"Research Note"

SOME MAXIMAL INEQUALITIES FOR RANDOM VARIABLES AND APPLICATIONS[*]

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Abstract – In this paper, we extend some famous maximal inequalities and obtain strong laws of large numbers for arbitrary random variables by use of these inequalities and martingale techniques.

1. INTRODUCTION

Let $\{X_n, n \geq 1\}$ be a sequence of random variables defined on the probability space (Ω, F, P). The convergence problem with probability one (*w.p.1*) for the sequence $\{\frac{1}{n}(\sum_{i=1}^{n} X_i - E(\sum_{i=1}^{n} X_i))\}$ has been studied by many authors for the sequence of independent random variables. The strong laws of large numbers and complete convergence for ND random variables were studied by Amini and Bozorgnia [1], [2]. In [3] we obtained some maximal inequalities under condition $E[X_n \mid F_{n-1}] = 0$ where $F_n = \sigma(X_1, X_2, \ldots, X_n)$ for all $n \geq 1$. In this paper, we extend some famous maximal inequalities by martingale techniques, and then by using these inequalities, we obtain strong laws of large numbers and some strong limit theorems for arbitrary random variables. To prove our main results we need the following lemma.

Lemma 1. ([4]) Let $\{X_n, F_n, n \geq 1\}$ be a submartingale and $\{b_n, n \geq 1\}$ be a sequence of positive nondecreasing real numbers, then for every $\varepsilon > 0$,

$$P[\max_{1 \leq k \leq n} \frac{|X_k|}{b_k} > \varepsilon] \leq \frac{1}{\varepsilon}(b_1^{-1} EX_1^+ + \sum_{k=2}^{n} b_k^{-1}(EX_k^+ - EX_{k-1}^+)),$$

Remark 1. Under the assumption of Lemma 1 for every $1 \leq m \leq n$, we have,

$$P[\max_{m \leq k \leq n} \frac{|X_k|}{b_k} > \varepsilon] = P[\max_{1 \leq k \leq n-m+1} \frac{|X_{k+m-1}|}{b_{k+m-1}} > \varepsilon] \leq \frac{1}{\varepsilon}(b_m^{-1} EX_m^+ + \sum_{k=2}^{n-m+1} b_{k+m-1}^{-1}(EX_{k+m-1}^+ - EX_{k+m-2}^+))$$

$$= \frac{1}{\varepsilon}(b_m^{-1} EX_m^+ + \sum_{k=m+1}^{n} b_k^{-1}(EX_k^+ - EX_{k-1}^+)).$$

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2. MAXIMAL INEQUALITIES

Hajek and Renyi [5] proved the following important inequality. Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables with $E(X_n) = 0, E(X_n^2) < \infty, n \geq 1$ *and* $\{b_n, n \geq 1\}$ is a positive nondecreasing sequence of real numbers, then for every $m < n$ *and* $\varepsilon > 0$,

$$P[\max_{m \leq k \leq n} \frac{|S_k|}{b_k} \geq \varepsilon] \leq \frac{1}{\varepsilon^2} (\sum_{k=m+1}^{n} \frac{\sigma_k^2}{b_k^2} + \sum_{k=1}^{m} \frac{\sigma_k^2}{b_m^2}),$$

where $\sigma_k^2 = Var(X_k)$.

This inequality has been studied by many authors. The latest literature is given by Liu, et.al. [6] for negative association random variables and Christofides [7] and [8]. We extend this inequality for arbitrary random variables.

Theorem 1. Let $\{X_n, n \geq 1\}$ be a sequence of random variables with $E(X_n) = 0, EX_n^2 < \infty, n \geq 1$, and $\{b_n, n \geq 1\}$ be a sequence of positive nondercreasing real numbers, then for every $\varepsilon > 0$,

i)
$$P[\max_{1 \leq k \leq n} \frac{|S_k|}{b_k} \geq \varepsilon] \leq \frac{\Lambda}{\varepsilon^\Upsilon} (\sum_{k=\backslash}^{n} \frac{\sigma_k^\Upsilon}{b_k^\Upsilon} + \Upsilon \sum_{k=\Upsilon}^{m} \frac{\sigma_k \sum_{i=1}^{k-\backslash} \sigma_i}{b_k^\Upsilon}).$$

ii)
$$P[\max_{m \leq k \leq n} \frac{|S_k|}{b_k} \geq \varepsilon] \leq \frac{8}{\varepsilon^2} (\sum_{k=1}^{m} \frac{\sigma_k^2}{b_m^2} + 2 \sum_{k=2}^{m} \frac{\sigma_k \sum_{i=1}^{k-1} \sigma_i}{b_m^2} + \sum_{k=m+1}^{n} \frac{\sigma_k^2}{b_k^2} + 2 \sum_{k=m+1}^{n} \frac{\sigma_k \sum_{i=1}^{k-1} \sigma_i}{b_j^2}).$$

