Iranian Journal of Science & Technology, Transaction A, Vol. 30, No. A2
Printed in The Islamic Republic of Iran, 2006

N-ARY HYPERGROUPS[*]

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Abstract – In this paper the class of *n*-ary hypergroups is introduced and several properties are found and examples are presented. *n*-ary hypergroups are a generalization of hypergroups in the sense of Marty. On the other hand, we can consider *n*-ary hypergroups as a good generalization of *n*-ary groups. We define the fundamental relation β^* on an *n*-ary hypergroup *H* as the smallest equivalence relation such that H/β^* is the *n*-ary group, and then some related properties are investigated.

Keywords – hypergroup, *n*-ary hypergroup, *n*-ary group, fundamental equivalence relation

1. INTRODUCTION

Hypergroup, which is based on the notion of hyperoperation, has been introduced by Marty in [1] and studied extensively by many mathematicians. For example, the connection between hypergraphs and hypergroups is studied by Corsini [2]. In [3], Corsini and Leoreanu described hypergroups associated with trees and in [4] some applications of hyperstructures in rough sets are given. The hypergroup theory both extends some well-known group results and introduces new topics, thus leading to a wide variety of applications, as well as to a broadening of the investigation fields. A comprehensive review of the theory of hyperstructures appears in [5-8].

The notion of an *n*-ary group was introduced by Dörnte [9], which is a natural generalization of the notion of a group. *n*-ary generalizations of algebraic structures is the most natural way for further development and deeper understanding of their fundamental properties. Since then many papers concerning various *n*-ary algebra have appeared in the literature, (for example see [10-15]).

In this paper, *n*-ary hypergroups are defined and considered. Examples of *n*-ary hypergroups are given and some of their properties described. *n*-ary hypergroups are a generalization of hypergroups in the sense of Marty. Also, we can consider *n*-ary hypergroups as a good generalization of *n*-ary groups. We define the fundamental relation β^* on an *n*-ary hypergroup *H* as the smallest equivalence relation such that H/β^* is the *n*-ary group, and then some related properties are investigated.

2. BASIC DEFINITIONS AND RESULTS

Let *H* be a non-empty set and *f* be a mapping $f : H \times H \longrightarrow \mathrm{P}^*(H)$, where $\mathrm{P}^*(H)$ is the set of all non-empty subsets of *H*. Then *f* is called a *binary hyperoperation* on *H*. We denote by H^n the cartesian product $H \times \cdots \times H$, where *H* appears *n* times. An element of H^n will be denoted by $(x_1, \cdots, x_n)$, where $x_i \in H$ for any i with $1 \leq i \leq n$. In general, a mapping $f : H^n \longrightarrow \mathrm{P}^*(H)$ is called an *n-ary*

[*]Received by the editor February 2, 2005 and in final revised form October 22, 2005
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hyperoperation and n is called the *arity of hyperoperation*.

Let f be an n-ary hyperoperation on H and $A_1,\cdots,A_n$ subsets of H. We define

$$f(A_1,\cdots,A_n)=\bigcup\left\{f(x_1,\cdots,x_n)\mid x_i\in A_i\ ,\ i=1,\cdots,n\right\}.$$

We shall use the following abbreviated notation: the sequence $x_i,x_{i+1},\cdots,x_j$ will be denoted by x_i^j. For $j<i$, x_i^j is the empty set. In this convention

$$f(x_1,\cdots,x_i,y_{i+1},\cdots y_j,z_{j+1},\cdots,z_n)$$

will be written as $f\left(x_1^i,y_{i+1}^j,z_{j+1}^n\right)$.

Definition 2. 1. A non-empty set H with an n-ary hyperoperation $f:H^n\longrightarrow \mathrm{P}^*(H)$ will be called an *n-ary hypergroupoid* and will be denoted by (H,f). An n-ary hypergroupoid (H,f) will be called an *n-ary semihypergroup* if and only if the following associative axiom holds:

$$f\left(x_1^{i-1},f\left(x_i^{n+i-1}\right),x_{x+i}^{2n-1}\right)=f\left(x_1^{j-1},f\left(x_j^{n+j-1}\right),x_{n+j}^{2n-1}\right)$$

for every $i,j\in\{1,2,\cdots,n\}$ and $x_1,x_2,\cdots,x_{2n-1}\in H$.

If for all $(a_1,a_2,\cdots,a_n)\in H^n$, the set $f(a_1,a_2,\cdots,a_n)$ is singleton, then f is called an *n-ary operation* and (H,f) is called an *n-ary groupoid* (resp. *n-ary semigroup*).

