SOME GENERALIZATIONS OF THE SEQUENCE SPACE a_p^r *

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Abstract – In the present paper, the sequence space $a^r(u, p)$ of a non-absolute type is introduced and it is proved that the space $a^r(u, p)$ is linearly isomorphic to the Maddox's space $\ell(p)$. Besides this, the basis is constructed and the α-, β- and γ-duals are computed for the space $a^r(u, p)$. Furthermore, some matrix mappings from $a^r(u, p)$ to some sequence spaces are characterized. The final section of the paper is devoted to some consequences related to the rotundity of the space $a^r(u, p)$.

Keywords – Paranormed sequence space, α-, β- and γ-duals, matrix mappings and rotundity of a sequence space

1. INTRODUCTION, DEFINITIONS AND NOTATION

By w, we denote the space of all real or complex sequences. Any vector subspace of w is called a *sequence space*. We write ℓ_∞, c and c_0 for the spaces of all bounded, convergent and null sequences, respectively. Also by *bs*, *cs*, ℓ_1 and ℓ_p, we denote the spaces of all bounded, convergent, absolutely and *p*-absolutely convergent series, respectively; where 1<p<∞.

A linear topological space X over the real field $\mathbb{R}$ is said to be a *paranormed space* if there is a subadditive function $g:X \rightarrow \mathbb{R}$ such that $g(\theta)=0$, $g(x)=g(-x)$ and scalar multiplication is continuous, i.e., $|\alpha_n - \alpha| \rightarrow 0$ and $g(x_n - x) \rightarrow 0$ imply $g(\alpha_n x_n - \alpha x) \rightarrow 0$ for all α's in $\mathbb{R}$ and all x's in X, where θ is the zero vector in the linear space X.

Assume here and after that $u=(u_k)$ be a sequence such that $u_k \neq 0$ for all $k \in \mathbb{N}$ and (p_k) be a bounded sequence of strictly positive real numbers with $\sup p_k = H$ and $L = max\{1, H\}$; where $\mathbb{N}=\{0, 1, 2, ...\}$. Then, the linear spaces $\ell(p)$ and $\ell_\infty(p)$ were defined by Maddox [1] (see also Simons [2] and Nakano [3]) as follows:

$$\ell(p) = \left\{ x = (x_k) \in w : \sum_k |x_k|^{p_k} < \infty \right\}, \, (0<p_k \leq H < \infty)$$

and

$$\ell_\infty(p) = \left\{ x = (x_k) \in w : \sup_{k \in \mathbb{N}} |x_k|^{p_k} < \infty \right\}$$

which are the complete spaces paranormed by

$$g_1(x) = \left(\sum_k |x_k|^{p_k} \right)^{1/L} \text{ and } g_2(x) = \sup_{k \in \mathbb{N}} |x_k|^{p_k/L} \text{ iff } inf\, p_k > 0,$$

respectively. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. We shall assume throughout that $0 < r < 1$, $p_k^{-1} + (p'_k)^{-1} = 1$ provided $1 < inf\, p_k \leq H < \infty$ and denote the collection of all finite subsets of $\mathbb{N}$ by $\mathcal{F}$.

Let λ, μ be any two sequence spaces and $A=(a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where n, $k \in \mathbb{N}$. Then, we say that A defines a *matrix mapping* from λ into μ, and we denote it by writing $A:\lambda \to \mu$, if for every sequence $x=(x_k) \in \lambda$ the sequence $Ax=\{(Ax)_n\}$, the *A-transform* of x, is in μ, where

$$(Ax)_n = \sum_k a_{nk} x_k , \ (n \in \mathbb{N}) \tag{1}$$

By $(\lambda:\mu)$, we denote the class of all matrices A such that $A:\lambda \to \mu$. Thus, $A \in (\lambda:\mu)$ if and only if the series on the right side of (1) converges for each $n \in \mathbb{N}$ and each $x \in \lambda$, and we have $Ax=\{(Ax)_n\}_{n \in \mathbb{N}} \in \mu$ for all $x \in \lambda$. A sequence x is said to be *A-summable* to α if Ax converges to α which is called the *A*-limit of x.

The main purpose of this paper, which is a continuation of Aydın and Başar [4-7], is to introduce the sequence space $a^r(u, p)$ of a non-absolute type which is defined in Section 2, below. Furthermore, the basis is constructed and the α-, β- and γ-duals are computed for the space $a^r(u, p)$. Besides this, the matrix mappings from the space $a^r(u, p)$ to some other sequence spaces are characterized. Finally, some results related to the rotundity of the space $a^r(u, p)$ are derived.

2. THE SEQUENCE SPACE $a^r(u, p)$ OF NON-ABSOLUTE TYPE

In this section, we define the complete paranormed linear sequence space $a^r(u, p)$ and give the basis for the space. Finally, we determine the α-, β- and γ-duals of the space $a^r(u, p)$.

The *matrix domain* λ_A of an infinite matrix A in a sequence space λ is defined by

$$\lambda_A = \{ x = (x_k) \in w : Ax \in \lambda \} \tag{2}$$

which is a sequence space. Choudhary and Mishra [8] have defined the sequence space $\overline{\ell(p)}$ which consists of all sequences such that *S*-transforms of them are in $\ell(p)$, where $S=(s_{nk})$ is defined by

$$s_{nk} = \begin{cases} 1 & , \ (0 \leq k \leq n) \\ 0 & , \ (k > n) \end{cases}$$

for all k, $n \in \mathbb{N}$. Başar and Altay [9] have recently examined the space *bs(p)*, which is formerly defined by Başar in [10], as the set of all series whose sequences of partial sums are in $\ell_\infty(p)$. Quite recently, Altay and Başar [11] have studied the sequence spaces $r^t(p)$ and $r_\infty^t(p)$ which are derived from the sequence spaces $\ell(p)$ and $\ell_\infty(p)$ of Maddox by the Riesz means R^t, respectively. With the notation of (2), the spaces $\overline{\ell(p)}$, *bs(p)*, $r^t(p)$ and $r_\infty^t(p)$ can be redefined by

$$\overline{\ell(p)} = [\ell(p)]_S , \ bs(p) = [\ell_\infty(p)]_S , \ r^t(p) = [\ell(p)]_{R^t} , \ r_\infty^t(p) = [\ell_\infty(p)]_{R^t} .$$

