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SOME NEW STABILITY AND BOUNDEDNESS RESULTS ON THE SOLUTIONS OF THE NONLINEAR VECTOR DIFFERENTIAL EQUATIONS OF SECOND ORDER*

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Abstract – In this paper, the stability and boundedness of solutions of a second order nonlinear vector differential equation are investigated. Our results include and improve some well-known results in the relevant literature.

Keywords – Boundedness, stability, differential equations of second order

1. INTRODUCTION

As is well-known, the investigation of qualitative properties of solutions, stability, instability, boundedness and asymptotic behavior of solutions and so on, of Hill equation

$$\ddot{x} + a(t)x = 0 \tag{1}$$

and Lienard equation

$$\ddot{x} + f(x,\dot{x})\dot{x} + g(x) = 0 \tag{2}$$

are very important problems in the theory and applications of differential equations. For example, it is significant in the investigation of stability and instability of the geodesic on Riemannian manifolds where Jacobi fields can be expressed in the form of the Hill equation system, [1]. This fact has been used by some physicists to study dynamics in Hamiltonian systems, [2]. Besides, equation (2) is frequently encountered as a mathematical model of most dynamics processes in electromechanical systems of physics and engineering, [3]. It should be noted that details of applications on equations (1) and (2) will not be given here. However, to the best of our knowledge, since the 1950s for the scalar cases, the problems of stability, boundedness and asymptotic behavior of solutions of these types of equations have been extensively studied by many authors; for the related works one can refer to Bellman [4], Bownds [5], Burton and Townsent [6], Burton and Grimmer [7, 8] Cantarelli [9], Hale [10], Heidel [11], Nápoles Valdés [12], Qian [13], Reissig et al [14], C.Tunc and E. Tunc [15], Yoshizawa [16], Zavarykin and Shakhtarin [17], Zhou and Jiang [18] and the references listed therein. In particular, some information about the works performed on the topic can be summarized as follows: In 1953, 1973 and 1969, respectively, Bellman [4], Bownds [5] and Hale [10] studied the asymptotic behavior of solutions of the well-known scalar Hill equation (1). In 1999, Yang [19], based on the theorem related to the qualitative

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behaviors of solutions of equation (1) established in Bellman [4], studied boundedness of solutions of an n-dimensional Hill equation system

$$\ddot{X} + A(t)X = 0\,, \tag{3}$$

where $X \in \Re^n$, $A(t) = (a_{ij}(t))$ is an $n \times n$- symmetric matrix function of time t. Later, in 2005, the authors in [20] derived two results related to the stability and uniform boundedness of the solutions of equation

$$\ddot{X} + A(t)F(X) = 0\,, \tag{4}$$

and a result concerning the boundedness of solutions of equations of the form

$$\ddot{X} + A(t)F(X) = P(t, X, \dot{X})\,. \tag{5}$$

This paper is concerned with a more general class of second-order nonlinear vector differential equations than those mentioned above by (1)-(5);

$$\ddot{X} + B(t)G(X,\dot{X})\dot{X} + A(t)F(X) = P(t, X, \dot{X})\,, \tag{6}$$

in which $X \in \Re^n$, $t \in \Re^+$ and $\Re^+ = [0,\infty)$; A, B and G are $n \times n$-symmetric matrix functions; $F : \Re^n \to \Re^n$, $P : \Re^+ \times \Re^n \times \Re^n \to \Re^n$ and $F(0) = 0$. It is assumed that the functions A, B, G, F and P are continuous. Moreover, the existence and the uniqueness of the solutions of equation (6) will be assumed (see Picard-Lindelof theorem in Rao [3]). It is worth mentioning that equation (6) represents the vector version for a system of real second order nonlinear differential equations of the form

$$\ddot{x}_i + \sum_{k=1}^{n}[b_{ij}(t)g_{jk}(x_1, x_2, ..., x_n; \dot{x}_1, \dot{x}_2, ..., \dot{x}_n)]\dot{x}_k + \sum_{k=1}^{n} a_{ik}(t)f_k(x_1, x_2, ..., x_n)$$
$$= p_i(t; x_1, x_2, ..., x_n; \dot{x}_1, \dot{x}_2, ..., \dot{x}_n)\,,\ (i,\ j = 1,\ 2, ..., n)\,.$$

