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Analytical solutions for the fractional nonlinear telegraph equation using a modified homotopy perturbation and separation of variables methods

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Abstract

In this paper, first a new homotopy perturbation method for solving a fractional order nonlinear telegraph equation is introduced. By applying the proposed method, the nonlinear equation is translated to linear equations for per iteration of homotopy perturbation method. Then, the obtained problems are solved with separation method. In the examples, it is illustrated that the exact solution is obtained in one iteration by conveniently separating source term of equation.

***Keywords**:* Fractional; Telegraph equation; Caputo derivative; homotopy perturbation; separation of variables; Mittag-Leffler

1. Introduction

The use of fractional telegraph equation has become increasingly popular in recent years. The fractional telegraph equation has recently been considered by several authors. (Ford et al. 2013) proposed a finite difference method for the two-parameter fractional telegraph equation and a stability condition of the numerical method is obtained. (Orsingher and Zhao (2003) discussed the numerical solution of the time-space fractional order telegraph equation.

(Orsingher and Beghin (2004) studied the fundamental solutions to time-fractional telegraph equations of order 2α. Recently, (Garg et al. (2013), considered space-time fractional telegraph equation with composite fractional derivative with respect to time and Riesz-Feller fractional derivative with respect to space.

The analytical solutions of fractional telegraph equation have been reported in literature. (Chen et al. (2008) proposed a method of separating variables for solving a linear time-fractional telegraph equation.

(Fino and Ibrahim (2013) proposed the analytical solutions of fractional telegraph equation under inhomogeneous Dirichlet and Neumann boundary conditions. The reproducing kernel theorem was used to solve the time-fractional telegraph equation with Robin boundary value conditions by (Jiang and Lin 2011).

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The homotopy perturbation method was first proposed by the Chinese mathematician (He 1999). Considerable research work has been conducted recently in applying the homotopy perturbation method to a class of linear and non-linear equations including (Chakraverty and Behera (2013), (Demir et al. (2013) and (Chowdhury et al. (2013). The Laplace transform method has been applied to a wide class of ordinary differential equations, partial differential equations, integral equations and integro-differential equations. In these problems it is necessary to calculate the Laplace transform and inverse Laplace transform of certain functions. The inverse of Laplace transform is usually difficult to compute by using the techniques of complex analysis, and there exist numerous numerical methods for its evaluation (Ouloin et al. 2013).

Fractional differential equations have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, biology, physics and engineering (Dimovski 1990, Samko et al. 1993). In (Zhao and Deng 2014), a novel predictor-corrector method, called Jacobian-predictor-corrector approach, for the numerical solutions of fractional ordinary differential equations, which are based on the polynomial interpolation, was presented. (Yang et al. (2013) proposed the Cantor-type cylindrical-coordinate method in order to investigate a family of local fractional differential operators on Cantor sets. In (Chalco-Cano, et al. 2013), the authors studied an initial value problem for a fractional differential equation using the Riemann-Liouville fractional derivative.

In this paper, a new modified homotopy perturbation technique is used to convert the fractional nonlinear telegraph equation to fractional linear telegraph in per iteration of modified homotopy perturbation method (MHPM). By using separation of variables the obtained linear problems are solved analytically.

2. Background theory

We begin by stating some preliminary definitions from fractional calculus. There exist different approaches to fractional derivatives (Dimovski 1990, Samko et al. 1993).

Definition 1. (Dimovski 1990) A function $f: R \to R^+$ is said to be in the space C_ν, with $\nu \in R$, if it can be written as $f(x) = x^p f_1(x)$ with $p > \nu$, $f_1(x) \in C[0,\infty)$ and it is said to be in the space C_ν^m if $f^{(m)} \in C_\nu$ for $m \in N \cup \{0\}$.

Definition 2. (Luchko and Gorenflo 1999) The Riemann-Liouville fractional integral of $f \in C_\nu$ with order $\alpha > 0$ and $\nu \geq -1$ is defined as:

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\,d\tau, \quad \alpha > 0, t > 0, \quad J^0 f(t) = f(t) \tag{1}$$

Definition3. (Podlubny 1998) The Riemann-Liouville fractional derivative of $f \in C_{-1}^m$ with order $\alpha > 0$and $m \in N \cup \{0\}$, is defined as:

$$D_t^\alpha f(t) = \frac{d^m}{dt^m} J^{m-\alpha} f(t), \quad m-1 < \alpha \leq m, \; m \in N. \tag{2}$$

Definition 4. (Podlubny 1998) The Caputo fractional derivative of $f \in C_{-1}^m$ with order $\alpha > 0$and $m \in N \cup \{0\}$, is defined as:

$$cD_t^\alpha f(t) = \begin{cases} J^{m-\alpha} f^{(m)}(t), m-1 < \alpha \leq m, m \in N \\ \dfrac{d^m f(t)}{dt^m}, & \alpha = m \end{cases} \tag{3}$$

Definition 5. (Podlubny 1998) A two-parameter Mittag-Leffler function is defined by the following series

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + \beta)}. \tag{4}$$

Definition 6. (Luchko and Gorenflo 1999) A multivariate Mittag-Leffler function is defined as

$$E_{(a_1,a_2,\cdots,a_n),b}(z_1, z_2, \cdots, z_n) = \sum_{k=0}^{\infty} \sum_{l_1+l_2+\ldots+l_n=k} \frac{k!}{l_1! \times l_2! \times \ldots \times l_n!} \frac{\prod_{i=1}^{n} z_i^{l_i}}{\Gamma(b + \sum_{i=1}^{n} a_i l_i)}, \tag{5}$$

where $b > 0$, $l_1, l_2, \cdots, l_n \geq$,$|z_i| < \infty, i = 1,2,\cdots,n.$

Definition 7. Let us define the Laplace-transform (LT) operator φ on a function $u(x,t), (t \geq 0)$ by

$$\phi\{u(x,t); t \to s\} = \int_0^\infty e^{-st} u(x,t)\,dt \tag{6}$$

and denote it by $\varphi\{u(x,t); t \mapsto s\} = L(u(x,t))$, where s is the LT parameter. For our purpose here, we shall take s to be real and positive.