Following Choudhary and Mishra [8], Başar and Altay [9], and Altay and Başar [11], we introduce the sequence space $a^r(u, p)$ as the set of all sequences such that their A^r-transforms are in the space $\ell(p)$, that is

$$a^r(u, p) = \left\{ x = (x_k) \in w : \sum_k \left| \frac{1}{k+1} \sum_{j=0}^{k} (1 + r^j)u_j x_j \right|^{p_k} < \infty \right\}, \quad (0 < p_k \leq H < \infty),$$

where A^r denotes the matrix $A^r = (a_{nk}^r)$ defined by

$$a_{nk}^r = \begin{cases} \dfrac{1 + r^k}{n+1} u_k & , \quad (0 \leq k \leq n) \\ 0 & , \quad (k > n) \end{cases}$$

for all $k, n \in \mathbb{N}$. It is trivial that in the cases $p_k = p$ for all $k \in \mathbb{N}$ and $(u_k) = e = (1, 1, 1, ...)$; the sequence space $a^r(u, p)$ is reduced to the sequence spaces $a_p^r(u)$ and a_p^r, respectively, where the sequence space a_p^r is introduced by Aydın and Başar [7]. With the notation of (2), we can redefine the space $a^r(u, p)$ as follows:

$$a^r(u, p) = [\ell(p)]_{A^r}.$$

Define the sequence $y = \{y_k(r)\}$, which will be frequently used, as the A^r-transform of a sequence $x = (x_k)$, i.e.,

$$y_k(r) = \sum_{j=0}^{k} \frac{1 + r^j}{k+1} u_j x_j; \quad (k \in \mathbb{N}). \tag{3}$$

Now, we may begin with the following theorem which is essential in the text:

Theorem 2. 1. $a^r(u, p)$ is the complete linear metric space paranormed by g defined by

$$g(x) = \left[\sum_k \left| \frac{1}{k+1} \sum_{j=0}^{k} (1 + r^j)u_j x_j \right|^{p_k} \right]^{1/L},$$

where $0 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$.

Proof: The linearity of $a^r(u, p)$ with respect to the coordinatewise addition and scalar multiplication follows from the inequalities which are satisfied for $x, z \in a^r(u, p)$, (see [12, p. 30]) and for any $\alpha \in \mathbb{R}$ (see [13]), respectively,

$$\left[\sum_k \left| \frac{1}{k+1} \sum_{j=0}^{k} (1 + r^j)u_j(x_j + z_j) \right|^{p_k} \right]^{1/L} \leq \left[\sum_k \left| \frac{1}{k+1} \sum_{j=0}^{k} (1 + r^j)u_j x_j \right|^{p_k} \right]^{1/L} \tag{4}$$

$$+ \left[\sum_k \left| \frac{1}{k+1} \sum_{j=0}^{k} (1 + r^j)u_j z_j \right|^{p_k} \right]^{1/L}$$

and

$$|\alpha|^{p_k} \leq \max\left\{ 1, |\alpha|^L \right\}. \tag{5}$$

It is clear that $g(\theta) = 0$ and $g(x) = g(-x)$ for all $x \in a^r(u, p)$. Additionally, the inequalities (4) and (5) yield the subadditivity of g and

$$g(\alpha x) \leq \max\left\{ 1, |\alpha| \right\} g(x).$$

Let $\{x^n\}$ be any sequence of the points $a^r(u, p)$ such that $g(x^n - x) \to 0$ and (α_n) also be any sequence of

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scalars such that $\alpha_n \to \alpha$ as $n \to \infty$. Then, since the inequality

$$g(x^n) \leq g(x) + g(x^n - x)$$

holds by the subadditivity of g, $\{g(x^n)\}$ is bounded and thus we have

$$g(\alpha_n x^n - \alpha x) = \left[\sum_k \left| \frac{1}{k+1} \sum_{j=0}^{k} (1 + r^j) u_j (\alpha_n x_j^n - \alpha x_j) \right|^{p_k} \right]^{1/L}$$

$$\leq |\alpha_n - \alpha| g(x^n) + |\alpha| g(x^n - x)$$

which tends to zero as $n \to \infty$. That is to say that the scalar multiplication is continuous. Hence, g is a paranorm on the space $a^r(u, p)$.

It remains to prove the completeness of the space $a^r(u, p)$. Let $\{x^i\}$ be any Cauchy sequence in the space $a^r(u, p)$, where

$$x^i = \left\{ x_0^{(i)}, x_1^{(i)}, x_2^{(i)}, \dots \right\}.$$

Then, for a given $\varepsilon > 0$ there exists a positive integer $n_0(\varepsilon)$ such that

$$g(x^i - x^j) < \varepsilon \tag{6}$$

for all $i, j > n_0(\varepsilon)$. Using the definition of g, we obtain for each fixed $k \in \mathbb{N}$ that

$$\left| (A^r x^i)_k - (A^r x^j)_k \right| \leq \left[\sum_k \left| (A^r x^i)_k - (A^r x^j)_k \right|^{p_k} \right]^{1/L} < \varepsilon$$

for every $i, j \geq n_0(\varepsilon)$ which leads us to the fact that

$$\left\{ (A^r x^0)_k, (A^r x^1)_k, (A^r x^2)_k, \dots \right\}$$

is a Cauchy sequence of real numbers for every fixed $k \in \mathbb{N}$. Since $\mathbb{R}$ is complete, it converges, say $(A^r x^i)_k \to (A^r x)_k$ as $i \to \infty$. Using these infinitely many limits $(A^r x)_0, (A^r x)_1, (A^r x)_2, \dots$, we define the sequence $\{(A^r x)_0, (A^r x)_1, (A^r x)_2, \dots\}$. From (6) for each $m \in \mathbb{N}$ and $i, j \geq n_0(\varepsilon)$

$$\sum_{k=0}^{m} \left| (A^r x^i)_k - (A^r x^j)_k \right|^{p_k} \leq g(x^i - x^j)^L < \varepsilon^L. \tag{7}$$

Take any $i \geq n_0(\varepsilon)$. First let $j \to \infty$ in (7) and after $m \to \infty$, to obtain $g(x^i - x) \leq \varepsilon$. Finally, taking $\varepsilon = 1$ in (7) and letting $i \geq n_0(1)$, we have by Minkowski's inequality for every fixed $m \in \mathbb{N}$, that

$$\left[\sum_{k=0}^{m} \left| (A^r x)_k \right|^{p_k} \right]^{1/L} \leq g(x^i - x) + g(x^i) \leq 1 + g(x^i)$$

which implies that $x \in a^r(u, p)$. Since $g(x^i - x) \leq \varepsilon$ for all $i \geq n_0(\varepsilon)$, it follows that $x^i \to x$ as $i \to \infty$, whence we have shown that $a^r(u, p)$ is complete.

Therefore, one can easily check that the absolute property does not hold on the space $a^r(u, p)$ that is $g(x) \neq g(|x|)$; where $|x| = (|x_k|)$. This says that $a^r(u, p)$ is the sequence space of the non-absolute type.