We consider throughout the paper, in place of equation (6), the equivalent differential system

$$\dot{X} = Y\,,$$
$$\dot{Y} = -B(t)G(X,Y)Y - A(t)F(X) + P(t, X, Y)\,, \tag{7}$$

which was obtained as usual by setting $\dot{X} = Y$ in equation (6).

Let $J_F(X)$ denote the linear operator from the vector function $F(X)$ to the matrix $J_F(X)$, that is,

$$J_F(X) = \left[\frac{\partial f_i}{\partial x_j}\right],\ (i,\ j = 1,\ 2,\ 3, ...,\ n)\,.$$

Other than these, it is assumed that $J_F(X)$ and the derivative $\frac{d}{dt}A(t) = \dot{A}(t)$ are exist and continuous. In addition, it is also supposed that all matrices given in the pairs $A(t)$, $J_F(X)$; $B(t)$, $G(X,Y)$ and $\dot{A}(t)$, $J_F(X)$ are symmetric and commute with each other.

The motivation for the present work has been inspired basically by the papers of Yang [19], Tunç and Şevli [20] and the references listed in those papers. It is worth mentioning once again that the result of Yang [19] was established about an n-dimensional linear vector differential equation, (3). He only proved a result on the boundedness of solutions of equation (3) without an example on the topic. In spite of the case in [19], the class of equation considered here is more complicated than the equation considered there. Next, as shown above, the authors in [20] established some sufficient conditions on the above mentioned subjects with a scalar example. However, we will prove here three results on the same and different topics with a vectorial example. Namely, by the first result, Theorem 3, we establish sufficient conditions which guarantee the stability of the trivial solution $X = 0$ of equation (6), when $P(t,X,Y) = 0$. In the same case, $P(t,X,Y) = 0$, we construct sufficient conditions, by Theorem 4, which ensure uniform-boundedness of the solutions of equation (6). Next, we prove a boundedness theorem, Theorem 5, for all solutions of equation (6) to be bounded. The boundedness result, which is stated by Theorem 5, is of the type in which the bounding constant depends on the solution in question as the same as in Yang [19]. It should also be pointed out that there is extensive literature on scalar differential equations of (1), (2) and vector differential equation (3), which are special cases of equation (6). However, we do not want to summarize details of the works on the related qualitative behaviors of equations (1), (2) and (3). Finally, in nearly all the above mentioned papers the Lyapunov's direct (or second) method [21] was used as a basic tool for proving the results established there. It is also worth mentioning that so far, perhaps the most effective method to determine the stability behavior of solutions of linear and non-linear differential equations is still the Lyapunov's direct (or second) method. The major advantage of this method is that stability in the large, instability and boundedness of solutions can be obtained without any prior knowledge of solutions. Today, this method is widely recognized as an excellent tool not only in the study of differential equations, but also in the theory of control systems, dynamical systems, systems with a time lag, power system analysis, time varying non-linear feedback systems, and so on. In the present paper, we also use the same method as a basic tool for verifying our main results.

Notations and definitions

The symbol $\langle X,Y \rangle$ corresponding to any pair X, Y in $\Re^n$ stands for the usual scalar product $\sum_{i=1}^{n} x_i y_i$, that is, $\langle X,Y \rangle = \sum_{i=1}^{n} x_i y_i$, thus $\|X\|^2 = \langle X,X \rangle$, and $\lambda_i(A)$, $(i = 1,2,\ldots,n)$, are the eigenvalues of the $n \times n$ - matrix A, $A = (a_{ij})$, $(i,\ j = 1,\ 2,\ldots,\ n)$. It is also well-known that a real symmetric matrix $A = (a_{ij})$, $(i,\ j = 1,\ 2,\ldots,\ n)$ is said to be positive definite if and only if the quadratic form $X^T AX$ is positive definite, where $X \in \Re^n$ and X^T denotes the transpose of X.