If $m=k(n-1)+1$, then the m-ary hyperoperation g given by

$$g\left(x_1^{k(n-1)+1}\right)=f\Big(\underbrace{f\Big(\cdots,f\Big(f}_{k}\left(x_1^n\right),x_{n+1}^{2n-1}\Big),\cdots\Big),x_{(k-1)(n-1)+2}^{k(n-1)+1}\Big)$$

will be denoted by $f_{(k)}$. In certain situations, when the arity of g does not play a crucial role, or when it will differ depending on additional assumptions, we write $f_{(\cdot)}$, to mean $f_{(k)}$ for some $k=1,2,\cdots$.

Definition 2. 2. An n-ary semihypergroup (H,f), in which the equation

$$b\in f\left(a_1^{i-1},x_i,a_{i+1}^n\right) \qquad (*)$$

has the solution $x_i\in H$ for every $a_1,\cdots,a_{i-1},a_{i+1},\cdots,a_n,b\in H$ and $1\le i\le n$, is called an *n-ary hypergroup*.

In Definition 2.2, if f is n-ary operation then the equation $(*)$ is as follows:

$$b=f\left(a_1^{i-1},x_i,a_{i+1}^n\right). \qquad (**)$$

In this case (H,f) is an *n-ary group*.

The important question is the solvability of the equation $(*)$. The classical n-ary semigroup is an n-ary group if and only if the equation $(**)$ is solvable at the place $i=1$ and $i=n$, or at least one place $1<i<n$, (see [12] or [13]). The following theorem shows that it is true for hypergroups.

Theorem 2. 3. *Let (H,f) be an n-ary semihypergroup. Then (H,f) is an n-ary hypergroup if and only if the equation $(*)$ is solvable at the place $i=1$ and $i=n$ or at least one place $1<i<n$.*

Proof: If $(*)$ is solvable at the place $i=1$ and $i=n$, then for every $a_1,\cdots,a_n,b\in H$ there exist $x_0,z_0\in H$ such that

$$b \in f(x_0, a_2^n) \text{ and } x_0 \in f\big(a_1^{n-1}, z_0\big).$$

Assume that $1 < j < n$ be arbitrary. Then

$$b \in f\big(f\big(a_1^{n-1}, z_0\big), a_2^n\big) = f\big(a_1^{j-1}, f\big(a_j^{n-1}, z_0, a_2^j\big), a_{j+1}^n\big).$$

Therefore there exists $x \in f\big(a_j^{n-1}, z_0, a_2^j\big)$ such that $b \in f\big(a_1^{j-1}, x, a_{j+1}^n\big)$.

Now, let (*) be solvable at place $1 < i < n$. Assume that $j < i$, then for every $a_1, \cdots, a_n, b \in H$ there exists $y_1 \in H$ such that

$$b \in f(a_1^{i-1}, y_1, f(\underbrace{a_1, \cdots, a_1}_{n-(i-j+1)}, a_{j+1}^{i+1}), a_{i+2}^n)$$

and so

$$b \in f(a_1^{j-1}, f(a_j^{i-1}, y_1, \underbrace{a_1, \cdots, a_1}_{n-(i-j+1)}), a_{j+1}^n).$$

Therefore there exists $x \in f\big(a_j^{i-1}, y_1, a_1, \cdots, a_1\big)$ such that $b \in f\big(a_1^{j-1}, x, a_{j+1}^n\big)$. If we choose $i < j$, then similarly we can prove that (*) is solvable.

Definition 2.2 is a generalization of Marty's formulation of axiom of a hypergroup. Let $\circ$ be a binary algebraic hyperoperation on *H*, then $(H, \circ)$ is called a *hypergroupoid*. A *hypergroup* is a hypergroupoid $(H, \circ)$ that satisfies:

1) $x \circ (y \circ z) = (x \circ y) \circ z$ for all $x, y, z \in H$,
2) $x \circ H = H \circ x = H$ for all $x \in H$.

The second condition is frequently used in the form: Given $a, b \in H$, there exist $u, v \in H$ such that $b \in a \circ u$ and $b \in v \circ a$.

Condition 2 can be formulated for *n*-ary hypergroups as follows:

$$f\big(H^{i-1}, x, H^{n-i}\big) = H$$

for all $x \in H$ and $i = 1, \cdots, n$.

Let (H, f) be an *n*-ary hypergroup, $a_2^{n-1} \in H$ be fixed and let $x \odot y = f\big(x, a_2^{n-1}, y\big)$. Then the hypergroupoid $(H, \odot)$ is a hypergroup and it is called a *retract of the n-ary hypergroup* (H, f).