Consequently, the LT of Mittag-Leffler function has the following form:

$$L(E_{\alpha,\beta}(t)) = \int_0^\infty e^{-st} E_{\alpha,\beta}(t)\,dt = \sum_{k=0}^{\infty} \frac{1}{s^{k+1}\Gamma(\alpha k + \beta)}. \tag{7}$$

Lemma 2.1. (Luchko and Gorenflo 1999) Let $\mu > \mu_1 > \mu_2 > \cdots > \mu_n \geq 0, m_i - 1 < \mu_i \leq m_i, m_i \in N_0 = N \cup \{0\}$, $d_i \in R, i = 1,2,\ldots,n.$Consider the initial value problem

$$\begin{cases} (D^\mu y)(x) - \sum_{i=1}^{n} \lambda_i (D^{\mu_i} y)(x) = g(x), \\ y^{(k)}(0) = c_k \in R, k = 0,1,\ldots,m-1, m-1 < \mu \leq m, \end{cases}$$

where the function g(x) is assumed to lie in C_{-1}, if $\mu \in N$, in C_{-1}^1, if $\mu \notin N$ and the unknown function $y(x)$ is to be determined in the space C_{-1}^m. This has solution

$$y(x) = y_g(x) + \sum_{k=0}^{m-1} c_k u_k(x), x \geq 0,$$

where

$$y_g(x) = \int_0^x t^{\mu-1} E_{(\cdot),\mu}(t) g(x-t)\,dt$$

and

$$u_k(x) = \frac{x^k}{k!} + \sum_{i=l_k+1}^{n} d_i x^{k+\mu-\mu_i} E_{(\cdot),k+1+\mu-\mu_i}(x), \quad k = 0,1,\ldots,m-1,$$

fulfills the initial conditions

$$u_k^{(1)}(0)=\delta_{kl}, k,l=0,1,\dots,m-1.$$

The function

$$E_{(\cdot),\sigma}(x)=E_{(\mu-\mu_1,\dots,\mu-\mu_n),\sigma}(d_1x^{\mu-\mu_1},\dots,d_nx^{\mu-\mu_n})$$

is a particular case of the multivariate Mittag-Leffler function (Luchko and Gorenflo 1999) and the natural numbers $l_k, k=0,1,\dots,m-1$, are determined from the condition

$$\begin{cases} m_{l_k} \geq k+1, \\ m_{l_k+1} \leq k. \end{cases}$$

In the case $m_i \leq k, i=1,2,\dots,n$, we set $l_k:=0$, and if $m_i \geq k+1, i=1,2,\dots,n$, then $l_k:=n$.

3. Modified homotopy perturbation method

The homotopy perturbation method is power and an effective method for solving nonlinear problems. There are several modification of this method.

In this paper, for solving the fractional nonlinear telegraph equation

$$D_t^{2\alpha}u(x,t)+aD_t^{\alpha}u(x,t)=K\frac{\partial^2u(x,t)}{\partial x^2}+h(u(x,t))+f(x,t), \tag{8}$$

with given initial and inhomogeneous boundary conditions, we first apply a proper transformation such as $u(x,t)=W(x,t)+V(x,t)$ for converting the inhomogeneous boundary conditions for homogeneous boundary conditions that result in

$$D_t^{2\alpha}W(x,t)+aD_t^{\alpha}W(x,t)=K\frac{\partial^2W(x,t)}{\partial x^2}+h\big(W(x,t)+V(x,t)\big)+\tilde{f}(x,t). \tag{9}$$

Now, for solving (9) we apply a MHPM as follow:

$$\begin{aligned} D_t^{2\alpha}W(x,t)+aD_t^{\alpha}W(x,t) &= K\frac{\partial^2W(x,t)}{\partial x^2}+ph(W(x,t)\\ &+V(x,t))\\ &+\tilde{f}_1(x,t)+p\tilde{f}_2(x,t), \end{aligned} \tag{10}$$

where $\tilde{f}_1(x,t)+\tilde{f}_2(x,t)=\tilde{f}(x,t)$, p is imbedding parameter that varies from zero to one. By assuming $W(x,t)=\sum_{i=0}^{\infty}W_i(x,t)p^i$ and substituting it in (10), we obtain

$$p^0:\begin{cases} D_t^{2\alpha}W_0(x,t)+aD_t^{\alpha}W_0(x,t) \\ =K\dfrac{\partial^2W_0(x,t)}{\partial x^2}+\tilde{f}_1(x,t), \\ W_0(x,0)=g_1(x), \\ (W_0)_t(x,0)=g_2(x), \\ 0\leq x\leq X, \\ W_0(0,t)=0,\ W_0(X,t)=0, \\ t\geq 0, \end{cases} \tag{11}$$

$$p^1:\begin{cases} D_t^{2\alpha}W_1(x,t)+aD_t^{\alpha}W_1(x,t) \\ =K\dfrac{\partial^2W_1(x,t)}{\partial x^2}+\tilde{f}_2(x,t)+A_0, \\ W_1(x,0)=0,\ (W_1)_t(x,0)=0, \\ 0\leq x\leq X, \\ W_1(0,t)=0,\ W_1(X,t)=0, \\ t\geq 0, \end{cases} \tag{12}$$

$$\vdots$$

$$p^k:\begin{cases} D_t^{2\alpha}W_k(x,t)+aD_t^{\alpha}W_k(x,t) \\ =K\dfrac{\partial^2W_k(x,t)}{\partial x^2}+A_{k-1}, \\ W_k(x,0)=0,\ (W_k)_t(x,0)=0, \\ 0\leq x\leq X, \\ W_k(0,t)=0,\ W_k(X,t)=0, \\ t\geq 0, \end{cases} \tag{13}$$

where A_k, $k=0,1,\dots$ are Adomian polynomials and are obtained as:

$$A_k=\frac{d^k}{dp^k}h\Big(\sum_{i=0}^{\infty}W_ip^i+V\Big)\Big|_{p=0}, k=0,1,\dots. \tag{14}$$

For more details see (Irandoust-Pakchin et al. 2013). All of the obtained problems in (11)-(13) are linear with homogeneous boundary conditions, so they can be solved by separation method.

The success of this method is based on the proper choice $\tilde{f}_1$ of source term of $\tilde{f}$. In the examples it has been shown that with a proper choice of the $\tilde{f}_1$, the solution can be obtained in two iterations of MHPM.