2. PRELIMINARIES

Throughout the paper, we require some preliminary results which we now state. Consider the non-autonomous differential system

$$\frac{dx}{dt} = F(t,x), \tag{8}$$

where x is an n-vector, $t \in [0,\infty)$. Suppose that $F(t,x)$ is continuous in (t,x) on $I \times D$, where D is a connected open set in $\Re^n$. Now, we shall dispose of the following theorems and the lemmas which will be required in the proof of our main results.

Theorem 1. Suppose that $F(t,0) = 0$ in (8) and there exists a Lyapunov function $V = V(t,x)$ defined on $0 \le t < \infty$, $\|x\| < H$, $H > 0$, which satisfies the following conditions;

(i) $V(t,0) = 0$,

(ii) $a(\|x\|) \le V(t,x)$, where $(CIP$ denotes the families of continuous increasing and positive definite functions).

(iii) $\dot{V}_{(8)}(t,x) \le 0$.

Then the solution $x(t) \equiv 0$ of system (8) is stable.

Proof: See Yoshizawa [16].

Theorem 2. Suppose that there exists a Lyapunov function $V = V(t,x)$ defined on $0 \le t < \infty$, $\|x\| \ge R$, where R may be large, which satisfies the following conditions;
(i) $a(\|x\|) \le V(t,x) \le b(\|x\|)$, where $a(r) \in CI$, $a(r) \to \infty$ as $r \to \infty$ and $b(r) \in CI$,
(CI denotes families of continuous increasing functions).
(ii) $\dot{V}_{(8)}(t,x) \le 0$.
Then the solutions of system (8) are uniform-bounded.

Proof: See Yoshizawa [16].

Lemma 1. Let A be a real symmetric $n \times n$ -matrix and
$a' \ge \lambda_i(A) \ge a > 0$ $(i = 1,\ 2, \dots,\ n)$, where a' and a are constants.
Then

$$a' \langle X, X \rangle \ge \langle AX, X \rangle \ge a \langle X, X \rangle$$

and

$$a'^2 \langle X, X \rangle \ge \langle AX, AX \rangle \ge a^2 \langle X, X \rangle.$$

Proof: See Mirsky [22].

Lemma 2. Let Q, D be any two real $n \times n$ commuting symmetric matrices $\Gamma = \Gamma(t, X, Y)$. Then
(i) The eigenvalues $\lambda_i(QD)$ and $(i = 1,\ 2, \dots,\ n)$ of the product matrix QD are real and satisfy
$\max\limits_{1 \le j,k \le n} \lambda_j(Q)\lambda_k(D) \ge \lambda_i(QD) \ge \min\limits_{1 \le j,k \le n} \lambda_j(Q)\lambda_k(D)$.
(ii) The eigenvalues $\lambda_i(Q + D)$ and $(i = 1,\ 2, \dots,\ n)$ of the sum of matrices Q and D are real and satisfy
$\left\{\max\limits_{1 \le j \le n} \lambda_j(Q) + \max\limits_{1 \le k \le n} \lambda_k(D)\right\} \ge \lambda_i(Q + D) \ge \left\{\min\limits_{1 \le j \le n} \lambda_j(Q) + \min\limits_{1 \le k \le n} \lambda_k(D)\right\}$,
where $\lambda_j(Q)$ and $\lambda_k(D)$ are, respectively, the eigenvalues of Q and D.

Proof: See Mirsky [22].

3. MAIN RESULTS

In the case $P \equiv 0$ the following results are established.