where $\sigma_k = \sqrt{Var(X_k)}$

Proof: Set $S_{1n} = \sum_{k=1}^{n} X_k^+$ and $S_{2n} = \sum_{k=1}^{n} X_k^-$. Since $E[S_{1n} | F_{n-1}] \geq S_{1(n-1)} . w.p.1.$ and $E[S_{2n} | F_{n-1}] \geq S_{2(n-1)} . w.p.1.$ Hence the sequences $\{S_{1n}, F_n, n \geq 1\}$ and $\{S_{2n}, F_n, n \geq 1\}$ are submartingales, where $F_n = \sigma(X_1, ..., X_n)$ for all $n \geq 1$. In addition, if h is any real convex and nondecreasing function, then $\{h(S_{1n}), F_n, n \geq 1\}$ and $\{h(S_{2n}), F_n, n \geq 1\}$ are also submartingales. Thus for every $\varepsilon > 0$, Lemma 1 implies that

i)
$$P[\max_{1 \leq k \leq n} \frac{S_{1k}}{b_k} \geq \varepsilon] \leq P[\max_{1 \leq k \leq n} \frac{(S_{1k})^2}{b_k^2} \geq \varepsilon^2] \leq \varepsilon^{-2} (b_1^{-2} E(X_1^+)^2 + \sum_{k=2}^{n} b_k^{-2} (E(S_{1k})^2 - E(S_{1(k-1)})^2))$$

$$\leq \frac{1}{\varepsilon^2} (\sum_{k=1}^{n} \frac{\sigma_k^2}{b_k^2} + 2 \sum_{k=2}^{n} \frac{\sigma_k \sum_{i=1}^{k-1} \sigma_i}{b_k^2}),$$

and similarly we have

$$P[\max_{1 \leq k \leq n} \frac{S_{2k}}{b_k} \geq \varepsilon] \leq \frac{1}{\varepsilon^2} (\sum_{k=1}^{n} \frac{\sigma_k^2}{b_k^2} + 2 \sum_{k=2}^{n} \frac{\sigma_k \sum_{i=1}^{k-1} \sigma_i}{b_k^2}),$$

Now, by $|S_n| \leq S_{1n} + S_{2n}$, we obtain

$$P[\max_{1 \leq k \leq n} \frac{|S_k|}{b_k} \geq \varepsilon] \leq P[\max_{1 \leq k \leq n} \frac{S_{1k}}{b_k} \geq \frac{\varepsilon}{2}] + P[\max_{1 \leq k \leq n} \frac{S_{2k}}{b_k} \geq \frac{\varepsilon}{2}] \leq \frac{8}{\varepsilon^2} (\sum_{k=1}^{n} \frac{\sigma_k^2}{b_k^2} + 2 \sum_{k=2}^{n} \frac{\sigma_k \sum_{i=1}^{k-1} \sigma_i}{b_k^2}).$$

ii) By Remark 1 and part i for $1 \leq m < n$,

$$P[\max_{m \leq k \leq n} \frac{|S_k|}{b_k} \geq \varepsilon] \leq P[\max_{m \leq k \leq n} \frac{S_{1k}}{b_k} \geq \frac{\varepsilon}{2}] + P[\max_{m \leq k \leq n} \frac{S_{2k}}{b_k} \geq \frac{\varepsilon}{2}]$$

$$\leq \frac{8}{\varepsilon^2} (\sum_{k=1}^{m} \frac{\sigma_k^2}{b_m^2} + 2 \sum_{k=2}^{m} \frac{\sigma_k \sum_{i=1}^{k-1} \sigma_i}{b_m^2} + \sum_{k=m+1}^{n} \frac{\sigma_k^2}{b_k^2} + 2 \sum_{k=m+1}^{n} \frac{\sigma_k \sum_{i=1}^{k-1} \sigma_i}{b_k^2}).$$

Hence, the proof is complete. In the following we assume $\sum_{i=1}^{0}\sigma_i = 0$ and $S_{1(0)} = S_{2(0)} = 0$.