4. Inhomogeneous fractional nonlinear telegraph equation with Dirichlet boundary condition

In this section, we determine the solution of the fractional nonlinear telegraph equation (8) with the initial and Dirichlet boundary conditions

$$u(x,0)=\varphi_1(x),\quad u_t(x,0)=\varphi_2(x),\\ 0\le x\le X,\ u(0,t)=\psi_1(t),\\ u(X,t)=\psi_2(t),\ t\ge 0. \tag{15}$$

In order to solve the problem with inhomogeneous boundary, first it should be transformed into an homogeneous boundary condition. For this purpose let

$u(x,t)=W(x,t)+V(x,t),$

where $W(x,t)$ is a new unknown function and

$$V(x,t)=\frac{\psi_2(t)-\psi_1(t)}{X}x+\psi_1(t), \tag{16}$$

that satisfies the boundary conditions

$$V(0,t)=\psi_1(t),\quad V(X,t)=\psi_2(t). \tag{17}$$

The function $W(x,t)$ is satisfied in problem with homogeneous boundary conditions:

$$\begin{cases} D_t^{2\alpha}W(x,t)+aD_t^{\alpha}W(x,t)= \\ K\dfrac{\partial^2 W(x,t)}{\partial x^2}+h(W+V)+\tilde{f}(x,t), \\ W(x,0)=g_1(x), W_t(x,0)=g_2(x), \\ 0\le x\le X, W(0,t)=0, W(X,t)=0, \end{cases} \tag{18}$$

where

$$\tilde{f}(x,t)=f(x,t)+\frac{x}{X}\left[(D_t^{2\alpha}\psi_1(t)-D_t^{2\alpha}\psi(t))\right.\\ \left.+a(D_t^{\alpha}\psi_1(t)-D_t^{\alpha}\psi_2(t))\right]-(D_t^{2\alpha}\psi_1(t)+D_t^{\alpha}\psi_1(t)) \tag{19}$$

and

$$g_1(x)=\phi_1(x)-\frac{x}{X}\left[\psi_2(0)-\psi_1(0)\right]-\psi_1(0),$$

$$g_2(x)=\varphi_2(x)-\frac{x}{X}\left[(\psi_2)_t(0)-(\psi_1)_t(0)\right]\\ -(\psi_1)_t(0). \tag{20}$$

For solving (18) we use the MHPM and obtain corresponding problems with homogeneous boundary conditions in (11)-(13). Now we consider the method of separation of variables for solving them. By assuming $W_0(x,t)=F_0(x)T_0(t)$ and substituting it in (11), we obtain an ordinary linear differential equation for $F_0(x)$:

$$F_0''(x)+\lambda^2F_0(x)=0,\ F_0(0)=F_0(X)=0 \tag{21}$$

and a fractional ordinary linear differential equation for $T_0(t)$ as follows:

$$D_t^{2\alpha}T_0+aD_t^{\alpha}T_0+K\lambda^2T_0=0. \tag{22}$$

The eigenvalues and corresponding eigenfunctions of Sturm-Liouvill (21) are

$$\lambda_n=\frac{n^2\pi^2}{X^2},\quad (F_0)_n(x)=\sin(\frac{n\pi x}{X})\\ n=1,2,\ldots \tag{23}$$

Now we seek a solution of the inhomogeneous problem in (11) of the form

$$W_0(x,t)=\sum_{n=1}^{\infty}(B_0)_n(t)\sin(\frac{n\pi x}{X}). \tag{24}$$

We assume that the series can be differentiated term by term. In order to determine $(B_0)_n(t)$, we expand $\tilde{f}_1(x,t)$ as a Fourier series by the eigenfunctions $\sin(\frac{n\pi x}{X})$ as follows

$$\tilde{f}_1(x,t)=\sum_{n=1}^{\infty}(\tilde{f}_1)_n(t)\sin(\frac{n\pi x}{X}), \tag{25}$$

Then

$$(\tilde{f}_1)_n(t)=\frac{2}{X}\int_0^x\tilde{f}_1(x,t)\sin(\frac{n\pi x}{X})dx. \tag{26}$$

Substituting (24) and (25) into (11) yields

$$\sum_{n=1}^{\infty}D^{2\alpha}(B_0)_n(t)\sin(\frac{n\pi x}{X})+\\ a\sum_{n=1}^{\infty}D^{\alpha}(B_0)_n(t)\sin(\frac{n\pi x}{X})\\ =\sum_{n=1}^{\infty}-K(\frac{n\pi}{X})^2\sin(\frac{n\pi x}{X})(B_0)_n(t)\\ +\sum_{n=1}^{\infty}(\tilde{f}_1)_n(t)\sin(\frac{n\pi x}{X}). \tag{27}$$

By orthogonal properties of $\sin(\frac{n\pi x}{X})$, we obtain

$$D_t^{2\alpha}(B_0)_n(t)+aD_t^{\alpha}(B_0)_n(t)\\ +K(\frac{n\pi}{X})^2(B_0)_n(t)=(\tilde{f}_1)_n(t), \tag{28}$$

where

$\sum_{n=1}^{\infty}(B_0)_n(0)\sin(\frac{n\pi x}{X}) = g_1(x),$ (29)

$$\sum_{n=1}^{\infty}(\frac{\partial B_0}{\partial t})_n(0)\sin(\frac{n\pi x}{X}) = g_2(x), \quad (30)$$

which yields

$$(B_0)_n(0) = \frac{2}{X}\int_0^x g_1(x)\sin(\frac{n\pi x}{X})dx,$$
$$(\frac{\partial B_0}{\partial t})_n(0) = \frac{2}{X}\int_0^x g_2(x)\sin(\frac{n\pi x}{X})dx, \quad (31)$$

For each value of n, (28) and (31) make up a fractional initial value problem.

According to lemma 2.1, the fractional initial value problem with $\mu = 2\alpha, \mu_1 = \alpha \le 1 = m_1$, $\mu_2 = 0 = m_2$, $\lambda_1 = -a$, $\lambda_2 = -K(\frac{n\pi}{X})^2$, $m = 2$ has the solution

$$(B_0)_n(t) = \int_0^t \tau^{\mu-1}E_{(\mu-\mu_1,\mu-\mu_2),1}(\lambda_1\tau^{\mu-\mu_1}, \lambda_2\tau^{\mu-\mu_2})$$
$$(\tilde{f}_1)_n(t-\tau)d\tau + (B_0)_n(0)\left[\frac{t^0}{0!} + \sum_{i=l_0+1}^{2}\lambda_i t^{0+\mu-\mu_i}\right.$$
$$\left.E_{(\mu-\mu_1,\mu-\mu_2),0+1+\mu-\mu_i}(\lambda_1 t^{\mu-\mu_1}, \lambda_2 t^{\mu-\mu_2})\right] \quad (32)$$
$$+(\frac{\partial B_0}{\partial t})_n(0)\left[\frac{t^1}{1!} + \sum_{i=1_1+1}^{2}\lambda_1 t^{1+\mu-\mu_i}\right.$$
$$\left.E_{(\mu-\mu_1,\mu-\mu_2),1+1+\mu-\mu_i}(\lambda_1 t^{\mu-\mu_1}, \lambda_2 t^{\mu-\mu_2})\right].$$