Theorem 3. In addition to the fundamental assumptions imposed on the functions A, B, G and F that appeared in (7) we suppose that there are positive constants a_0, b_0, a and b such that
(i) The matrices A, $\dot{A}$ and B are symmetric, and $\lambda_i(A(t)) \ge a_0$, $\lambda_i(\dot{A}(t)) \le 0$ and $\lambda_i(B(t)) \ge b_0$ for all $t \in [0, \infty)$, $(i = 1, 2, \dots, n)$.
(ii) $G(X,Y)$ is symmetric and $\lambda_i(G(X,Y)) \ge b$ for all X, $Y \in \Re^n$, $(i = 1, 2, \dots, n)$.
(iii) $J_F(X)$ is symmetric and $\lambda_i(J_F(X)) \ge a$ for all $X \in \Re^n$, $(i = 1, 2, \dots, n)$.
Then the trivial solution $X = 0$ of equation (6) is stable.
Now, throughout our three main results, we will use, as a basic tool, a continuously differentiable Lyapunov function $\Gamma = \Gamma(t, X, Y)$, which is defined by:

$$\Gamma := \int_0^1 \langle A(t) F(\sigma X), X \rangle d\sigma + \frac{1}{2} \langle Y, Y \rangle. \tag{9}$$

Remark 1. It should be noted that Theorem 3 improves the results of Yang [19] and Tunç and Şevli [20], and includes that in [20, Theorem 3].

Proof: Because of $F(0)=0$, it is clear that $\Gamma(t,0,0)=0$. Now, since

$$F(0)=0\,,\ \frac{\partial}{\partial\sigma}F(\sigma X)=J_F(\sigma X)X\,,$$

then

$$F(X)=\int_0^1 J_F(\sigma X)X d\sigma\,. \tag{10}$$

Hence, assumptions (i) and (iii) of Theorem 3 and (10) show that

$$\int_0^1 \langle A(t)F(\sigma X),X\rangle d\sigma=\int_0^1\int_0^1 \langle\sigma_1 A(t)J_F(\sigma_1\sigma_2 X)X,X\rangle d\sigma_2 d\sigma_1$$

$$\geq \int_0^1\int_0^1 \langle\sigma_1 a_0 aX,X\rangle d\sigma_2 d\sigma_1=\frac{a_0 a}{2}\langle X,X\rangle=\frac{a_0 a}{2}\|X\|^2\,.$$

In view of (9) and the above inequality, it follows that

$$\Gamma\geq\frac{1}{2}\left(a_0 a\|X\|^2+\|Y\|^2\right). \tag{11}$$

Thus, it is clear that the function Γ defined by (9) is positive definite. Along any solution (X,Y) of system (7) it follows from (9) and (7) that

$$\dot\Gamma=\frac{d}{dt}\Gamma(t,X,Y)=-\langle A(t)F(X),Y\rangle-\langle B(t)G(X,Y)Y,Y\rangle$$
$$+\frac{d}{dt}\int_0^1\langle A(t)F(\sigma X),X\rangle d\sigma\,. \tag{12}$$

Clearly,

$$\frac{d}{dt}\int_0^1\langle A(t)F(\sigma X),X\rangle d\sigma=\int_0^1\sigma\langle A(t)J_F(\sigma X)Y,X\rangle d\sigma+\int_0^1\langle A(t)F(\sigma X),Y\rangle d\sigma$$
$$+\int_0^1\left\langle \dot A(t)F(\sigma X),X\right\rangle d\sigma$$
$$=\int_0^1\sigma\frac{\partial}{\partial\sigma}\langle A(t)F(\sigma X),Y\rangle d\sigma+\int_0^1\langle A(t)F(\sigma X),Y\rangle d\sigma$$
$$+\int_0^1\left\langle \dot A(t)F(\sigma X),X\right\rangle d\sigma$$
$$=\sigma\langle A(t)F(\sigma X),Y\rangle\Big|_0^1+\int_0^1\left\langle \dot A(t)F(\sigma X),X\right\rangle d\sigma$$
$$=\langle A(t)F(X),Y\rangle+\int_0^1\left\langle \dot A(t)F(\sigma X),X\right\rangle d\sigma \tag{13}$$