A sequence space λ with a linear topology is called a *K-space*, provided each of the maps $p_i : \lambda \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$; where $\mathbb{C}$ denotes the complex field. A *K*-space λ is called an *FK-space* provided λ is a complete linear metric space. An *FK*-space whose topology is normable is called a *BK-space*. Now, we may give the following:

Theorem 2. 2. $a_p^r(u)$ is the linear space under the coordinatwise addition and scalar multiplication, which is the *BK*-space with the norm

$$\|x\| = \left[\sum_k \left| \frac{1}{k+1} \sum_{j=0}^{k} (1 + r^j)u_j x_j \right|^p \right]^{1/p} ;$$

where $1 \le p < \infty$.

Proof: Because the first part of the theorem is a routine verification, we omit the detail. Since ℓ_p is the *BK*-space with respect to its usual norm (see [12, pp. 217-218]) and A^r is a normal matrix, Theorem 4.3.2 of Wilansky [14, p. 61] gives the fact that $a_p^r(u)$ is the *BK*-space, where $1 \le p < \infty$.

Theorem 2. 3. The sequence space $a^r(u, p)$ of the non-absolute type is linearly isomorphic to the space $\ell(p)$, where $0 < p_k \le H < \infty$ for all $k \in \mathbb{N}$.

Proof: To prove the theorem, we should show the existence of a linear bijection between the spaces $a^r(u, p)$ and $\ell(p)$. With the notation of (3), define the transformation T from $a^r(u, p)$ to $\ell(p)$ by $x \mapsto y = Tx$. The linearity of T is trivial. Further, it is obvious that $x = \theta$ whenever $Tx = \theta$ and hence T is injective.

Let $y = (y_k) \in \ell(p)$ and define the sequence $x = \{x_k(r)\}$ by

$$x_k(r) = \sum_{j=k-1}^{k} (-1)^{k-j} \frac{1+j}{(1+r^k)u_k} y_j ; \quad (k \in \mathbb{N}).$$

Then, we have

$$g(x) = \left[\sum_k \left| \frac{1}{k+1} \sum_{j=0}^{k} (1 + r^j)u_j x_j \right|^{p_k} \right]^{1/L} = \left(\sum_k |y_k|^{p_k} \right)^{1/L} = g_1(y) < \infty.$$

Thus, we have that $x \in a^r(u, p)$ and consequently T is surjective and is paranorm preserving. Hence, T is a linear bijection and this tells us that the spaces $a^r(u, p)$ and $\ell(p)$ are linearly isomorphic, as desired.

Let us suppose that $1 < p_k \le s_k$ for all $k \in \mathbb{N}$. Then, it is known that $\ell(p) \subset \ell(s)$ which leads us to the immediate consequence that $a^r(u, p) \subset a^r(u, s)$.

We first define the concept of the Schauder basis for a paranormed sequence space and next give the theorem exhibiting the basis of the sequence space $a^r(u, p)$.

Let (λ, g) be a paranormed space. A sequence (b_k) of the elements of λ is called a *basis* for λ if and only if, for each $x \in \lambda$, there exists a unique sequence (α_k) of scalars such that

$$g\left(x - \sum_{k=0}^{n} \alpha_k b_k \right) \to 0, \quad (n \to \infty).$$

The series $\Sigma \alpha_k b_k$, which has the sum x, is then called the expansion of x with respect to (b_k), and written as $x = \Sigma \alpha_k b_k$.

Theorem 2. 4. Let $0 < p_k \le H < \infty$ and $\lambda_k(r) = (A^r x)_k$ for all $k \in \mathbb{N}$. Define the sequence $b^{(k)}(r) = \left\{ b_n^{(k)}(r) \right\}_{n \in \mathbb{N}}$ of the elements of the space $a^r(u, p)$ by

$$b_n^{(k)}(r) = \begin{cases} (-1)^{n-k} \dfrac{1+k}{(1+r^n)u_n} & , \quad (k \le n \le k+1) \\[2mm] 0 & , \quad (n < k \ or \ n > k+1) \end{cases} \tag{8}$$

for every fixed $k \in \mathbb{N}$. Then, the sequence $\{b^{(k)}(r)\}_{k \in \mathbb{N}}$ is a basis for the space $a^r(u, p)$ and any $x \in a^r(u, p)$ has a unique representation of the form

$$x = \sum_k \lambda_k(r) b^{(k)}(r). \tag{9}$$

Proof: It is clear that $\{b^{(k)}(r)\}_{k \in \mathbb{N}} \subset a^r(u, p)$, since

$$A^r b^{(k)}(r) = e^{(k)} \in \ell(p), \quad (k \in \mathbb{N}); \tag{10}$$

for $0 < p_k \le H < \infty$; where $e^{(k)}$ is the sequence whose only non-zero term is a 1 in k^{th} place for each $k \in \mathbb{N}$.

Let $x \in a^r(u, p)$ be given. For every non-negative integer m, we put

$$x^{[m]} = \sum_{k=0}^m \lambda_k(r) b^{(k)}(r). \tag{11}$$

Then, we obtain by applying A^r to (11) with (10), that

$$A^r x^{[m]} = \sum_{k=0}^m \lambda_k(r) A^r b^{(k)}(r) = \sum_{k=0}^m (A^r x)_k e^{(k)}$$

and

$$\left\{ A^r \left(x - x^{[m]} \right) \right\}_i = \begin{cases} 0 & , \quad (0 \le i \le m) \\ (A^r x)_i & , \quad (i > m) \end{cases} ; \quad (i, m \in \mathbb{N}).$$

Given $\varepsilon > 0$, then there is an integer m_0 such that

$$\left[\sum_{i=m}^\infty |(A^r x)_i|^{p_k} \right]^{1/L} < \frac{\varepsilon}{2}$$

for all $m \ge m_0$. Hence,

$$g(x - x^{[m]}) = \left[\sum_{i=m}^\infty |(A^r x)_i|^{p_k} \right]^{1/L} \le \left[\sum_{i=m_0}^\infty |(A^r x)_i|^{p_k} \right]^{1/L} < \varepsilon$$

for all $m \ge m_0$, which proves that $x \in a^r(u, p)$ is represented as in (9).

Let us show the uniqueness of the representation for $x \in a^r(u, p)$ given by (9). Suppose, on the contrary, that there exists a representation $x = \Sigma_k \mu_k(r) b^{(k)}(r)$. Since the linear transformation T from $a^r(u, p)$ to $\ell(p)$, used in the proof of Theorem 2.3 is continuous, we have at this stage that

$$(A^r x)_n = \sum_k \lambda_k(r)(A^r b^{(k)}(r))_n = \sum_k \mu_k(r) e_n^{(k)} = \mu_n(r); \quad (n \in \mathbb{N})$$

which contradicts the fact that $(A^r x)_n = \lambda_n(r)$ for all $n \in \mathbb{N}$. Hence, the representation (9) of $x \in a^r(u, p)$ is unique. This completes the proof.