Hence we get the solution of the initial boundary value problem (11) in the form

$$W_0(x,t) = \sum_{n=1}^{\infty}(B_0)_n(t)\sin(\frac{n\pi x}{X})$$
$$= \sum_{n=1}^{\infty}\sin(\frac{n\pi x}{X})\left\{\int_0^t \tau^{2\alpha-1}E_{(\alpha,2\alpha),2\alpha}(-a\tau^{\alpha}, -(\frac{n\pi}{X})^2\tau^{2\alpha})\right.$$
$$(\tilde{f}_1)_n(t-\tau)d\tau + (B_0)_n(0)\left[\frac{t^0}{0!} + \sum_{i=l_0+1}^{2}\lambda_i t^{0+2\alpha-\mu_i}\right.$$
$$\left.E_{(\alpha,2\alpha),0+1+2\alpha-\mu_i}(-at^{\alpha}, -(\frac{n\pi}{X})^2 t^{2\alpha})\right] \quad (33)$$
$$+(\frac{\partial B_0}{\partial t})_n(0)\left[\frac{t^1}{1!} + \sum_{i=1_1+1}^{2}\lambda_i t^{1+2\alpha-\mu_i}E_{(\alpha,2\alpha),1+1+2\alpha-\mu_i}\right.$$
$$\left.\left.(-at^{\alpha}, -(\frac{n\pi}{X})^2 t^{2\alpha})\right]\right\}.$$

In similar way, we can obtain W_k, $k = 1,2,...$ from (12) and (13). Note that in calculating W_{k+1}, the value of A_k is known from previous stages. Then, in spite of the main problem of telegraph, all of the problems in (11)-(13) are linear and hence solving them is simple with respect to the main problem.

Note that an important observation that can be made here is that the success of the proposed methods depends mainly on the proper choice of the functions $\tilde{f}_1$and $\tilde{f}_2$. Furthermore, this proper selection of the components $\tilde{f}_1$ and $\tilde{f}_2$ may provide the solution only in two iterations of MHPM.

In this section, the solution of the fractional nonlinear telegraph equation (8) is determined with the initial and Neumann boundary conditions

$$u(x,0) = \phi_1(x), \quad u_t(x,0) = \phi_2(x),$$
$$0 \le x \le X, \quad (34)$$
$$u_x(0,t) = \psi_1(t), \quad u_x(X,t) = \psi_2(t), \; t \ge 0$$

where $\phi_1(x)$, $\phi_2(x)$, $\psi_1(t)$, $\psi_2(t)$ are as defined in section 4.

In order to solve the problem assume that

$$u(x,t) = \widetilde{W}(x,t) + \tilde{V}(x,t),$$

where $\widetilde{W}(x,t)$ is a new unknown function and

$$\tilde{V}(x,t) = \frac{\psi_2(t) - \psi_1(t)}{2X}x^2 + \psi_1(t)x, \quad (35)$$

that fulfills the boundary conditions

$$\tilde{V}_x(0,t) = \psi_1(t), \; \tilde{V}_x(X,t) = \psi_2(t). \quad (36)$$

The function $\widetilde{W}(x,t)$ is satisfied in the problem with homogeneous boundary conditions:

$$\begin{cases} D_t^{2\alpha}\tilde{W}(x,t) + aD_t^{\alpha}\tilde{W}(x,t) = \\ k\,\frac{\partial^2\tilde{W}(x,t)}{\partial x^2} + h(\tilde{W} + \tilde{V}) + \tilde{f}(x,t), \\ \tilde{W}(x,0) = g_1(x), \tilde{W}_t(x,0) = g_2(x), \\ 0 \le x \le X, \\ \tilde{W}_x(0,t) = 0, \tilde{W}_x(X,t) = 0, \\ t \ge 0, \end{cases} \quad (37)$$

where

$$\tilde{f}(x,t) = f(x,t) + \frac{x^2}{2X}\left[(D_t^{2\alpha}\psi_1(t) - D_t^{2\alpha}\psi_2(t)) + \right. \quad (38)$$
$$\left.a(D_t^{\alpha}\psi_1(t) - D_t^{\alpha}\psi_2(t))\right] - (D_t^{2\alpha}\psi_1(t) + D_t^{\alpha}\psi_1(t))$$

and

$$g_1(x) = \phi_1(x) - \frac{x^2}{2X}[\psi_2(0) - \psi_1(0)] - \psi_1(0), \quad (39)$$
$$g_2(x) = \phi_2(x) - \frac{x^2}{2X}[(\psi_2)_t(0) - (\psi_1)_t(0)] - (\psi_1)_t(0).$$

For solving (37) we use MHPM

$$D_t^{2\alpha}\tilde{W}(x,t)+aD_t^{\alpha}\tilde{W}(x,t)= K\frac{\partial^2\tilde{W}(x,t)}{\partial x^2}+ph(\tilde{W}+\tilde{V}) +\tilde{f}_1(x,t)+p\tilde{f}_2(x,t). \quad (40)$$

By assuming $\tilde{W}(x,t)=\sum_{i=0}^{\infty}\tilde{W}_i(x,t)\,p^i$ and substituting it in (40), we again obtain fractional linear problems (11)-(13) with homogeneous boundary conditions.

With a similar manner in section 4, we solve the corresponding homogeneous equation in (11) by the method of separation of variables.