Substituting estimate (13) into (12) we obtain

$$\dot{\Gamma} = - \langle B(t)G(X,Y)Y, Y \rangle + \int_0^1 \langle \dot{A}(t)F(\sigma X), X \rangle d\sigma \tag{14}$$

In view of assumptions (i) and (ii) of Theorem 3, Lemma 1 and (14), it is clear that

$$\dot{\Gamma} \leq - \langle b_0 b Y, Y \rangle = - b_0 b \|Y\|^2 \leq 0 \ .$$

Hence, by a simple extension of Theorem 1 just stated on equation (8), it can be easily followed that the origin $X = 0$ is stable.
This completes the proof of Theorem 3.

Example: Le us take in (7), for $n = 2$, that $A(t) = B(t) = I$, where I is $n \times n$ identity matrix,

$$G = \begin{bmatrix} 1 + x^2 + y^2 & x^2 \\ x^2 & 1 + x^2 + y^2 \end{bmatrix}$$

and

$$F = \begin{bmatrix} x + x^3 \\ y + y^3 \end{bmatrix}.$$

Clearly, G is a symmetric matrix. Then, by an easy calculation, we obtain eigenvalues of the matrices G and $J_F(X)$, $J_F(X)$ is obtained from the function F, as follows:

$$\lambda_1(G) = 1 + y^2, \ \lambda_2(G) = 1 + 2x^2 + y^2$$

and

$$J_F(X) = \begin{bmatrix} 1 + 3x^2 & 0 \\ 0 & 1 + 3y^2 \end{bmatrix}.$$

Next, it follows that $\lambda_1(J_F) = 1 + 3x^2$ and $\lambda_2(J_F) = 1 + 3y^2$, and $\lambda_i(G) \geq 1 = b$ and $\lambda_i(J_F) \geq 1 = a$, $(i = 1, 2)$. Thus, all the conditions of Theorem 3 are satisfied. It should be noted that, when G and F reduce to the linear case, our conclusions are also valid.

Theorem 4. In addition to the fundamental assumptions imposed on the functions A, B, G and F that appeared in (7), we assume that there exist some positive constants a_0, a_0', b_0, a and b and η such that the following conditions are fulfilled:
(i) The matrices A, B and $\dot{A}$ are symmetric, and $a_0' \geq \lambda_i(A(t)) \geq a_0, \lambda_i(\dot{A}(t)) \leq 0$ and $\lambda_i(B(t)) \geq b_0$ for all $t \in [0, \infty)$, $(i = 1, 2, ..., n)$.
(ii) $G(X,Y)$ is symmetric and $\lambda_i(G(X,Y)) \geq b$ for all X, $Y \in \Re^n$, $(i = 1, 2, ..., n)$.
(iii) $J_F(X)$ is symmetric and $\eta \geq \lambda_i(J_F(X)) - a \geq 0$ for all $X \in \Re^n$, $(i = 1, 2, ..., n)$.
Then all solutions of system (7) are uniformly bounded.

Remark 2. It should be noted that Theorem 4 improves and includes the results of Yang [19] and Tunç and Şevli [20, Theorem 4].

Proof: Clearly, the assumptions of Theorem 4 yield that

$$\int_0^1 \langle A(t)F(\sigma X), X\rangle d\sigma = \int_0^1\int_0^1 \langle \sigma_1 A(t) J_F(\sigma_1\sigma_2 X)X, X\rangle d\sigma_2 d\sigma_1$$

$$\leq \int_0^1\int_0^1 \langle \sigma_1 a_0'(a+\eta)X, X\rangle d\sigma_2 d\sigma_1 = \left(\frac{a_0'(a+\eta)}{2}\right)\|X\|^2 . \tag{15}$$

In view of (9), (11) and (15), it follows that

$$K_1\left(\|X\|^2 + \|Y\|^2\right) \leq V_0(t,X,Y) \leq K_2(\|X\|^2 + \|Y\|^2),$$

where $K_1 = \min\left\{\frac{a_0 a}{2}, \frac{1}{2}\right\}$ and $K_2 = \max\left\{\frac{1}{2}, \frac{a_0'(a+\eta)}{2}\right\}$.