3. THE α-, β- AND γ-DUALS OF THE SPACE $a^r(u, p)$

In this section, we state and prove the theorems determining the α-, β- and γ-*duals* of the sequence space

$a^r(u, p)$ of a non-absolute type.

The set $S(\lambda, \mu)$ defined by

$$S(\lambda, \mu) = \left\{ z = (z_k) \in w : xz = (x_k z_k) \in \mu \ \textit{for all} \ x \in \lambda \right\} \tag{12}$$

is called the *multiplier space* of the sequence spaces λ and μ. With the notation of (12), the α-, β- and γ-duals of a sequence space λ, which are respectively denoted by λ^α, λ^β and λ^γ, are defined by

$$\lambda^\alpha = S(\lambda, \ell_1), \ \lambda^\beta = S(\lambda, cs) \ \text{and} \ \lambda^\gamma = S(\lambda, bs).$$

Because the case (i) may be established in a similar way to the proof of case (ii), we omit the detail of that case and give the proof only for case (ii) in Theorems 3.4 and 3.5, below. We begin with quoting the lemmas which are needed in proving Theorems 3.4-3.6.

Lemma 3. 1. [15, Theorem 5.1.0 with $q_n=1$] (i) Let $1<p_k\leq H<\infty$ for all $k\in\mathbb{N}$. Then, $A\in(\ell(p):\ell_1)$ if and only if there exists an integer $B>1$ such that

$$\sup_{N\in\mathcal{F}} \sum_k \left| \sum_{n\in N} a_{nk} B^{-1} \right|^{p_k'} < \infty. \tag{13}$$

(ii) Let $0<p_k\leq 1$ for all $k\in\mathbb{N}$. Then, $A\in(\ell(p):\ell_1)$ if and only if

$$\sup_{N\in\mathcal{F}} \sup_{k\in\mathbb{N}} \left| \sum_{n\in N} a_{nk} \right|^{p_k} < \infty. \tag{14}$$

Lemma 3. 2. [16, Theorem 1 (i)-(ii)] (i) Let $1<p_k\leq H<\infty$ for all $k\in\mathbb{N}$. Then, $A\in(\ell(p):\ell_\infty)$ if and only if there exists an integer $B>1$ such that

$$\sup_{n\in\mathbb{N}} \sum_k \left| a_{nk} B^{-1} \right|^{p_k'} < \infty. \tag{15}$$

(ii) Let $0<p_k\leq 1$ for all $k\in\mathbb{N}$. Then, $A\in(\ell(p):\ell_\infty)$ if and only if

$$\sup_{n,k\in\mathbb{N}} |a_{nk}|^{p_k} < \infty. \tag{16}$$

Lemma 3. 3. [16, Corollary for Theorem 1] Let $0<p_k\leq H<\infty$ for all $k\in\mathbb{N}$. Then, $A\in(\ell(p):c)$ if and only if (15), (16) hold, and

$$\lim_{n\to\infty} a_{nk} = \beta_k, \ (k\in\mathbb{N}) \tag{17}$$

also holds.

Let $N_k^* = N\cap\{n\in\mathbb{N} : k\leq n\leq k+1\}$ for $N\in\mathcal{F}$, $B\in\{n\in\mathbb{N}:n\geq 2\}$ and $\Delta x_k=x_k-x_{k+1}$ for all $k\in\mathbb{N}$. Define the sets $e_1^r(p)$, $e_2^r(p)$, $e_3^r(p)$, $e_4^r(p)$, $e_5^r(p)$ and $e_6^r(p)$ as follows:

$$e_1^r(p) = \left\{ a = (a_k) \in w : \sup_{N\in\mathcal{F}} \sup_{k\in\mathbb{N}} \left| \sum_{n\in N_k^*} (-1)^{n-k} \frac{1+k}{(1+r^n)u_n} a_n \right|^{p_k} < \infty \right\},$$

$$e_2^r(p) = \bigcup_{B>1} \left\{ a = (a_k) \in w : \sup_{N\in\mathcal{F}} \sum_k \left| \sum_{n\in N_k^*} (-1)^{n-k} \frac{1+k}{(1+r^n)u_n} a_n B^{-1} \right|^{p_k'} < \infty \right\},$$

$$e_3^r(p) = \left\{ a = (a_k) \in w : \sup_{k \in \mathbb{N}} \left| \varDelta \left[\frac{a_k}{(1+r^k)u_k} \right] (k+1) \right|^{p_k} < \infty \right\},$$

$$e_4^r(p) = \left\{ a = (a_k) \in w : \left(\left[\frac{k+1}{(1+r^k)u_k} a_k \right]^{p_k} \right) \in \ell_\infty \right\},$$

$$e_5^r(p) = \bigcup_{B>1} \left\{ a = (a_k) \in w : \sum_k \left| \Delta \left[\frac{a_k}{(1+r^k)u_k} \right] (k+1)B^{-1} \right|^{p_k'} < \infty \right\},$$

$$e_6^r(p) = \bigcup_{B>1} \left\{ a = (a_k) \in w : \left(\left[\frac{1+k}{(1+r^k)u_k} a_k B^{-1} \right]^{p_k'} \right) \in \ell_\infty \right\};$$

where $\varDelta[a_k/(1+r^k)u_k] = a_k/(1+r^k)u_k - a_{k+1}/(1+r^{k+1})u_{k+1}$ for all $k \in \mathbb{N}$.

Theorem 3. 4. (i) Let $0 < p_k \leq 1$ for all $k \in \mathbb{N}$. Then, $\{a^r(u, p)\}^\alpha = e_1^r(p)$.

(ii) Let $1 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then, $\{a^r(u, p)\}^\alpha = e_2^r(p)$.

Proof: Let us take any $a = (a_n) \in w$. We easily derive with (3) that

$$a_n x_n = \sum_{k=n-1}^{n} (-1)^{n-k} \frac{1+k}{(1+r^n)u_n} a_n y_k = (Cy)_n, \quad (n \in \mathbb{N}), \tag{18}$$

where $C = (c_{nk}^r)$ is defined by

$$c_{nk}^r = \begin{cases} (-1)^{n-k} \dfrac{1+k}{(1+r^n)u_n} a_n, & (n-1 \leq k \leq n) \\ 0, & (0 \leq k < n-1 \; or \; k > n) \end{cases}$$

for all $k, n \in \mathbb{N}$. Thus, we observe by combining (18) with condition (13) of Lemma 3.1(i) that $ax = (a_n x_n) \in \ell_1$ whenever $x = (x_k) \in a^r(u, p)$ if and only if $Cy \in \ell_1$ whenever $y = (y_k) \in \ell(p)$. This leads us to the desired result that $\{a^r(u, p)\}^\alpha = e_2^r(p)$.