In this situation the eigenvalues and eigenfunctions:

$$\lambda_n=\frac{n^2\pi^2}{X^2},\ (F_0)_n(x)=\cos(\frac{n\pi x}{X}),n=1,2,.... \quad (41)$$

Now we seek a solution of the inhomogeneous problem in (11) of the form

$$\tilde{W}_0(x,t)=\sum_{n=1}^{\infty}(B_0)_n(t)\cos(\frac{n\pi x}{X}). \quad (42)$$

In order to determine $(B_0)_n(t)$, we expand $\tilde{f}_1(x,t)$ as a Fourier series by the eigenfunctions $\cos(\frac{n\pi x}{X})$ as follows

$$\tilde{f}_1(x,t)=\sum_{n=1}^{\infty}(\tilde{f}_1)_n(t)\cos(\frac{n\pi x}{X}), \quad (43)$$

Then

$$(\tilde{f}_1)_n(t)=\frac{2}{X}\int_0^x\tilde{f}_1(x,t)\cos(\frac{n\pi x}{X})dx. \quad (44)$$

Substituting (42)-(43) into (11) yields

$$\sum_{n=1}^{\infty}D^{2\alpha}(B_0)_n(t)\cos(\frac{n\pi x}{X})+a\sum_{n=1}^{\infty}D^{\alpha}(B_0)_n(t)\cos(\frac{n\pi x}{X}) =\sum_{n=1}^{\infty}-k(\frac{n\pi}{X})^2\cos(\frac{n\pi x}{X})(B_0)_n(t)+\sum_{n=1}^{\infty}(\tilde{f}_1)_n(t)\cos(\frac{n\pi x}{X}) \quad (45)$$

By orthogonality properties of $\cos(\frac{n\pi x}{X})$, we get

$$D_t^{2\alpha}(B_0)_n(t)+aD_t^{\alpha}(B_0)_n(t) +K(\frac{n\pi}{X})^2(B_0)_n(t)=(\tilde{f}_1)_n(t), \quad (46)$$

where

$$\sum_{n=1}^{\infty}(B_0)_n(0)\cos(\frac{n\pi x}{X})=g_1(x), \quad (47)$$

$$\sum_{n=1}^{\infty}(\frac{\partial B_0}{\partial t})_n(0)\cos(\frac{n\pi x}{X})=g_2(x) \quad (48)$$

which yields

$$(B_0)_n(0)=\frac{2}{X}\int_0^x g_1(x)\cos(\frac{n\pi x}{X})dx, \qquad (\frac{\partial B_0}{\partial t})_n(0)=\frac{2}{X}\int_0^x g_2(x)\cos(\frac{n\pi x}{X})dx. \quad (49)$$

For each value of n, (46) and (49) make up a fractional initial value problem.

According to lemma 2.1, the fractional initial value problem with $\mu=2\alpha,\mu_1=\alpha\le 1=m_1$, $\mu_2=0=m_2$, $\lambda_1=-a$, $\lambda_2=-K(\frac{n\pi}{X})^2$, $m=2$ has the solution

$$\begin{aligned}(B_0)_n(t)&=\int_0^t\tau^{\mu-1}E_{(\mu-\mu_1,\mu-\mu_2),1}(\lambda_1\tau^{\mu-\mu_1},\lambda_2\tau^{\mu-\mu_2})(\tilde{f}_1)_n(t-\tau)d\tau+\\&(B_0)_n(0)\left[\frac{t^0}{0!}+\sum_{i=l_0+1}^{2}\lambda_i t^{0+\mu-\mu_i}E_{(\mu-\mu_1,\mu-\mu_2),0+1+\mu-\mu_i}(\lambda_1 t^{\mu-\mu_1},\lambda_2 t^{\mu-\mu_2})\right]\\&+(\frac{\partial B_0}{\partial t})_n(0)\left[\frac{t^1}{1!}+\sum_{i=l_1+1}^{2}\lambda_i t^{1+\mu-\mu_i}E_{(\mu-\mu_1-\mu_2),1+1+\mu-\mu_i}(\lambda_1 t^{\mu-\mu_1},\lambda_2 t^{\mu-\mu_2})\right]\end{aligned} \quad (50)$$

Hence we get the solution of the initial boundary value problem (11) in the form

$$\begin{aligned}\tilde{W}_0(x,t)&=\sum_{n=1}^{\infty}(B_0)_n(t)\cos(\frac{n\pi x}{X})\\&=\sum_{n=1}^{\infty}\cos(\frac{n\pi x}{X})\Big\{\int_0^t\tau^{2\alpha-1}E_{(\alpha,2\alpha),2\alpha}(-a\tau^{\alpha},-(\frac{n\pi}{X})^2\tau^{2\alpha})\\&(\tilde{f}_1)_n(t-\tau)d\tau+(B_0)_n(0)\left[\frac{t^0}{0!}+\sum_{i=l_0+1}^{2}\lambda_i t^{0+2\alpha-\mu_i}\right.\\&\left.E_{(\alpha,2\alpha),0+1+2\alpha-\mu_i}(-at^{\alpha},-(\frac{n\pi}{X})^2t^{2\alpha})\right]+(\frac{\partial B_0}{\partial t})_n(0)\\&\left[\frac{t^1}{1!}+\sum_{i=l_1+1}^{2}\lambda_i t^{1+2\alpha-\mu_i}E_{(\alpha,2\alpha),1+1+2\alpha-\mu_i}(-at^{\alpha},-(\frac{n\pi}{X})^2t^{2\alpha})\right]\Big\}.\end{aligned} \quad (51)$$

In a similar way, we can obtain $\tilde{W}_k,\ k=1,2,...$ from (12)-(13). Note that in calculating $\tilde{W}_{k+1}$ the value of A_k is known from previous stages. So, in spite of the main problem of telegraph, all of the problems in (11)-(13) are linear with homogeneous boundary conditions.

5. Examples

In this section, we consider two examples with different initial and boundary conditions and source term. We show that by suitable selecting f_1 and f_2 in source term, the exact solution is obtained in two iterations of MHPM.

Example 1.
Consider the fractional nonlinear telegraph equation

$$\begin{aligned} &D_t^{2\alpha}u(x,t)+aD_t^{\alpha}u(x,t) \\ &\quad = K\frac{\partial^2 u(x,t)}{\partial x^2}+\sin(u(x,t))+f(x,t), \end{aligned} \tag{52}$$

with the initial and Dirichlet boundary conditions

$$\begin{aligned} &u(x,0)=\sin(5\pi x)+1, \quad u_t(x,0)=x, \quad 0\le x\le 1, \\ &u(0,t)=1, \quad u(1,t)=t+1, \quad t\ge 0, \end{aligned} \tag{53}$$

where

$$\begin{aligned} &f(x,t) \\ &=\sin(5\pi x)\left[2\frac{\Gamma(\gamma_1+1)}{\Gamma(\gamma_1+1-2\alpha)}t^{\gamma_1-2\alpha}\right. \\ &+5a\frac{\Gamma(\gamma_1+1)}{\Gamma(\gamma_1+1-\alpha)}t^{\gamma_1-\alpha}+K(5\pi)^2(2t^{\gamma_1}+5t^{\gamma_2}+1) \\ &\left.+2\frac{\Gamma(\gamma_2+1)}{\Gamma(\gamma_2+1-2\alpha)}t^{\gamma_2-2\alpha}+5a\frac{\Gamma(\gamma_2+1)}{\Gamma(\gamma_2+1-\alpha)}t^{\gamma_2-\alpha}\right] \\ &+x^2\left(\frac{t^{1-2\alpha}}{\Gamma(2-2\alpha}+\frac{t^{1-\alpha}}{\Gamma(2-\alpha}\right)-\sin((2t^{\gamma_1}+5t^{\gamma_2}+1) \\ &\times\sin(5\pi x)+tx+1). \end{aligned}$$