Example 2. 4. Let $H = \{x, y, z\}$ be a set with a 3-ary hyperoperation *f* as follows:

$f(x,x,x) = x$	$f(y,y,x) = \{x,z\}$	$f(z,x,x) = z$
$f(x,x,y) = y$	$f(y,y,y) = \{y,z\}$	$f(z,x,y) = \{y,z\}$
$f(x,x,z) = z$	$f(y,y,z) = H$	$f(z,x,z) = \{x,y\}$
$f(x,y,x) = y$	$f(y,x,x) = y$	$f(z,y,x) = \{y,z\}$
$f(x,y,y) = \{x,z\}$	$f(y,x,y) = \{x,z\}$	$f(z,y,y) = H$
$f(x,y,z) = \{y,z\}$	$f(y,x,z) = \{y,z\}$	$f(z,y,x) = H$
$f(x,z,x) = z$	$f(y,z,x) = \{y,z\}$	$f(z,z,x) = \{x,y\}$
$f(x,z,y) = \{y,z\}$	$f(y,z,y) = H$	$f(z,z,y) = H$
$f(x,z,z) = \{x,y\}$	$f(y,z,z) = H$	$f(z,z,z) = \{y,z\}.$

For every $x_i \in H$ $(i = 1, \cdots, 5)$, we have

$$f(f(x_1, x_2, x_3), x_4, x_5) = f(x_1, f(x_2, x_3, x_4), x_5) = f(x_1, x_2, f(x_3, x_4, x_5))$$

i.e., *f* is associative, and it is easy to see that *f* is a 3-ary hypergroup.

Let (H,f) be an *n*-ary hypergroup. If the value of $f(x_1,x_2,\cdots,x_n)$ is independent on the permutation of elements $x_1,x_2,\cdots,x_n$, then (H,f) is called a *commutative n-ary hypergroup*.

The element $a \in H$ is called a *scalar* if

$$\left|f\left(x_1^i,a,x_{i+2}^n\right)\right| = 1$$

for all $x_1,\cdots,x_i,x_{i+2},\cdots,x_n \in H$.

Element *e* of an *n*-ary hypergroup (H,f) is called a *neutral* (*identity*) element if

$$f(\underbrace{e,\cdots,e}_{i-1},x,\underbrace{e,\cdots,e}_{n-i})$$

includes *x*, for all $x \in H$ and all $1 \le i \le n$.

Lemma 2. 5. *Let (H,f) be a commutative n-ary hypergroup and $a \in H$ a scalar element such that $f(a,e,\cdots,e) = a$ for some $e \in H$. Then e is a neutral element.*

Proof: We have

$$f(f(x,a,\underbrace{e,\cdots,e}_{n-2},\underbrace{e,\cdots,e}_{n-1}) = f(x,f(a,\underbrace{e,\cdots,e}_{n-1},\underbrace{e,\cdots,e}_{n-2}) = f(x,a,\underbrace{e,\cdots,e}_{n-2}).$$

Since every element of *H* is representable in the form $f(x,a,e,\cdots,e)$ and *f* is commutative, this means that *e* is a neutral element.

It is to be noted that in Lemma 2.4, the condition $f(a,x,\cdots,x) = a$ can be replaced by the condition $f(x,\cdots,x,a,x,\cdots,x) = a$, where *a* appears at one fixed place $i = 1,\cdots,n$.

Proposition 2. 6. *If the set of all scalar neutral elements of a given commutative n-ary hypergroup is non-empty, then it is an n-ary group.*

Proof: To prove that the set N_H of all scalar neutral elements is closed under the hyperoperation *f*, let $a = f(e_1^n)$, where $e_1,\cdots,e_n \in N_H$. Then

$$\begin{aligned}
f(\underbrace{a,\cdots,a}_{i-1},x,\underbrace{a,\cdots,a}_{n-i}) &= f(\underbrace{f(e_1^n),\cdots,f(e_1^n)}_{i-1},x,\underbrace{f(e_1^n),\cdots,f(e_1^n)}_{n-i}) \\
&= f(\underbrace{e_1,\cdots,e_1}_{n-1},f(\underbrace{e_2,\cdots,e_2}_{n-1},f(\cdots,f(\underbrace{e_{n-1},\cdots,e_{n-1}}_{n-1},f(\underbrace{e_n,\cdots,e_n}_{n-1},x))\cdots))) \\
&= f(\underbrace{e_1,\cdots,e_1}_{n-1},f(\underbrace{e_2,\cdots,e_2}_{n-1},f(\cdots,f(\underbrace{e_{n-1},\cdots,e_{n-1}}_{n-1},x)\cdots))) \\
&= \cdots = f(\underbrace{e_1,\cdots,e_1}_{n-1},x) = x,
\end{aligned}$$

which proves that an element $a = f(e_1^n)$ is neutral. Therefore, N_H is closed under *f*. Also, for all $e_2,\cdots,e_n,e \in N_H$, the equation $e = f(x,e_2^n)$ has the solution

$$x = f_{(\cdot)}(e,\underbrace{e_n,\cdots,e_n}_{n-2},\underbrace{e_{n-1},\cdots,e_{n-1}}_{n-2},\cdots,\underbrace{e_3,\cdots,e_3}_{n-2},\underbrace{e_2,\cdots,e_2}_{n-2})$$

which is contained in N_H.

