(r,w)-\beta_r}(|\nabla u(x)|^p - \frac{m}{3}u^3(x)) > r$.

Then, there exists $\rho \in R$ such that

$$\operatorname{Sup}_{\lambda\ge 0}\inf_{u\in X}(\Phi(u) + \lambda\Psi(u) + \rho\lambda) < \inf_{u\in X}\sup_{\lambda\ge 0}(\Phi(u) + \lambda\Psi(u) + \rho\lambda).$$

Proof: From (ii) we obtain $B(r,w) - \beta_r \notin \{v \in R^+; \frac{1}{p}\inf_{|u(x)|\le v}(|\nabla u(x)|^p - \frac{m}{3}u^3(x)) \le r\}$. Moreover $\inf_{|u(x)|\le B(r,w)-\beta_r}\{v \in R^+; \frac{1}{p}\inf_{|u(x)|\le v}(|\nabla u(x)|^p - \frac{m}{3}u^3(x)) \le r\} \ge B(r,w) - \beta_r$ In fact, arguing by contradiction, we assume that there is $v_1 \in R^+$ such that

$$\frac{1}{p}\inf_{|u(x)|\le v_1}(|\nabla u(x)|^p - \frac{m}{3}u^3(x)) \le r \text{ and } v_1 \langle A(r,w) - \beta_r, \text{so}$$

$\frac{1}{p}\inf_{|u(x)|\le v_1}(|\nabla u(x)|^p - \frac{m}{3}u^3(x)) \le \frac{1}{p}\inf_{|u(x)|\le v_1}(|\nabla u(x)|^p - \frac{m}{3}u^3(x)) \le r$ and this is a contradiction. Further, taking into account that the function $v \to \frac{1}{p}\inf_{|u(x)|\le v}(|\nabla u(x)|^p - \frac{m}{3}u^3(x))$ is continuous in $[0,+\infty[$, one has

$$\inf\{v \in R^+; \frac{1}{p}\inf_{|u(x)|\le v}(|\nabla u(x)|^p - \frac{m}{3}u^3(x)) \le r\} > B(r,w) - \beta_r.$$

Therefore, inf $\{\frac{1}{p}\|u\| \in R^+; \frac{1}{p}\inf_{|u(x)|\le\frac{1}{p}||u||}(|\nabla u(x)|^p - \frac{m}{3}u^3(x)) \le r\} + \beta_r > B(r,w)$. So, we have inf $\{\|u\|^p \in R^+; \Phi(u) \le r\} > A(r,w)$. Using (5) one has $\inf\left\{\int_\Omega g(x,u)dx; \Phi(u) \le r\right\} > A(r,w)$, namely

$\inf_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u) > r\dfrac{\Psi(w)}{\Phi(w)}$, and, taking into account Theorem B with choice $-J=\psi, u_0=0$ and $u_1=w$, we obtain $\operatorname{Sup}_{\lambda\geq 0}\inf_{u\in X}(\Phi(u)+\lambda\Psi(u)+\rho\lambda) < \inf_{u\in X}\sup_{\lambda\geq 0}(\Phi(u)+\lambda\Psi(u)+\rho\lambda)$.

Theorem 3. 2. Assume that there exist $r\in R,\ w\in X$ such that:

$$\textbf{(i)}\ 0<r<\Phi(w),\ \textbf{(ii)}\ \frac{1}{p}\inf_{|u(x)|\leq B(r,w)}(|\nabla u(x)|^p-\frac{m}{3}u^3(x))>r.$$

Then, there exists $\rho\in R$ such that

$$\operatorname{Sup}_{\lambda\geq 0}\inf_{u\in X}(\Phi(u)+\lambda\Psi(u)+\rho\lambda) < \inf_{u\in X}\sup_{\lambda\geq 0}(\Phi(u)+\lambda\Psi(u)+\rho\lambda).$$

Proof: Taking into account that $\beta_r\geq 0$ one has
$\operatorname{Inf}(|\nabla u(x)|^p-\frac{m}{3}u^3(x))\geq\inf(|\nabla u(x)|^p-\frac{m}{3}u^3(x))>r$ and the conclusion follows from Theorem 3.1.