Corollary 1. Under the assumptions of Theorem 1 we have

i)
$$P[\max_{1\le k\le n}\frac{|S_k|}{b_n}\ge\varepsilon]\le\frac{8}{\varepsilon^2}(\sum_{k=1}^{n}\frac{\sigma_k^2}{b_k^2}+2\sum_{k=2}^{n}\frac{\sigma_k\sum_{i=1}^{k-1}\sigma_i}{b_k^2}).$$

ii)
$$P[\max_{m\le k\le n}\frac{|S_k|}{b_n}\ge\varepsilon]\le\frac{8}{\varepsilon^2}(\sum_{k=m+1}^{n}\frac{\sigma_k^2}{b_k^2}+\sum_{k=1}^{m}\frac{\sigma_k^2}{b_m^2}+2\sum_{k=2}^{m}\frac{\sigma_k\sum_{i=1}^{k-1}\sigma_k}{b_m^2}+2\sum_{k=1}^{n}\frac{\sigma_k\sum_{i=1}^{k-1}\sigma_i}{b_k^2}).$$

Remark 2. We have the following inequalities

i)
$$ES_{1k}^2 - ES_{1(k-1)}^2 = E(X_k^+)^2 + 2E(X_k^+\sum_{j=1}^{k-1}X_j^+)\le\sigma_k^2 + 2\sigma_k\sum_{j=1}^{k-1}\sigma_j;$$

the inequality holds by Cauchy- Schwartz' inequality. Similarly

$$ES_{2k}^2 - ES_{2(k-1)}^2 \le \sigma_k^2 + 2\sigma_k\sum_{j=1}^{k-1}\sigma_j.$$

ii)
$$\sum_{k=1}^{n}\frac{\sigma_k^2}{b_k^2}+2\sum_{k=2}^{n}\frac{\sigma_k\sum_{i=1}^{k-1}\sigma_i}{b_k^2}\le\sum_{k=1}^{n}\frac{1}{b_k^2}(\sigma_k^2+2\sigma_k\sum_{i=1}^{k-1}\sigma_i)\le 2\sum_{k=1}^{n}\frac{\sigma_k\sum_{i=1}^{k}\sigma_i}{b_k^2}.$$

iii) If $\sum_{k=1}^{\infty}\frac{\sigma_k\sum_{i=1}^{k}\sigma_i}{b_k^2}$ converges, then series $\sum_{k=1}^{\infty}\frac{\sigma_k^2}{b_k^2}$ and $\sum_{k=1}^{\infty}\frac{\sigma_k\sum_{i=1}^{k-1}\sigma_i}{b_k^2}$ are converged.

The following corollary is an extension of Kolmogorov's inequality for arbitrary random variables.

Corollary 2. Let $\{X_n, n\ge 1\}$ be a sequence of random variables with $E(X_n)=0, EX_n^2<\infty, n\ge 1$ and $b_k=1,\ k\ge 1$. Then for every $\varepsilon>0$,

$$P[\max_{1\le k\le n}|S_k|\ge\varepsilon]\le\frac{8}{\varepsilon^2}(\sum_{k=1}^{n}\sigma_k^2+2\sum_{k=2}^{n}\sigma_k\sum_{i=1}^{k-1}\sigma_i).).$$

3. SOME STRONG LIMIT THEOREMS

In this section, we use the results of section 2 to prove some strong limit theorems for arbitrary random variables.

Theorem 2. Let $\{X_n, n\ge 1\}$ be a sequence of random variables and $\{b_n, n\ge 1\}$ be a sequence of positive nondecreasing real numbers. If $\sum_{k=1}^{\infty}\frac{\sigma_k\sum_{i=1}^{k}\sigma_i}{b_k^2}<\infty$. then for every $0<\beta<2$,

i) $E(\sup_n(\frac{|S_n|}{b_n})^\beta)<\infty$, and ii) if $b_n\to\infty$, as $n\to\infty$, then $\frac{S_n}{b_n}\to 0$, w.p.1.

where $S_n=\sum_{k=1}^{n}(X_k-E(X_k)),\quad\sigma_n^2=Var(X_n)$ and $\sigma_n=\sqrt{Var(X_n)},\ n\ge 1$.

Proof: i) Note

$$E(\sup_n(\frac{|S_n|}{b_n})^\beta)<\infty\Leftrightarrow\int_1^{\infty}P[\sup_n(\frac{|S_n|}{b_n})>t^{\frac{1}{\beta}}]dt<\infty.$$

By Theorem 1, we get

$$\int_1^\infty P[\sup_n \frac{|S_n|}{b_n} > t^{\frac{1}{\beta}}]dt \le \wedge \int_1^\infty t^{\frac{-\gamma}{\beta}}(\sum_{n=1}^\infty \frac{\sigma_n^\gamma}{b_n^\gamma} + \gamma \sum_{n=1}^\infty \frac{\sigma_n \sum_{i=1}^{n-1}\sigma_i}{b_n^\gamma})dt = C(\sum_{n=1}^\infty \frac{\sigma_n^\gamma}{b_n^\gamma} + \gamma \sum_{n=1}^\infty \frac{\sigma_n \sum_{i=1}^{n-1}\sigma_i}{b_n^\gamma}) < \infty.$$

where $0 < C < \infty$. The second inequality holds by Remark 2.