Definition 2. 7. Let (H,f) be an *n*-ary hypergroup and *B* be a non-empty subset of *H*. Then *B* is an *n-ary* *subhypergroup* of *H* if the following conditions hold:

1) B is closed under the *n*-ary hyperoperation *f*, i.e., for every $(x_1,\cdots,x_n)\in B^n$ implies that $f(x_1,\cdots,x_n)\subseteq B$.

2) Equation $b\in f\left(b_1^{i-1},x_i,b_{i+1}^n\right)$ has the solution $x_i\in B$ for every $b_1,\cdots,b_{i-1},b_{i+1},\cdots,b_n,b\in B$ and $1\le i\le n$.

Definition 2. 8. Let (A,f) and (B,g) be two *n*-ary hypergroups. A *homomorphism* from *A* to *B* is a mapping $\varphi : A\longrightarrow B$ such that

$$\varphi(f(a_1,\cdots,a_n)) = g(\varphi(a_1),\cdots,\varphi(a_n))$$

holds for all $a_1,\cdots,a_n\in A$.

If φ is injective, then it is called an *embedding*. The map φ is an *isomorphism* if φ is injective and onto. We say that *A* is *isomorphic* to *B*, denoted by $A\cong B$, if there is an isomorphism from *A* to *B*.

Theorem 2. 9. *Let* (A,f) *and* (B,g) *be two n-ary hypergroups and* $\varphi : A\longrightarrow B$ *a homomorphism. Then*

1) *If S is an n-ary subhypergroup of A, then* $\varphi(S)$ *is an n-ary subhypergroup of B,*

2) *If K is an n-ary subhypergroup of B such that* $\varphi^{-1}(K)\neq\phi$*, then* $\varphi^{-1}(K)$ *is an n-ary subhypergroup of A.*

Proof: 1) Suppose that $y_1,\cdots,y_n\in\varphi(S)$. Then there exist $x_1,\cdots x_n\in S$ such that $\varphi(x_i)=y_i$ for all $1\le i\le n$. We have $\varphi(f(x_1,\cdots,x_n))\subseteq\varphi(S)$ and so $g(\varphi(x_1),\cdots,\varphi(x_n))\subseteq\varphi(S)$ or $g(y_1,\cdots,y_n)\subseteq\varphi(S)$. Therefore the first condition of Definition 2.7 is satisfied. For the second condition of Definition 2.7, we consider the equation $b\in g\left(b_1^{i-1},x_i,b_{i+1}^n\right)$ for all $b_1,\cdots,b_{i-1},b_{i+1},\cdots,b_n,b\in\varphi(S)$.
Then there exist $a_1,\cdots,a_{i-1},a_{i+1},\cdots,a_n,a\in S$ such that $\varphi(a)=b$ and $\varphi(a_i)=b_i$. Since *S* is an *n*-ary subhypergroup of *A*, the equation

$$a\in f\left(a_1^{i-1},y_i,a_{i+1}^n\right)$$

has a solution $y_i\in S$. From the equation $a\in f\left(a_1^{i-1},y_i,a_{i+1}^n\right)$ we obtain the equation $\varphi(a)\in\varphi\left(f\left(a_1^{i-1},y_i,a_{i+1}^n\right)\right)$ or $b\in g\left(b_1^{i-1},\varphi(y_i),b_{i+1}^n\right)$. Therefore the equation $b\in g\left(b_1^{i-1},x_i,b_{i+1}^n\right)$ has the solution $\varphi(y_i)$.

2) The proof of this part is similar to (1).

3. QUOTIENT *N*-ARY HYPERGROUPS

Let (H,f) be an *n*-ary hypergroup. An equivalence relation θ on *H* is called *compatible* if $a_1\theta b_1,\cdots,a_n\theta b_n$, then for all $a\in f(a_1,\cdots,a_n)$ there exists $b\in f(b_1,\cdots,b_n)$ such that $a\theta b$. An equivalence relation θ is called *strongly compatible* if $a_1\theta b_1,\cdots,a_n\theta b_n$ implies that $a\theta b$ for all $a\in f(a_1,\cdots,a_n)$ and $b\in f(b_1,\cdots,b_n)$.

Theorem 3. 1. *Let* (H,f) *be an n-ary hypergroup and* θ *a compatible relation on H. Then* $(H/\theta,f|_\theta)$ *is an n-ary hypergroup where*

$$f|_\theta\ (\theta(a_1),\cdots,\theta(a_n)) = \{\theta(a)\mid a\in f(a_1,\cdots,a_n)\}.$$