Proposition 3. 3. The following assertions are equivalent:
(a) there are $r\in R,\ w\in X$ such that:

$$\text{(i)}\ 0<r<\Phi(w),\ \text{(ii)}\ \frac{1}{p}\inf_{|u(x)|\leq B(r,w)}(|\nabla u(x)|^p-\frac{m}{3}u^3(x))>r.$$

(b) there are $\rho\in R,\ w\in X$ such that:

$$\text{(j)}\ 0<\rho<\Psi(w),\ \text{(jj)}\frac{1}{p}\inf_{|u(x)|\leq\frac{1}{p}\sqrt[p]{\rho}}(|\nabla u(x)|^p-\frac{m}{3}u^3(x))>C(\rho,w).$$

Proof: We put $\rho=A(r,w)$. We obtain $r=C(\rho,w)$ and $B(r,w)=\frac{1}{p}\sqrt[p]{\rho}$. Hence, (a) and (b) are equivalent.

Theorem 3. 4. Assume that there exist $\rho\in R,\ w\in X$ such that:

$$\text{(j)}\ 0<\rho<\Psi(w),\text{(jj)}\ \frac{1}{p}\inf_{|u(x)|\leq\frac{1}{p}\sqrt[p]{\rho}}(|\nabla u(x)|^p-\frac{m}{3}u^3(x))>C(\rho,w).$$

Then, there exists $\rho\in R$ such that

$$\operatorname{Sup}_{\lambda\geq 0}\inf_{u\in X}(\Phi(u)+\lambda\Psi(u)+\rho\lambda) < \inf_{u\in X}\sup_{\lambda\geq 0}(\Phi(u)+\lambda\Psi(u)+\rho\lambda).$$

Proof: It follows from Theorem 3.1 and Proposition 3.3.

We now want to point out a simple consequence of Theorem 3.1:

Theorem 3. 5. Let $f:R\to R$ be a positive continuous function. Put $g(t)=\int_0^t f(\xi)d\xi$ for all $t\in R$ and assume that there exist $k_0\in R^+$ such that:

$$\text{(i)}\int_\Omega mu^p dx<0,\ \text{(ii)}\ \frac{1}{p}\inf_{|u(x)|\leq\frac{1}{p}\left(\frac{3k_0^{p-2}\int_\Omega g(x)dx}{3k_0^{p-3}-\int mx^3dx}\right)^{1/p}}(|\nabla u(x)|^p-\frac{m}{3}u^3(x))>k_0^p.$$

Then, there exists $\rho\in R$ such that

$$\operatorname{Sup}_{\lambda\geq 0}\inf_{u\in X}(\Phi(u)+\lambda\Psi(u)+\rho\lambda) < \inf_{u\in X}\sup_{\lambda\geq 0}(\Phi(u)+\lambda\Psi(u)+\rho\lambda).$$

Proof: We put $u(x) = x$ in Ω and $w(x) = k_0 u(x)$. for every $x \in \Omega$. We obtain $\Phi(w) = \frac{k_0^p}{p}(1 - \int_\Omega mx^p dx)$ and $\Psi(w) = \frac{1}{k_0}\int_\Omega g(x)dx.$ Now, we put $r = k_0^p / p.$ So, one has

$B(r,w) = \frac{1}{p}(\frac{3{k_0}^{p-2}\int_\Omega g(x)dx}{3{k_0}^{p-3} - \int_\Omega mx^3 dx})^{1/p}$, that is, the conclusion follows from Theorem 3.2.

Example 3. 6. Consider the problem

$$\begin{cases} -u''(x) + \lambda(96x^2) = -xu^2(x), \\ u(0) = u(1) = 0. \end{cases} \tag{6}$$

By choosing $k_0 = \frac{1}{25}$, $p = 2,$ and $m(x) = -x$ all assumptions of Theorem 3.5 are satisfied. So there exists $\rho \in R$ such that

$$\operatorname{Sup}_{\lambda \geq 0} \inf_{u \in \mathrm{X}} (\Phi(u) + \lambda\Psi(u) + \rho\lambda) < \inf_{u \in X} \sup_{\lambda \geq 0} (\Phi(u) + \lambda\Psi(u) + \rho\lambda).$$

REFERENCES

1. Anello, G. & Cordaro, G. (2002). Positive infinitely many arbitrarily small solutions for the diric- hlet problem involving the p-Laplacian. *Proc. Roy. Soc. Edinburgh Sect. A* (511-519).
2. Bonanno, G. & Livrea, R. (2003). Multiplicity theorems for the Dirichelt problem involving the p-Laplacian. *J. Nonlinear Analysis, 54*, 1-7.
3. Ricceri, B. (2000). On a three critical points theorem. *Arch. Math. (Basel), 75*, 220-226.
4. Bonanno, G. (2002). A minimax inequality and its applications to ordinary differential equations, *J. Math. Anal. Appl*., *270*, 210-229.