Theorem 3. 5. (i) Let $0 < p_k \leq 1$ for all $k \in \mathbb{N}$. Then, $\{a^r(u, p)\}^\beta = e_3^r(p) \cap e_4^r(p)$.
(ii) Let $1 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then, $\{a^r(u, p)\}^\beta = e_5^r(p) \cap e_6^r(p)$.

Proof: Take any $a = (a_k) \in w$ and consider the equation obtained with (3) that

$$\sum_{k=0}^{n} a_k x_k = \sum_{k=0}^{n} \left[\sum_{i=k-1}^{k} (-1)^{k-i} \frac{i+1}{(1+r^k)u_k} y_i \right] a_k \tag{19}$$

$$= \sum_{k=0}^{n-1} \varDelta \left[\frac{a_k}{(1+r^k)u_k} \right] (k+1)y_k + \frac{n+1}{(1+r^n)u_n} a_n y_n = (Dy)_n; \quad (n \in \mathbb{N}),$$

where $D = (d_{nk}^r)$ is defined by

$$d^r_{nk} = \begin{cases} \Delta\left[\dfrac{ak}{(1+r^k)u_k}\right](k+1) & , \quad (0 \le k \le n-1) \\[3ex] \dfrac{n+1}{(1+r^n)u_n}a_n & , \qquad (k=n) \\[3ex] 0 & , \qquad (k > n) \end{cases} \tag{20}$$

for all $k, n \in \mathbb{N}$. Thus, we deduce from Lemma 3.3 with (19) that $ax=(a_k x_k) \in cs$ whenever $x=(x_k) \in a^r(u, p)$, if and only if $Dy \in c$ whenever $y=(y_k) \in \ell(p)$. Therefore, we derive from (15) and (17) that

$$\sum_k \left| \Delta\left[\frac{a_k}{(1+r^k)u_k}\right](k+1)B^{-1} \right|^{p'_k} < \infty$$

and

$$\sup_{k \in \mathbb{N}} \left| \frac{k+1}{(1+r^k)u_k} a_k B^{-1} \right|^{p'_k} < \infty.$$

This shows that $\{a^r(u, p)\}^\beta = e^r_5(p) \cap e^r_6(p)$.

Theorem 3. 6. (i) Let $0<p_k\le 1$ for all $k\in\mathbb{N}$. Then, $\{a^r(u, p)\}^\gamma = e^r_3(p) \cap e^r_4(p)$.
(ii) Let $1<p_k\le H<\infty$ for all $k\in\mathbb{N}$. Then, $\{a^r(u, p)\}^\gamma = e^r_5(p) \cap e^r_6(p)$.

Proof: We see from Lemma 3.2 with (19) that $ax=(a_k x_k) \in bs$ whenever $x=(x_k) \in a^r(u, p)$ if and only if $Dy \in \ell_\infty$ whenever $y=(y_k) \in \ell(p)$, where $D=(d^r_{nk})$ is defined by (20). Therefore, we respectively obtain from (16) and (15) that $\{a^r(u, p)\}^\gamma = e^r_3(p) \cap e^r_4(p)$ for $p_k\le 1$, $\{a^r(u, p)\}^\gamma = e^r_5(p) \cap e^r_6(p)$ for $p_k>1$ and this completes the proof.

4. MATRIX MAPPINGS ON THE SPACE $a^r(u, p)$

In this section, we characterize some matrix mappings on the space $a^r(u, p)$. Theorem 4.1 gives the exact conditions of the general case $0<p_k\le H<\infty$ by combining the cases $0<p_k\le 1$ and $1<p_k\le H<\infty$. We consider only the case $1<p_k\le H<\infty$ and leave the proof of the case $0<p_k\le 1$ to the reader because it may be proven in a similar fashion.

We write for brevity that

$$\Delta\left[\frac{a_{nk}}{(1+r^k)u_k}\right] = \frac{a_{nk}}{(1+r^k)u_k} - \frac{a_{n,k+1}}{(1+r^{k+1})u_{k+1}}$$

for all $k, n \in \mathbb{N}$.

Theorem 4. 1. (i) Let $1<p_k\le H<\infty$ for all $k\in\mathbb{N}$. Then, $A \in(a^r(u, p): \ell_\infty)$ if and only if there exists an integer $B>1$ such that

$$C(B) = \sup_{n \in \mathbb{N}} \sum_k \left| \Delta\left[\frac{a_{nk}}{(1+r^k)u_k}\right](k+1)B^{-1} \right|^{p'_k} < \infty, \tag{21}$$

$$\left\{\left[\left|\frac{k+1}{(1+r^k)u_k}a_{nk}B^{-1}\right|\right]^{p'_k}\right\}_{k\in\mathbb{N}} \in \ell_\infty ; \ (n \in \mathbb{N}). \tag{22}$$

(ii) Let $0<p_k\leq1$ for all $k\in\mathbb{N}$. Then, $A\in(a^r(u,\ p):\ \ell_\infty)$ if and only if

$$\sup_{k,n\in\mathbb{N}}\left|\Delta\left[\frac{a_{nk}}{(1+r^k)u_k}\right](k+1)\right|^{p_k} < \infty, \tag{23}$$

$$\left\{\left[\left|\frac{k+1}{(1+r^k)u_k}a_{nk}\right|\right]^{p_k}\right\}_{k\in\mathbb{N}} \in \ell_\infty ; \ (n \in \mathbb{N}). \tag{24}$$

Proof: Let $A\in(a^r(u,\ p):\ \ell_\infty)$ and $1<p_k\leq H<\infty$ for all $k\in\mathbb{N}$. Then, Ax exists for every $x\in a^r(u,\ p)$, and this implies that $\{a_{nk}\}_{k\in\mathbb{N}}\in\{a^r(u,\ p)\}^\beta$ for each $n\in\mathbb{N}$. Now, the necessities of (21) and (22) are immediate.