Proof: We shall use the following abbreviated notation: the sequence $\theta(a_i),\theta(a_{i+1}),\cdots,\theta(a_j)$ will be denoted by $\theta_{a_i}^{a_j}$. Since θ is a compatible relation, then we conclude that $f|_\theta$ is well-defined. We show that $f|_\theta$ is associative. We have

$$f\mid_\theta\left(\theta_{a_1}^{a_{i-1}}, f\mid_\theta\left(\theta_{a_i}^{a_{n+i-1}}\right), \theta_{a_{n+i}}^{a_{2n-1}}\right) = \bigcup\left\{ f\mid_\theta\left(\theta_{a_1}^{a_{i-1}}, \theta(y), \theta_{a_{n+i}}^{a_{2n-1}}\right) \mid y \in f\left(a_i^{n+i-1}\right)\right\}$$

$$= \left\{ \theta(a) \mid a \in f\left(a_1^{i-1}, y, a_{n+i}^{2n-1}\right), y \in f\left(a_i^{n+i-1}\right)\right\}$$

$$= \left\{ \theta(a) \mid a \in f\left(a_1^{i-1}, f\left(a_i^{n+i-1}\right), a_{n+i}^{2n-1}\right)\right\}$$

$$= \left\{ \theta(a) \mid a \in f\left(a_1^{j-1}, f\left(a_j^{n+j-1}\right), a_{n+j}^{2n-1}\right)\right\}$$

$$= f\mid_\theta\left(\theta_{a_1}^{a_{j-1}}, f\mid_\theta\left(a_j^{n+j-1}\right), \theta_{a_{n+j}}^{a_{2n-1}}\right).$$

Therefore $f\mid_\theta$ is associative. Now, we consider the equation

$$\theta(b) \in f\mid_\theta\left(\theta_{a_1}^{a_{i-1}}, \theta(x_i), \theta_{a_{i+1}}^{a_n}\right) \qquad (*)$$

for every $a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n, b \in H$. Since H is an n-ary hypergroup, the equation $b \in f\left(a_1^{i-1}, x_i, a_{i+1}^n\right)$ has the solution $x_i \in H$, and so $\theta(x_i)$ is a solution for (*).

The natural map $\pi : H \longrightarrow H/\theta$, where $\pi(x) = \theta(x)$ is an onto homomorphism.

Definition 3. 2. Let (A, f) and (B, g) be two n-ary hypergroups and let $\varphi : A \longrightarrow B$ be a homomorphism. Then the *kernel* φ, written $ker\varphi$, is defined by

$$ker\varphi = \left\{ (a,b) \in A^2 \mid \varphi(a) = \varphi(b)\right\}.$$

It is easy to see that $ker\varphi$ is a compatible relation.

Theorem 3. 3. *Let (A, f) and (B, g) be two n-ary hypergroups and let $\varphi : A \longrightarrow B$ be a homomorphism. Then there exists a compatible relation θ on A and a monomorphism $\psi : A/\theta \longrightarrow B$ such that $\psi \circ \pi = \varphi$.*

Proof: We consider $\theta = ker\varphi$. Now, let $\theta(a) \in A/\theta$ and define $\psi(\theta(a)) = \varphi(a)$.

Theorem 3. 4. *Let ρ and θ be compatible relations on an n-ary hypergroup (H, f) such that $\rho \subseteq \theta$. Then there exists a compatible relation μ on $\left(H/\rho, f\mid_\rho\right)$ such that $(H/\rho)/\mu$ is isomorphic to H/θ.*

Proof: We consider the map $\varphi : H/\rho \longrightarrow H/\theta$ by $\varphi(\rho(x)) = \theta(x)$. Since $\rho \subseteq \theta$, φ is well-defined. Clearly φ is a homomorphism. Now, by Theorem 3.3, there exists a compatible relation μ and a monomorphism $\psi : (H/\rho)/\mu \longrightarrow H/\theta$ such that $\psi \circ \pi = \varphi$, and so ψ is an isomorphism.

The diagonal relation Δ on H is the set $\left\{ (a,a) \mid a \in H\right\}$ and the full relation H^2 is denoted by ∇. The set of all equivalence relations on a set H, with ⊆ as the partial ordering, is a complete lattice. Let θ_1 and θ_2 be two equivalence relations on H. It is clear that $\theta_1 \wedge \theta_2 = \theta_1 \cap \theta_2$. Also, we have

$$\theta_1 \vee \theta_2 = \theta_1 \bigcup (\theta_1 \circ \theta_2) \bigcup (\theta_1 \circ \theta_2 \circ \theta_1) \bigcup (\theta_1 \circ \theta_2 \circ \theta_1 \circ \theta_2) \bigcup \cdots.$$

We suppose that analogous results on other products of hyperstructures can be obtained [7], [16].