ii) For every $\varepsilon > 0$, Theorem 1 implies that

$$P[\sup_{k \ge m}\frac{|S_k|}{b_k} > \varepsilon] \le \frac{8}{\varepsilon^2}(\sum_{k=1}^m \frac{\sigma_k^2}{b_m^2} + 2\sum_{k=2}^m \frac{\sigma_k \sum_{i=1}^{k-1}\sigma_i}{b_m^2} + \sum_{k=m+1}^\infty \frac{\sigma_k^2}{b_k^2} + 2\sum_{k=m+1}^\infty \frac{\sigma_k \sum_{i=1}^{k-1}\sigma_i}{b_k^2}),$$

Therefore $\sum_{k=1}^\infty \frac{\sigma_k \sum_{i=1}^k \sigma_i}{b_k^2} < \infty$, and Kronecker's Lemma yield $\lim_{m\to\infty} P[\sup_{k\ge m}\frac{|S_k|}{b_k} > \varepsilon] = 0$. Hence Proposition 5.6 in [9] completes the proof.

Corollary 3. Under the assumptions of Theorem 2, if $\sup_n(\sigma_n \sum_{k=1}^n \sigma_k) < \infty$, then for every $\alpha > \frac{1}{2}$, $\varepsilon > \cdot$ and $m \ge 1$.

i) $$P[\sup_{k\ge m}\frac{|S_k|}{k^\alpha} > \varepsilon] \le \frac{16}{\varepsilon^2} \times \frac{2\alpha}{2\alpha-1} \times m^{1-2\alpha} \sup_n(\sigma_n \sum_{k=1}^n \sigma_k).$$

ii) $\frac{S_n}{n^\alpha} \to 0, w.p.1.$ as $n \to \infty$, and for every $0 < \beta < 2, E\sup_n(\frac{|S_n|}{b_n})^\beta < \infty$.

Theorem 3. Let $\{X_n, n \ge 1\}$ be a sequence of random variables, if $\sum_{k=1}^\infty \sigma_k \sum_{i=1}^k \sigma_i < \infty$ and $E(X_n) = 0$. Then $\sum_{k=1}^\infty X_k$ converges *w.p.1*.

Proof: By Corollary 2 for every $\varepsilon > 0$,

$$P[\sup_{k\ge 1}|S_{n+k} - S_n| > \varepsilon] = \lim_{m\to\infty} P[\max_{1\le k\le m}|S_{n+k} - S_n| > \varepsilon] \le \frac{8}{\varepsilon^2}(\sum_{k=n+1}^\infty \sigma_k^2 + 2\sum_{k=n+1}^\infty \sigma_k \sum_{i=1}^{k-1}\sigma_i).$$

Since by Remark 2 the left hand side of the above inequality tends to zero when $n \to \infty$, Lemma 7.1 in [9] implies that $\sum_{k=1}^\infty X_k$ converges *w.p.*1.

Example: If $\{X_n, n \ge 1\}$ is a sequence of *i.i.d.* random variables with distribution $U[0,1]$, taking $Y_n = \prod_{k=1}^n X_k$ and $\sigma_n = \sqrt{Var(Y_n)}$, we have $\sigma_n < (\frac{1}{\sqrt{3}})^n$, $\sum_{n=1}^\infty EY_n < \infty$, and

$$\sum_{k=1}^\infty \sigma_k \sum_{i=1}^k \sigma_i < \frac{1}{\sqrt{3}-1} \cdot (\sum_{k=1}^\infty 3^{-\frac{k}{2}} - \sum_{k=1}^\infty 3^{-k}) < \infty.$$

Hence these and Theorem 3 prove that $\sum_{n=1}^\infty \prod_{k=1}^n X_k$, converges *w.p.1*.

The following Corollary, which is an extension of Kolmogorov's Theorem, provides strong law of large numbers for arbitrary random variables.

Corollary 4. Let $\{X_n, n \ge 1\}$ be a sequence of random variables with $E(X_n) = \mu_n$ and $Var(X_n) = \sigma_n^2$ for all $n \ge 1$. If $\sum_{k=1}^\infty \frac{\sigma_k \sum_{i=1}^k \sigma_i}{k^2} < \infty$, then $\frac{1}{n}\sum_{k=1}^n (X_k - \mu_k) \to 0$. *w.p.1*

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