In order to solve the problem, we first transform it into a homogeneous boundary condition. For this purpose, let

$$u(x,t)=W(x,t)+V(x,t)=W(x,t)+tx+1,$$

where $W(x,t)$ is a new unknown function and is satisfied in problem with homogeneous boundary conditions:

$$\begin{aligned} &D_t^{2\alpha}W(x,t)+aD_t^{\alpha}W(x,t) \\ &=k\frac{\partial^2 W(x,t)}{\partial x^2}+\sin(W+V)+\tilde{f}(x,t), \\ &W(x,0)=\sin(5\pi x),\ W_t(x,0)=0,\ 0\le x\le 1, \\ &\qquad W(0,t)=0, W(1,t)=0 \ . \end{aligned} \tag{54}$$

where

$$\begin{aligned} &\tilde{f}(x,t) \\ &=\sin(5\pi x)\left[2\frac{\Gamma(\gamma_1+1)}{\Gamma(\gamma_1+1-2\alpha)}t^{\gamma_1-2\alpha}\right. \\ &+5a\frac{\Gamma(\gamma_1+1)}{\Gamma(\gamma_1+1-\alpha)}t^{\gamma_1-\alpha} \\ &+K(5\pi)^2(2t^{\gamma_1}+5t^{\gamma_2}+1)+2\frac{\Gamma(\gamma_2+1)}{\Gamma(\gamma_2+1-2\alpha)}t^{\gamma_2-2\alpha} \\ &\left.+5a\frac{\Gamma(\gamma_2+1)}{\Gamma(\gamma_2+1-\alpha)}t^{\gamma_2-\alpha}\right] \\ &-\sin((2t^{\gamma_1}+5t^{\gamma_2}+1)\sin(5\pi x)+tx+1). \end{aligned}$$

For solving (54), we apply MHPM

$$\begin{aligned} &D_t^{2\alpha}W(x,t)+a\,D_t^{\alpha}W(x,t) \\ &=K\frac{\partial^2 W(x,t)}{\partial x^2}+ \\ &+p\sin(W+V) \\ &+\tilde{f}_1(x,t)+p\,\tilde{f}_2(x,t). \end{aligned} \tag{55}$$

By assuming $W(x,t)=\sum_{i=0}^{\infty}W_i(x,t)\,p^i$ and substituting it in (55), we obtain (11)-(12). Where $\tilde{f}(x,t)=\tilde{f}_1(x,t)+\tilde{f}_2(x,t)$ and

$$\begin{aligned} &\tilde{f}_1(x,t) \\ &=\sin(5\pi x)\left[2\frac{\Gamma(\gamma_1+1)}{\Gamma(\gamma_1+1-2\alpha)}t^{\gamma_1-2\alpha}\right. \\ &+5\alpha\frac{\Gamma(\gamma_1+1)}{\Gamma(\gamma_1+1-\alpha)}t^{\gamma_i-\alpha}+K(5\pi)^2(2t^{\gamma_1}+5t^{\gamma_2}+1) \\ &\left.+2\frac{\Gamma(\gamma_2+1)}{\Gamma(\gamma_2+1-2\alpha)}t^{\gamma_2-2\alpha}+5a\frac{\Gamma(\gamma_2+1)}{\Gamma(\gamma_2+1-\alpha)}t^{\gamma_2-\alpha}\right] \\ &\tilde{f}_2(x,t)=-\sin((2t^{\gamma_1}+5t^{\gamma_2}+1)\sin(5\pi x)+tx+1) \end{aligned}$$

and

$$g_1(x)=\sin(5\pi x),\quad g_2(x)=0.$$

We solve the corresponding homogeneous equation in (11) by the method of separation of variables. With similar calculation in section 4 we obtain a Sturm-Liouville problem and an ordinary linear differential equation with respect to x and t respectively. The eigenvalues and eigenfunctions of obtained Sturm-Liouville are,

$$\begin{aligned} &\lambda_n=n^2\pi^2,\ (F_0)_n(x) \\ &=\sin(n\pi x),\ n=1,2,\ldots \end{aligned} \tag{56}$$

Now we seek a solution of the inhomogeneous problem in (11) of the form

$$W_0(x,t)=\sum_{n=1}^{\infty}(B_0)_n(t)\sin(n\pi x). \qquad (57)$$

By substituting this in (11) and using of arguments in section 4 we have

$$\begin{aligned}&\sum_{n=1}^{\infty}D^{2\alpha}(B_0)_n(t)\sin(n\pi x)\\&+a\sum_{n=1}^{\infty}D^{\alpha}(B_0)_n(t)\sin(n\pi x)\\&=\sum_{n=1}^{\infty}-K(n\pi)^2\sin(n\pi x)(B_0)_n(t)\\&+\sum_{n=1}^{\infty}(\tilde{f}_1)_n(t)\sin(n\pi x),\end{aligned} \qquad (58)$$

where

$$(f_1)_n(t)=\frac{2}{1}\int_0^1\tilde{f}_1(x,t)\sin(n\pi x)\,dx=\begin{cases}H(t), & n=5\\ 0, & n\neq 5\end{cases}$$

with

$$\begin{aligned}H(t)&=2\frac{\Gamma(\gamma_1+1)}{\Gamma(\gamma_1+1-2\alpha)}t^{\gamma_1-2\alpha}\\&+5a\frac{\Gamma(\gamma_1+1)}{\Gamma(\gamma_1+1-\alpha)}t^{\gamma_1-\alpha}\\&+K(5\pi)^2(2t^{\gamma_1}+5t^{\gamma_2}+1)\\&2\frac{\Gamma(\gamma_2+1)}{\Gamma(\gamma_2+1-2\alpha)}t^{\gamma_2-2\alpha}\\&+5a\frac{\Gamma(\gamma_2+1)}{\Gamma(\gamma_2+1-\alpha)}t^{\gamma_2-\alpha}.\end{aligned}$$