Definition 3. 5. Let (A_1, f_1) and (A_2, f_2) be two n-ary hypergroups. Define the direct hyperproduct $(A_1 \times A_2, f_1 \times f_2)$ to be the n-ary hypergroup whose universe is the set $A_1 \times A_2$ and such that for $a_i \in A_1$, $a_i' \in A_2$, $1 \le i \le n$,

$$(f_1 \times f_2)((a_1,a_1'),\cdots,(a_n,a_n')) = \left\{ (a,a') \mid a \in f_1(a_1,\cdots,a_n), a' \in f_2(a_1',\cdots,a_n') \right\}.$$

The mapping $\pi_i : A_1 \times A_2 \longrightarrow A_i$, $i = 1,2$, defined by $\pi_i((a_1,a_2)) = a_i$, is called the *projection map* on the *i*th coordinate of $A_1 \times A_2$. For $i = 1,2$, the mapping $\pi_i : A_1 \times A_2 \longrightarrow A_i$ is an onto homomorphism. Furthermore, we have

i) $ker\,\pi_1 \bigcap ker\,\pi_2 = \Delta$,

ii) $ker\,\pi_1$ and $ker\,\pi_2$ permute,

iii) $ker\,\pi_1 \wedge ker\,\pi_2 = \nabla$,

where

$$ker\,\pi_i = \left\{((a_1,a_2),(b_1,b_2)) \mid \pi_i(a_1,a_2) = \pi_i(b_1,b_2)\right\}, \qquad i = 1,2.$$

Note that

$$((a_1,a_2),(b_1,b_2)) \in ker\,\pi_i \quad \Leftrightarrow \quad \pi_i((a_1,a_2)) = \pi_i((b_1,b_2)) \quad \Leftrightarrow \quad a_i = b_i.$$

Thus $ker\,\pi_1 \bigcap ker\,\pi_2 = \Delta$. Also, if (a_1,a_2), (b_1,b_2) are any two elements of $A_1 \times A_2$, then

$$(a_1,a_2)\ ker\pi_1\ (a_1,b_2),$$

$$(a_1,b_2)\ ker\pi_2\ (b_1,b_2),$$

so $\nabla = ker\,\pi_1 \circ ker\,\pi_2$. But, then $ker\,\pi_1$ and $ker\,\pi_2$ permute, and their joining is ∇.

Definition 3. 6. Let (H,f) be an *n*-ary hypergroup. A compatible relation θ on *H* is a *factor compatible relation* if there is a compatible relation θ^* on *H* such that $\theta \bigcap \theta^* = \Delta$, $\theta \wedge \theta^* = \nabla$ and θ permutes with θ^*.

The pair θ, θ^* is called a *pair of factor compatible relations* on *H.*

Theorem 3. 7. *If* θ, θ^* *is a pair of factor compatible relations on H, then*

$$H \cong H/\theta \times H/\theta^*$$

under the map $\psi(a) = \left(\theta(a),\theta^*(a)\right)$.

Proof: If $a,b \in H$ and $\psi(a) = \psi(b)$, then $\theta(a) = \theta(b)$ and $\theta^*(a) = \theta^*(b)$, so $(a,b) \in \theta \bigcap \theta^*$; hence $a = b$. This means that ψ is injective. Next, given $a,b \in H$, there is $c \in H$ such that $a\theta c$ and $c\theta^* b$, hence $\psi(c) = \left(\theta(c),\theta^*(c)\right) = \left(\theta(a),\theta^*(b)\right)$, so ψ is onto. Finally, for $a_1,\cdots,a_n \in H$, we show that

$$\psi(f(a_1,\cdots,a_n)) = (f\,|_\theta \times f\,|_{\theta^*})(\psi(a_1),\cdots,\psi(a_n)).$$

We have

$$\begin{aligned}\psi(f(a_1,\cdots,a_n)) &= \left\{ \psi\langle a\rangle \mid a \in f(a_1,\cdots,a_n) \right\} \\ &= \left\{ \left(\theta\langle a\rangle,\theta^*\langle a\rangle\right) \mid a \in f(a_1,\cdots,a_n) \right\} \\ &\subseteq \left\{ \left(\theta\langle a\rangle,\theta^*(b)\right) \mid a \in f(a_1,\cdots,a_n), b \in f(a_1,\cdots,a_n) \right\} \\ &\subseteq f|_\theta\ (\theta(a_1),\cdots,\theta(a_n)) \times f|_{\theta^*}\ (\theta^*(a_1),\cdots,\theta^*(a_n)) \\ &= (f|_\theta \times f|_{\theta^*})\big((\theta(a_1),\theta^*(a_1)),\cdots,(\theta(a_n),\theta^*(a_n))\big) \\ &= (f|_\theta \times f|_{\theta^*})(\psi(a_1),\cdots,\psi(a_n))\end{aligned}$$

and so $\psi(f(a_1,\cdots,a_n)) \subseteq (f|_\theta \times f|_{\theta^*})(\psi(a_1),\cdots,\psi(a_n))$.