The remainder of the proof proceeds just as in the proof of Theorem 3, and therefore we omit the details of proof.

Next, in the case $P \neq 0$, the following result is established.

Theorem 5. Assume that all the assumptions of Theorem 3 hold and also that the function P satisfies

$$\|P(t,X,Y)\| \leq (A + \|Y\|)\theta(t),$$

where $\theta(t)$ is a non-negative and continuous function of *t* and satisfies $\int_0^t \theta(s)ds \leq B < \infty$ for all $t \geq 0$, A and B are positive constants. Then there exists a positive constant K such that any solution $(X(t), Y(t))$ of system (7) determined by

$$X(0) = X_0,\ Y(0) = Y_0$$

satisfies for all $t \geq 0$,

$$\|X(t)\| \leq K,\ \|Y(t)\| \leq K .$$

Remark 3. It should be noted that Theorem 5 improves and includes the result of Yang [19]. Namely, our assumptions are less restrictive than those established by Yang [19] because of (2) and the assumptions $K\|X\|^2 \leq X^T A(t)X \leq \bar{K}\|X\|^2$, where K and $\bar{K}$ are positive constants, and $X^T \dot{A}(t)X \geq 0$, which were established in [19]. Theorem 5 improves and includes the result established in [20, Theorem 5].

Proof: Consider the function Γ defined by (9) again. Then, under the assumptions of Theorem 5 it can be easily seen that

$$\Gamma \geq K_1(\|X\|^2 + \|Y\|^2), \tag{16}$$

where $K_1 = \min\left\{\frac{a_0 a}{2}, \frac{1}{2}\right\}$. Similarly, since $P \neq 0$, then along any solution $(X(t), Y(t))$ of system (7), the conclusion concerning the total derivative of the function Γ with respect to t can be revised as follows:

$$\begin{aligned}\dot{\Gamma} &\leq -b_0 b\|Y\|^2 + \langle Y, P(t,X,Y)\rangle \\ &\leq \|Y\|\ \|P(t,X,Y)\| \\ &\leq \|Y\|\ (A + \|Y\|)\theta(t),\end{aligned} \tag{17}$$

because of $b_0 > 0$, $b > 0$ and the assumption $\|P(t,X,Y)\| \leq (A + \|Y\|)\theta(t)$ of Theorem 5.

In view of the inequality

$$\|Y\| \le 1 + \|Y\|^2$$

it follows from (17) that

$$\dot{\Gamma} \le \left[A + (A+1)\|Y\|^2 \right]\theta(t)$$

Hence

$$\dot{\Gamma} \le K_3\left[1 + \|Y\|^2\right]\theta(t)\,, \tag{18}$$

where $K_3 = A + 1$.
The assumptions of Theorem 5 and (16) show that

$$\dot{\Gamma} \le K_3\theta(t) + K_4\Gamma\theta(t)\,, \tag{19}$$

where $K_4 = K_3 / K_1$. Integration of both sides of (18) from 0 to t $(t \ge 0)$ leads to the inequality

$$\Gamma(t) - \Gamma(0) \le K_3\int_0^t \theta(s)ds + K_4\int_0^t \Gamma(s)\theta(s)ds\,.$$

On putting $K_5 = \Gamma(0) + K_3 B$, we obtain that

$$\Gamma(t) \le K_5 + K_4\int_0^t \Gamma(s)\theta(s)ds\,.$$

Gronwall-Reid-Bellman inequality, (see [3]), yields

$$\Gamma(t) \le K_5 \exp\left(K_4\int_0^t \theta(s)ds \right).$$

The proof of Theorem 5 is now complete.

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