By orthogonal properties of $\sin(\frac{n\pi x}{X})$, from (58) we have

$$\begin{aligned}&D_t^{2\alpha}(B_0)_n(t)+aD_t^{\alpha}(B_0)_n(t)\\&+K(n\pi)^2(B_0)_n(t)=(\tilde{f}_1)_n(t),\end{aligned} \qquad (59)$$

where

$$\sum_{n=1}^{\infty}(B_0)_n(0)\sin(n\pi x)=\sin(5\pi x), \qquad (60)$$

$$\sum_{n=1}^{\infty}(\frac{\partial B_0}{\partial t})_n(0)\sin(n\pi x)=0, \qquad (61)$$

$$(B_0)_n(0)=2\int_0^1\sin(5\pi x)\sin(n\pi x)dx=\begin{cases}1, & n=5,\\ 0, & n\neq 5,\end{cases} \qquad (62)$$

$$(\frac{\partial B_0}{\partial t})_n(0)=0.$$

For each value of n, (59) and (62) make up a fractional initial value problem. According to lemma 2.1, the fractional of this problem has the solution

$$\begin{aligned}&(B_0)_n(t)\\&=\int_0^t\tau^{2\alpha-1}E_{(\alpha,2\alpha),2\alpha}\left(-a\tau^{\alpha},-k(n\pi)^2\tau^{2\alpha}\right)\\&(\tilde{f}_1)_n(t-\tau)d\tau+(B_0)_n(0)\end{aligned} \qquad (63)$$

$$\begin{aligned}&=\int_0^t\tau^{2\alpha-1}E_{(\alpha,2\alpha),2\alpha}(-a\tau^{\alpha},-k(n\pi)^2\tau^{2\alpha}\\&\times\begin{cases}H(t-\tau), & n=5,\\ & \qquad d\tau\\ 0, & n\neq 5,\end{cases}\\&+\begin{cases}1, & n=5,\\ 0, & n\neq 5.\end{cases}\end{aligned}$$

Taking the Laplace transform from both sides of (63), we get

$$L\left[(B_0)_n(t)\right]=0, \quad n\neq 5 \qquad (64)$$

and

$$L\left[(B_0)_5(t)\right] \qquad (65)$$

$$\begin{aligned}&=L\left[\int_0^t\tau^{2\alpha-1}E_{(\alpha,2\alpha),2\alpha}(-a\tau^{\alpha},-K(5\pi)^2\tau^{2\alpha}\,H(t-\tau)d\tau\right]\\&+L[1]\\&=L\left[E_{(\alpha,2\alpha),2\alpha}(-at^{\alpha},-K(5\pi)^2t^{2\alpha})\right]L[H(t)]+\frac{1}{s}\\&=L\left[\sum_{k=0}^{\infty}\sum_{l_1=0}^{k}\binom{k}{l_1}\frac{(-at^{\alpha})^{l_1}(-K(5\pi)^2t^{2\alpha})^{k-l_1}}{\Gamma(2\alpha+l_1\alpha+(k-l_1)2\alpha)}\right]\\&\times L[H(t)]+\frac{1}{s}\\&=(\sum_{K=0}^{\infty}\sum_{L_1=0}^{K}\binom{k}{l_1}\frac{(-a)^{l_1}(-k(5\pi)^2)^{k-l_1}}{s^{2\alpha+2\alpha k}})\\&\times L[H(t)]+\frac{1}{s}\end{aligned}$$

$$= (\frac{1}{s^{2\alpha}}\sum_{k=0}^{\infty}(\frac{(-K(5\pi)^2)}{s^{2\alpha}}(\frac{as^{\alpha}}{K(5\pi)^2}+1))^k$$
$$\times L[H(t)] + \frac{1}{s}$$
$$= \frac{1}{s^{2\alpha}} \times \frac{1}{1+\frac{K(5\pi)^2}{s^{2\alpha}}(\frac{as^{\alpha}}{K(5\pi)^2}+1)}$$
$$\times (2\frac{\Gamma(\gamma_1+1)}{s^{\gamma_1-2\alpha+1}} + 2a\frac{\Gamma(\gamma_1+1)}{s^{\gamma_1-\alpha+1}}$$
$$+5\frac{\Gamma(\gamma_2+1)}{s^{\gamma_2-2\alpha+1}} + 5a\frac{\Gamma(\gamma_2+1)}{s^{\gamma_2-\alpha+1}} + K(5\pi)^2$$
$$\left[2\frac{\Gamma(\gamma_1+1)}{s^{\gamma_1+1}} + 5\frac{\Gamma(\gamma_2+1)}{s^{\gamma_2+1}}\right]) + \frac{1}{s}$$
$$= \frac{2\Gamma(\gamma_1+1)}{s^{\gamma_1+1}} + \frac{5\Gamma(\gamma_2+1)}{s^{\gamma_2+1}} + \frac{1}{s}.$$

From (64) and (65), we get

$$(B_0)_n(t) = \begin{cases} 2t^{\gamma_1}+5t^{\gamma_2}+1 & n=5, \\ 0, & n \neq 5. \end{cases}$$

Therefore, we have

$$\tilde{W}_0(x,t) = (2t^{\gamma_1}+5t^{\gamma_2}+1)\sin(5\pi x).$$

Again by arguments in section 4, we have

$$(\tilde{W})_i(x,t) \equiv 0, i = 1, 2, \ldots$$

Then, the exact solution of the fractional nonlinear telegraph equation given in example 1 is

$$u(x,t) = (2t^{\gamma_1}+5t^{\gamma_2}+1)\sin(5\pi x) + tx + 1.$$

Example 2. Consider the fractional nonlinear telegraph equation

$$D_t^{2\alpha}u(x,t) + aD_t^{\alpha}u(x,t) = K\frac{\partial^2 u(x,t)}{\partial x^2} + u^2(x,t) + f(x,t), \tag{66}$$

with the initial and Neumann boundary conditions

$$u(x,0) = \cos(3\pi x) + x,\ u_t(x,0) = x^2,\ 0 \le x \le 1, \quad u_x(0,x) = 1,\ \ u_x(1,t) = 2t+1, \quad t \ge 0 \tag{67}$$

and

$$f(x,t) = \cos(3\pi x)\left[\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-2\alpha)}t^{\beta-2\alpha} + a\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}t^{\beta-\alpha}\right.$$
$$\left. + K(3\pi)^2(t^{\beta}+1)\right] + \cos(5\pi x)\left[\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-2\alpha)}l^{\gamma-2\alpha} + a\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)}t^{\gamma-\alpha}\right.$$
$$\left. + K(3\pi)^2(t^{\beta})\right] + x^2(\frac{t^{1-2\alpha}}{\Gamma(2-2\alpha)} + \frac{t^{1-\alpha}}{\Gamma(2-\alpha)})$$
$$-\left[(t^{\beta}+1)\cos(3x\pi) + t^{\gamma}\cos(5x\pi) + tx^2 + x\right]^2.$$