Conversely, suppose that $\left(\theta\langle x\rangle,\theta^*(y)\right) \in (f|_\theta \times f|_{\theta^*})(\psi(a_1),\cdots,\psi(a_n))$, then $\left(\theta\langle x\rangle,\theta^*(y)\right) \in \left\{ \left(\theta\langle a\rangle,\theta^*(b)\right) \mid \theta\langle a\rangle \in f|_\theta\ (\theta(a_1),\cdots,\theta(a_n))\ ,\ \theta^*(b) \in f|_{\theta^*}\ \left(\theta^*(a_1),\cdots,\theta^*(a_n)\right) \right\}$. Now there exists $c \in H$ such that $x\theta c$ and $c\theta^* y$, and so $(\theta\langle x\rangle,\theta(y)) = \left(\theta\langle c\rangle,\theta^*\langle c\rangle\right)$ where $c \in f(a_1,\cdots,a_n)$. Therefore $(\theta\langle x\rangle,\theta(y)) \in \psi(f(a_1,\cdots,a_n))$.

4. FUNDAMENTAL N-ARY GROUPS

If (H,f) is an *n*-ary hypergroup, then $\hat{\beta}$ denotes the transitive closure of the relation $\beta = \bigcup_{k\geq 1} \beta_k$, where β_1 is the diagonal relation, i.e., $\beta_1 = \left\{ (x,x) \mid x \in H \right\}$ and for every integer $k > 1$, β_k is the relation defined as follows:

$$x\beta_k y \text{ if and only if } \{x,y\} \subseteq f_{(\cdot)},$$

where $f_{(\cdot)}$ means that $f_{(k)}$ for some $k = 1,2,\cdots$. When $x\beta_1 y$ (i.e., $x = y$) then we write $\{x,y\} \subseteq f_{(0)}$, we define β^* as the smallest equivalence relation such that the quotient $\left(H/\beta^*, f/\beta^*\right)$ is an *n*-ary group, where H/β^* is the set of all equivalence classes. The β^* is called *fundamental equivalence relation*. The equivalence relation β^* was first introduced on hypergroups by Koskas [17] and studied mainly by Corsini [6] concerning hypergroups, Vougiouklis [16] and Davvaz [7] concerning H_v-structures.

Theorem 4. 1. *The fundamental relation β^* is the transitive closure of the relation β, i.e.,* ($\beta^* = \hat{\beta}$).

Proof: First we show that the quotient set $H/\hat{\beta}$ is an *n*-ary semigroup. The *n*-ary operation $f/\hat{\beta}$ in $H/\hat{\beta}$ is defined in the usual manner:

$$f/\hat{\beta}\left(\hat{\beta}(x_1),\cdots,\hat{\beta}(x_n)\right) = \left\{ \hat{\beta}(y) \mid y \in \left(\hat{\beta}(x_1),\cdots,\hat{\beta}(x_n)\right) \right\}$$

for all $x_1,\cdots,x_n \in H$. Suppose $a_1 \in \hat{\beta}(x_1),\cdots,a_n \in \hat{\beta}(x_n)$. Then we have
$a_1\hat{\beta}x_1$ if there exist $x_{11},\cdots,x_{1m_1+1}$ with $x_{11} = a_1$, $x_{1m_1+1} = x_1$ such that

$$\left\{ x_{1i_1}, x_{1i_1+1} \right\} \subseteq f_{(k_1)} \qquad (0 < i_1 \leq m_1)$$

$a_n\hat{\beta}x_n$ if there exist $x_{n1},\cdots,x_{nm_n+1}$ with $x_{n1} = a_n$, $x_{nm_n+1} = x_n$ such that

$$\left\{ x_{ni_n}, x_{ni_n+1} \right\} \subseteq f_{(k_n)} \qquad (0 < i_n \leq m_n).$$

Therefore, we obtain

$$f\left(\{x_{1i_1},x_{1i_1+1}\},x_{21},\cdots,x_{n1}\right) \subseteq f_{(k_1)} \qquad 1 \le i_1 \le m_1,$$
$$f\left(x_{1m_1+1},\{x_{2i_2},x_{2i_2+1}\},\cdots,x_{n1}\right) \subseteq f_{(k_2)} \qquad 1 \le i_2 \le m_2,$$
$$\vdots \qquad\qquad \vdots$$
$$f\left(x_{1m_1},,x_{21m_2+1},\cdots,\{x_{ni_n},x_{ni_n+1}\}\right) \subseteq f_{(k_n)} \qquad 1 \le i_n \le m_n.$$

So, every element $z \in f(x_{11},x_{21},\cdots,x_{n1}) = f(a_1,a_2,\cdots,a_n)$ is equivalent to every element $t \in f\left(x_{1m_1+1},x_{2m_2+1},\cdots,x_{nm_n+1}\right) = f(x_1,x_2,\cdots,x_n)$. Therefore

$$f/\hat{\beta}\left(\hat{\beta}(x_1),\cdots,\hat{\beta}(x_n)\right)$$

is singleton. So we can write $f/\hat{\beta}\left(\hat{\beta}(x_1),\cdots,\hat{\beta}(x_n)\right) = \hat{\beta}(y)$ for all $y \in f\left(\hat{\beta}(x_1),\cdots,\hat{\beta}(x_n)\right)$.