Conversely, suppose that the conditions (21) and (22) hold, and take any $x\in a^r(u,\ p)$. In this situation, since $\{a_{nk}\}_{k\in\mathbb{N}}\in\{a^r(u,\ p)\}^\beta$ for every $n\in\mathbb{N}$, the A-transform of x exists. Consider the following equality obtained by using relation (3) that

$$\sum_{k=0}^{m} a_{nk}x_k = \sum_{k=0}^{m-1}\Delta\left[\frac{a_{nk}}{(1+r^k)u_k}\right](k+1)y_k + \frac{m+1}{(1+r^m)u_m}a_{nm}y_m \tag{25}$$

for all $m,\ n\in\mathbb{N}$. Taking into account the hypothesis we derive from (25) as $m\to\infty$ that

$$\sum_{k} a_{nk}x_k = \sum_{k}\Delta\left[\frac{a_{nk}}{(1+r^k)u_k}\right](k+1)y_k ; \ (n \in \mathbb{N}). \tag{26}$$

Now, by combining (26) and the inequality which holds for any $B>0$ and any complex numbers $a,\ b$

$$|ab| \leq B\left(\left|aB^{-1}\right|^{p'} + |b|^p\right),$$

where $p>1$ and $p^{-1}+p'^{-1}=1$ (see [16]), one can easily see that

$$\sup_{n\in\mathbb{N}}\left|\sum_{k} a_{nk}x_k\right| \leq \sup_{n\in\mathbb{N}}\sum_{k}\left|\Delta\left[\frac{a_{nk}}{(1+r^k)u_k}\right](k+1)\right||y_k|$$

$$\leq B\left(C(B) + g_1^B(y)\right) < \infty.$$

This completes the proof of the part (i).

Theorem 4. 2. Let $0<p_k\leq H<\infty$ for all $k\in\mathbb{N}$. Then, $A\in(a^r(u,\ p):c)$ if and only if (21)-(24) hold, and there is a sequence (α_k) of the scalars such that

$$\lim_{n\to\infty}\Delta\left[\frac{a_{nk}}{(1+r^k)u_k}\right](k+1) = \alpha_k ; \ (k \in \mathbb{N}). \tag{27}$$

Proof: Let $A\in(a^r(u,\ p):c)$ and $0<p_k\leq H<\infty$ for all $k\in\mathbb{N}$. Then, since the inclusion $c\subset\ell_\infty$ holds, the necessities of (21) and (22) are immediately obtained from part (i) of Theorem 4.1.

To prove the necessity of (27), consider the sequence $b^{(k)}(r)$ defined by (8), which is in the space $a^r(u,\ p)$ for every fixed $k\in\mathbb{N}$. Because the A-transform of every $x\in a^r(u,\ p)$ exists and is in c by the hypothesis,

$$Ab^{(k)}(r) = \left\{\Delta\left[\frac{a_{nk}}{(1+r^k)u_k}\right](k+1)\right\}_{n\in\mathbb{N}} \in c$$

for every fixed $k \in \mathbb{N}$ which shows the necessity of (27).

Conversely, suppose that the conditions (21), (22) and (27) hold, and take any $x=(x_k)$ in the space $a^r(u, p)$. Then, Ax exists. We observe for all $m, n \in \mathbb{N}$ that

$$\sum_{k=0}^{m}\left| \Delta\left[\frac{a_{nk}}{(1+r^k)u_k}\right](k+1)B^{-1}\right|^{p_k'} \leq \sup_{n\in\mathbb{N}}\sum_{k}\left| \Delta\left[\frac{a_{nk}}{(1+r^k)u_k}\right](k+1)B^{-1}\right|^{p_k'}$$

which gives the fact by letting $m, n \to \infty$ with (21) and (27) that

$$\lim_{m,n\to\infty}\sum_{k=0}^{m}\left| \Delta\left[\frac{a_{nk}}{(1+r^k)u_k}\right](k+1)B^{-1}\right|^{p_k'} \leq \sup_{n\in\mathbb{N}}\sum_{k}\left| \Delta\left[\frac{a_{nk}}{(1+r^k)u_k}\right](k+1)B^{-1}\right|^{p_k'} < \infty.$$

This shows that $\sum_{k}\left|\alpha_k B^{-1}\right|^{p_k'} < \infty$ and so $(\alpha_k)\in\{a^r(u, p)\}^\beta$, which implies that the series $\Sigma_k\alpha_k x_k$ converges for all $x \in a^r(u, p)$.

Let us now consider the equality obtained from (26) with $a_{nk}-\alpha_k$ instead of a_{nk}

$$\sum_{k}(a_{nk} - \alpha_k)x_k = \sum_{k}b_{nk}y_k, \ (n \in \mathbb{N}); \tag{28}$$

where $B=(b_{nk})$ is defined by

$$b_{nk} = \Delta\left[\frac{a_{nk} - \alpha_k}{(1 + r^k)u_k}\right](k + 1)$$

for all $k, n \in \mathbb{N}$. Therefore, we have at this stage from Lemma 3. 3 with $\beta_k=0$ for all $k \in \mathbb{N}$ that the matrix B belongs to the class $(\ell(p) : c_0)$ of infinite matrices. Thus, we see by (28) that

$$\lim_{n\to\infty}\sum_{k}(a_{nk} - \alpha_k)x_k = 0. \tag{29}$$

(29) means that $Ax \in c$ whenever $x \in a^r(u, p)$ and this is what we wished to prove.

If the sequence space c is replaced by the space c_0, then Theorem 4.2 is reduced.

Corollary 4. 3. Let $0<p_k\leq H<\infty$ for all $k\in\mathbb{N}$. Then, $A \in (a^r(u, p):c_0)$ if and only if (21)-(24) hold, and (27) also holds with $\alpha_k=0$ for all $k\in\mathbb{N}$.

Now, we may give our basic lemma which is useful for deriving the characterizations of the certain matrix classes via Theorems 4.1, 4.2 and Corollary 4.3.

Lemma 4. 4. [17, Lemma 5.3] Let λ, μ be any two sequence spaces, A be an infinite matrix and B a triangle matrix. Then, $A \in(\lambda:\mu_B)$ if and only if $BA \in(\lambda:\mu)$.

It is trivial that Lemma 4.4 has several consequences, some of which give the necessary and sufficient conditions of matrix mappings between the sequence spaces generated by the matrix A^r. Indeed, combining Lemma 4.4 with Theorems 4.1, 4.2 and Corollary 4.3, one can easily derive the following results:

Corollary 4. 5. Let $A=(a_{nk})$ be an infinite matrix and define the matrix $C=(c_{nk})$ by

$$c_{nk} = \sum_{j=0}^{n}\binom{n}{j}(1 - s)^{n-j}s^j a_{jk}$$

for all $k, n \in \mathbb{N}$. Then, the necessary and sufficient conditions in order for A belong to any one of the classes

$(a^r(u, p): e_\infty^s)$, $(a^r(u, p): e_c^s)$, and $(a^r(u, p): e_0^s)$ are obtained from the respective ones in Theorems 4.1, 4.2 and Corollary 4.3 by replacing the entries of the matrix *A* by those of the matrix *C*; where $0<s<1$, e_∞^s and e_c^s, e_0^s respectively denote the spaces of all sequences whose Euler transforms of order *s* are in the spaces ℓ_∞ and *c*, c_0 and have recently been studied by Altay, Başar & Mursaleen [18] and, Altay & Başar [19], respectively.

Corollary 4. 6. Let $A=(a_{nk})$ be an infinite matrix and $t=(t_k)$ be a sequence of positive numbers and define the matrix $C=(c_{nk})$ by

$$c_{nk} = \frac{1}{T_n}\sum_{j=0}^{n} t_j a_{jk}$$

for all k, $n \in \mathbb{N}$; where $T_n = \Sigma_{k=0}^{n} t_k$ for all $n \in \mathbb{N}$. Then, the necessary and sufficient conditions in order for *A* belong to any one of the classes $(a^r(u, p): r_\infty^t)$, $(a^r(u, p): r_c^t)$, and $(a^r(u, p): r_0^t)$ are obtained from the respective ones in Theorems 4.1, 4.2 and Corollary 4.3 by replacing the entries of the matrix *A* by those of the matrix *C*; where r_∞^t, r_c^t and r_0^t are the Banach spaces derived with $p_k=1$ for all $k \in \mathbb{N}$ from the paranormed spaces $r_\infty^t(p)$, $r_c^t(p)$ and $r_0^t(p)$.

Since the spaces r_∞^t, r_c^t and r_0^t reduce in the case $t=e$ to the Cesàro sequence spaces X_∞, $\tilde{c}$ and $\tilde{c}_0$ of non-absolute type, Corollary 4.6 also includes the characterizations of the classes $(a^r(u, p):X_\infty)$, $(a^r(u, p): \tilde{c})$ and $(a^r(u, p): \tilde{c}_0)$, as a special case; where $\tilde{c}$ and $\tilde{c}_0$ are the Cesàro sequence spaces of all sequences whose C_1-transforms are in the sequence spaces *c* and c_0, and have recently been studied by Şengönül & Başar [20].

Corollary 4. 7. Let $A=(a_{nk})$ be an infinite matrix and define the matrices $C=(c_{nk})$ and $D=(d_{nk})$ by $c_{nk}=a_{nk}-a_{n+1,k}$ and $d_{nk}=a_{nk}-a_{n-1,k}$ for all k, $n \in \mathbb{N}$. Then, the necessary and sufficient conditions in order for *A* belong to any one of the classes $(a^r(u, p): \ell_\infty(\Delta))$, $(a^r(u, p):c(\Delta))$ and $(a^r(u, p):c_0(\Delta))$, and $(a^r(u, p):bv_\infty)$ are obtained from the respective ones in Theorems 4.1, 4.2 and Corollary 4.3 by replacing the entries of the matrix *A* by those of the matrices *C* and *D*, where $\ell_\infty(\Delta)$, $c(\Delta)$, $c_0(\Delta)$ denote the difference spaces of all bounded, convergent, null sequences and were introduced by Kızmaz [21], while bv_∞ also denotes the space of all sequences $x=(x_k)$ such that $(x_k-x_{k-1}) \in \ell_\infty$ and has recently been studied by Başar & Altay [17].

Corollary 4. 8. Let $A=(a_{nk})$ be an infinite matrix and define the matrices $C=(c_{nk})$ and $D=(d_{nk})$ by $c_{nk}=\Sigma_{j=0}^{n} (1+r^k)a_{jk}/(j+1)$ and $d_{nk}=c_{nk}-c_{n-1,k}$ for all k, $n \in \mathbb{N}$. Then, the necessary and sufficient conditions in order for *A* belong to any one of the classes $(a^r(u, p): a_\infty^r)$, $(a^r(u, p): a_c^r)$, $(a^r(u, p): a_0^r)$ and $(a^r(u, p): a_c^r(\Delta))$, $(a^r(u, p): a_0^r(\Delta))$ are obtained from the respective ones in Theorems 4.1, 4.2 and Corollary 4.3 by replacing the entries of the matrix *A* by those of the matrices *C* and *D*; where $a_c^r(\Delta)$ and $a_0^r(\Delta)$ denote the difference spaces of all sequences $x=(x_k)$ such that (x_k-x_{k-1}) is in the spaces a_c^r and a_0^r, respectively, and have recently been examined by Aydın & Başar [6].

Corollary 4. 9. Let $A=(a_{nk})$ be an infinite matrix and define the matrix $C=(c_{nk})$ by $c_{nk}=\Sigma_{j=0}^{n} a_{jk}$ for all k, $n \in \mathbb{N}$. Then, the necessary and sufficient conditions in order for *A* belong to any one of the classes $(a^r(u, p):bs)$, $(a^r(u, p) :cs)$ and $(a^r(u, p):cs_0)$ are obtained from the respective ones in Theorems 4.1, 4.2 and Corollary 4.3 by replacing the entries of the matrix *A* by those of the matrix *C*, where cs_0 denotes the set of those series converging to zero.

5. THE ROTUNDITY OF THE SPACE $a^r(u, p)$

Among many geometric properties, the rotundity of Banach spaces is one of the most important topics in functional analysis. For details, the reader may refer to [22], [23] and [24]. In this section, we characterize the rotundity of space $a^r(u, p)$ and emphasize some results related to this concept.

By $S(X)$ and $B(X)$, we denote the unit sphere and unit ball of a Banach space X, respectively. A point $x \in S(X)$ is called an *extreme point* if $2x=y+z$ implies $y=z$ for every $y, z \in S(X)$.

A Banach space X is said to be *rotund* (*strictly convex*) if every point of $S(X)$ is an extreme point.

Let X be a real vector space. A functional $\sigma:X \rightarrow [0, \infty)$ is called a *modular* if

(i) $\sigma(x)=0$ if and only if $x=\theta$;

(ii) $\sigma(\alpha x)=\sigma(x)$ for all scalars α with $|\alpha|=1$;

(iii) $\sigma(\alpha x+\beta y)\leq\sigma(x)+\sigma(y)$ for all $x, y \in X$ and $\alpha, \beta \geq 0$ with $\alpha+\beta=1$.

The modular σ is called *convex* if

(iv) $\sigma(\alpha x+\beta y)\leq\alpha\sigma(x)+\beta\sigma(y)$ for all $x, y \in X$ and $\alpha, \beta > 0$ with $\alpha+\beta=1$.