Moreover, since *f* is associative, it is obvious that $f/\hat{\beta}$ is associative, and consequently, $H/\hat{\beta}$ is an *n*-ary semigroup.

Now, let θ be an equivalence relation on *H* such that H/θ is an *n*-ary semigroup. Denote $\theta\langle a\rangle$ the class of *a*. Then for all $x_1,\cdots x_n \in H$, $f|_\theta\ (\theta(x_1),\cdots,\theta(x_n)) = \theta(y)$ for all $y \in f(\theta(x_1),\cdots,\theta(x_n))$. But also, for every $x_1,\cdots x_n \in H$ and $A_i \subseteq \theta(x_i)$ $(i = 1,\cdots,n)$ we have

$$f|_\theta\ (\theta(x_1),\cdots,\theta(x_n)) = \theta(f(x_1,\cdots,x_n)) = \theta(f(A_1,\cdots,A_n)).$$

Therefore $\theta\langle x\rangle = \theta\left(f_{(k)}\right)$ for all $k \ge 0$ and for all $x \in f_{(k)}$. So for every $a \in H$, $x \in \beta\langle a\rangle$ implies $x \in \theta\langle a\rangle$. But θ is transitively closed, so we obtain $x \in \hat{\beta}\langle a\rangle$ implies $x \in \theta\langle a\rangle$. Hence, the relation $\hat{\beta}$ is the smallest equivalence relation on *H* such that $H/\hat{\beta}$ is an *n*-ary semigroup, i.e., $\hat{\beta} = \beta^*$.

Theorem 4. 2. *β^* is a strongly compatible relation.*

Proof: If $a_1\beta^*b_1,\cdots,a_n\beta^*b_n$, then $\beta^*(a_1) = \beta^*(b_1),\cdots,\beta^*(a_n) = \beta^*(b_n)$. For every $a \in f(a_1,\cdots,a_n)$ and $b \in f(b_1,\cdots,b_n)$ we have

$$\begin{aligned}\beta^*\langle a\rangle &= \beta^*(f(a_1,\cdots,a_n))\\ &= f/\beta^*\left(\beta^*(a_1),\cdots,\beta^*(a_n)\right)\\ &= f/\beta^*\left(\beta^*(b_1),\cdots,\beta^*(b_n)\right)\\ &= \beta^*(f(b_1,\cdots,b_n))\\ &= \beta^*(b).\end{aligned}$$

Theorem 4. 3. *Let (A,f) and (B,g) be two n-ary hypergroups and let β_A^*, β_B^* and $\beta_{\mathrm{A\times B}}^*$ be fundamental equivalence relations on A, B and $A \times B$ respectively. Then*

$$A \times B/\beta_{A\times B}^* \cong A/\beta_A^* \times B/\beta_B^*.$$

Proof: First we define the relation $\tilde{\beta}$ on $A \times B$ as follows:

$$(a_1,b_1)\tilde{\beta}(a_2,b_2) \quad \Leftrightarrow \quad a_1\beta_A^*a_2 \text{ and } b_1\beta_B^*b_2.$$

$\tilde{\beta}$ is an equivalence relation. We define *h* on $A \times B/\tilde{\beta}$ as follows:

$$h\left(\tilde{\beta}(a_1,b_1),\cdots,\tilde{\beta}(a_n,b_n)\right) = \tilde{\beta}(a,b)$$

for all $a \in f\left(\beta_A^*(a_1),\cdots,\beta_A^*(a_n)\right)$, $b \in g\left(\beta_B^*(b_1),\cdots,\beta_B^*(b_n)\right)$. Since *f g* are associative, we see that *h* is associative, and consequently, $A \times B/\tilde{\beta}$ is an *n*-ary semigroup. Now let θ be an equivalence relation on $A \times B$ such that $A \times B/\theta$ is an *n*-ary group. Similar to the proof of Theorem 4.1, we get

$$(a_1, b_1)\tilde{\beta}(a_2, b_2) \quad \Rightarrow \quad (a_1, b_1)\theta(a_2, b_2).$$

Therefore the relation $\tilde{\beta}$ is the smallest equivalence relation on $A \times B$ such that $A \times B/\tilde{\beta}$ is an *n*-ary group, i.e., $\tilde{\beta} = \beta^*_{A\times B}$. Now we consider the map $\varphi : A/\beta^*_A \times B/\beta^*_B \longrightarrow A \times B/\beta^*_{A\times B}$ by

$$\varphi\left(\beta^*_A(a), \beta^*_B(b)\right) = \beta^*_{A\times B}(a, b).$$

It is easy to see that φ is an isomorphism.

Acknowledgements-The authors are extremely grateful to the referees for their valuable comments and suggestions for improving the paper. The first author would like to thank Yazd University for their financial support.

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