A modular σ on X is called

(a) r*ight continuous* if $\lim_{\alpha\rightarrow 1^+}\sigma(\alpha x)=\sigma(x)$ for all $x \in X_\sigma$,

(b) l*eft continuous* if $\lim_{\alpha\rightarrow 1^-}\sigma(\alpha x)=\sigma(x)$ for all $x \in X_\sigma$,

(c) c*ontinuous* if it is both right and left continuous;

where

$$X_\sigma = \left\{ x \in X : \lim_{\alpha \rightarrow 0^+} \sigma(\alpha x) = 0 \right\}.$$

For $a^r(u, p)$, we define

$$\sigma_p(x) = \sum_k \left| \frac{1}{k+1} \sum_{j=0}^{k} (1 + r^j)u_j x_j \right|^{p_k}.$$

If $p_k \geq 1$ for all $k \in \mathbb{N}$, by the convexity of the function $t \mapsto |t|^{p_k}$ for each $k \in \mathbb{N}$, one can see that σ_p is a convex modular on $a^r(u, p)$. We consider $a^r(u, p)$ equipped with the *Luxemburg norm* given by

$$\|x\| = \inf\left\{\alpha > 0 : \sigma_p\left(\frac{x}{\alpha}\right) \leq 1\right\}.$$

It is easy to show that $a^r(u, p)$ is a Banach space with this norm. Now, we may give the proposition without proof concerning some basic properties for modular σ_p:

Proposition 5. 1. The modular σ_p on $a^r(u, p)$ satisfies the following properties:

(i) If $0<\alpha\leq 1$, then $\alpha^M\sigma_p(x/\alpha)\leq\sigma_p(x)$ and $\sigma_p(\alpha x)\leq\alpha\sigma_p(x)$,

(ii) If $\alpha\geq 1$, then $\sigma_p(x)\leq\alpha^M\sigma_p(x/\alpha)$,

(iii) If $\alpha\geq 1$, then $\sigma_p(x)\leq\alpha\sigma_p(x/\alpha)$,

(iv) σ_p is continuous.

Now, we may establish some relationships between the modular σ_p and the Luxemburg norm on $a^r(u,p)$.

Proposition 5. 2. For any $x \in a^r(u, p)$, we have

(i) If $\|x\|<1$, then $\sigma_p(x)\leq\|x\|$,

(ii) If $\|x\|>1$, then $\sigma_p(x)\geq\|x\|$,

(iii) $\|x\|=1$ if and only if $\sigma_p(x)=1$,

(iv) $\|x\|<1$ if and only if $\sigma_p(x)<1$,

(v) $\|x\|>1$ if and only if $\sigma_p(x)>1$.

Proof: (i) Let $\varepsilon>0$ be such that $0<\varepsilon<1-\|x\|$. By the definition of $\|\cdot\|$, there exists an $\alpha>0$ such that $\|x\|+\varepsilon>\alpha$ and $\sigma_p(x)\leq1$. From Proposition 5.1 (i) and (ii), we have

$$\sigma_p(x) \leq \sigma_p\left[\left(\|x\|+\varepsilon\right)\frac{x}{\alpha}\right] \leq \left(\|x\|+\varepsilon\right)\sigma_p\left(\frac{x}{\alpha}\right) \leq \|x\|+\varepsilon.$$

Since ε is arbitrary, we have (i).

   (ii) If we choose $\varepsilon>0$ such that $0<\varepsilon<1-\|x\|^{-1}$, then $1<(1-\varepsilon)\|x\|<\|x\|$. Combining the definition of $\|\cdot\|$ and Proposition 5.1 (i), we have

$$1 < \sigma_p\left[\frac{x}{(1-\varepsilon)\|x\|}\right] \leq \frac{1}{(1-\varepsilon)\|x\|}\sigma_p(x),$$

so $(1-\varepsilon)\|x\|<\sigma_p(x)$ for all $\varepsilon\in(0,\ 1-\|x\|^{-1})$. This implies that $\|x\|<\sigma_p(x)$.

   Since σ_p is continuous, (iii) directly follows from Theorem 1.4 of [24].
   (iv) follows from (i) and (iii).
   (v) follows from (iii) and (iv).

Theorem 5.1. The space $a^r(u, p)$ is rotund if and only if $p_k>1$ for all $k\in\mathbb{N}$.

Proof: Necessity. Let $a^r(u, p)$ be rotund and choose $k_0\in\mathbb{N}$ such that $p_{k_0} = 1$. Take $x=(1/2,\ -1/(1+r),\ 0,\ 0,\ 0,\ ...)$ and $y=(0,\ 2/(1+r),\ -2/(1+r^2),\ 0,\ 0,\ 0,\ ...)$. Then, $x\neq y$ and

$$\sigma_p(x) = \sigma_p(y) = \sigma_p\left(\frac{x+y}{2}\right) = 1.$$

By Proposition 5.3 (iii); $x, y, (x+y)/2 \in S(a^r(u, p))$ so that $a^r(u, p)$ is not rotund, a contradiction.

Sufficiency. Let $x\in S(a^r(u, p))$ and $y,\ z\in S(a^r(u, p))$ with $x=(y+z)/2$. By convexity of σ_p and Proposition 5.3 (iii), we have

$$1 = \sigma_p(x) \leq \frac{\sigma_p(y) + \sigma_p(z)}{2} \leq \frac{1}{2} + \frac{1}{2} = 1$$

which gives that $\sigma_p(y)=\sigma_p(z)=1$ and

$$\sigma_p(x) = \frac{\sigma_p(y) + \sigma_p(z)}{2}. \tag{30}$$

Further, we have by (30) that

$$\sum_k\left|\frac{1}{k+1}\sum_{j=0}^{k}(1+r^j)u_jx_j\right|^{p_k} = \frac{1}{2}\sum_k\left|\frac{1}{k+1}\sum_{j=0}^{k}(1+r^j)u_jy_j\right|^{p_k}$$

$$+ \frac{1}{2}\sum_k\left|\frac{1}{k+1}\sum_{j=0}^{k}(1+r^j)u_jz_j\right|^{p_k}.$$

Since $x=(y+z)/2$, we have

$$\sum_k\left|\frac{1}{2(k+1)}\sum_{j=0}^{k}(1+r^j)u_j(y_j+z_j)\right|^{p_k} = \frac{1}{2}\sum_k\left|\frac{1}{k+1}\sum_{j=0}^{k}(1+r^j)u_jy_j\right|^{p_k}$$

$$+\frac{1}{2}\sum_k \left|\frac{1}{k+1}\sum_{j=0}^k (1+r^j)u_j z_j\right|^{p_k}.$$

This implies that

$$\left|\frac{1}{2(k+1)}\sum_{j=0}^k (1+r^j)(y_j+z_j)\right|^{p_k} = \frac{1}{2}\left|\frac{1}{k+1}\sum_{j=0}^k (1+r^j)y_j\right|^{p_k} \tag{31}$$

$$+\frac{1}{2}\left|\frac{1}{k+1}\sum_{j=0}^k (1+r^j)z_j\right|^{p_k}$$

for all $k \in \mathbb{N}$. Since the function $t \mapsto |t|^{p_k}$ is strictly convex for all $k \in \mathbb{N}$, it follows by (31) that $y_k = z_k$ for all $k \in \mathbb{N}$. Hence $y = z$, that is, $a^r(u, p)$ is rotund.

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