By assuming

$$u(x,t) = \tilde{W}(x,t) + \tilde{V}(x,t) = \tilde{W}(x,t) + tx^2 + x,$$

we get

$$D_t^{2\alpha}\tilde{W}(x,t) + aD_t^{\alpha}\tilde{W}(x,t) = K\frac{\partial^2\tilde{W}(x,t)}{\partial x^2} + (\tilde{W}(x,t) + tx^2 + x)^2 + \tilde{f}(x,t). \tag{68}$$

$$\tilde{W}(x,0) = \cos(3\pi x),\ \tilde{W}_t(x,0) = 0\ ,$$
$$\tilde{W}_x(0,t) = 0\ ,\ \tilde{W}_x(1,t) = 0, \quad t \ge 0,$$

where

$$\tilde{f}(x,t) = \cos(3\pi x)\left[\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-2\alpha)}t^{\beta-2\alpha}\right.$$
$$\left. + a\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}t^{\beta-\alpha} + K(3\pi)^2(t^{\beta}+1)\right]$$
$$+\cos(5\pi x)\left[\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-2\alpha)}t^{\gamma-2\alpha} + a\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)}t^{\gamma-\alpha}\right.$$
$$\left. + K(3\pi)^2(t^{\beta})\right]$$
$$-\left[t^{\beta}+1)\cos(3x\pi) + t^{\gamma}\cos(5x\pi)\right.$$
$$\left. + tx^2 + x\right]^2.$$

For solving (68) we use MHPM

$$D_t^{2\alpha}\tilde{W}(x,t) + aD_t^{\alpha}\tilde{W}(x,t) = K\frac{\partial^2\tilde{W}(x,t)}{\partial x^2} + ph(\tilde{W}+\tilde{V}) + \tilde{f}_1(x,t) + p\tilde{f}_2(x,t), \tag{69}$$

By assuming $\tilde{W}(x,t) = \sum_{i=0}^{\infty}\tilde{W}_i(x,t)p^i$ and substituting it in (69), we again obtain corresponding (11)-(13), where

$$g_1(x) = \cos(3\pi x),\ \ g_2(x) = 0$$

and

$$\begin{aligned}\tilde{f}_1(x,t) &= \cos(3\pi x)\left[\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-2\alpha)}t^{\beta-2\alpha}\right.\\ &+\left[\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}t^{\beta-\alpha}+K(3\pi)^2(t^\beta+1)\right]\\ &+\cos(5\pi x)\left[\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-2\alpha)}t^{\gamma-2\alpha}\right.\\ &\left.+a\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)}t^{\gamma-\alpha}+K(3\pi)^2(t^\beta)\right],\\ \tilde{f}_2(x,t) &= -\left[(t^\beta+1)\cos(3x\pi)+t^\gamma\cos(5x\pi)\right.\\ &\left.+tx^2+x\right]^2.\end{aligned}$$

Similar progress in example 1, we first apply separation method for the corresponding homogeneous equation in (11). The eigenvalue and eigenfunction of obtained Sturm-Liouville problem are

$$\lambda_n = n^2\pi^2,\ \ (F_0)_n(x) = \cos(n\pi x),\ \ n = 1,2,\ldots. \tag{70}$$

By assuming

$$\widetilde{W}_0(x,t) = \sum_{n=1}^{\infty}(B_0)_n(t)\cos(n\pi x). \tag{71}$$

and substituting in (11) we get

$$\begin{aligned}&D_t^{2\alpha}(B_0)_n(t)+aD_t^{\alpha}(B_0)_n(t)\\ &+k(n\pi)^2(B_0)_n(t)=(\tilde{f}_1)_n(t),\end{aligned} \tag{72}$$

where

$$\sum_{n=1}^{\infty}(B_0)_n(0)\cos(n\pi x) = \cos(3\pi x), \tag{73}$$

$$\sum_{n=1}^{\infty}\left(\frac{\partial B_0}{\partial t}\right)_n(0)\cos(n\pi x) = 0, \tag{74}$$

which yields

$$(B_0)_n(0)=\frac{2}{1}\int_0^1\cos(3\pi x)\cos(n\pi x)\,dx=\begin{cases}5, & n=3,\\ 0, & n\neq 3,\end{cases}$$

$$(B'_0)_n(0) = 0, \tag{75}$$

where

$$\begin{aligned}(f_1)_n(t) &= 2\int_0^1\tilde{f}_1(x,t)\cos(n\pi x)\,dx\\ &=\begin{cases}H(t), & n=3,\\ G(t), & n=5,\\ 0, & n\neq 3,5,\end{cases}\end{aligned}$$

with

$$\begin{aligned}H(t) &= \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-2\alpha)}t^{\beta-2\alpha}\\ &+a\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}t^{\beta-\alpha}+k(3\pi)^2(t^\beta+1),\\ G(t) &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-2\alpha)}t^{\gamma-2\alpha}+a\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)}t^{\gamma-\alpha}\\ &+k(5\pi)^2(t^\gamma).\end{aligned}$$

By apply lemma 2.1, we have

$$\begin{aligned}(B_0)_n(t) &= \int_0^t \tau^{2\alpha-1}E_{(\alpha,2\alpha),2\alpha}(-a\tau^\alpha,-K(n\pi)^2\tau^{2\alpha})\\ &\times(\tilde{f}_1)_n(t-\tau)d\tau+(B_0)_n(0)\\ &=\int_0^t \tau^{2\alpha-1}E_{(\alpha,\,2\alpha)\,,\,2\alpha}(-a\tau^\alpha,-K(n\pi)^2\tau^{2\alpha}\\ &\times\begin{cases}H(t-\tau), & n=3,\\ G(t-\tau), & n=5,\quad d\tau\\ 0, & n\neq 3,5\end{cases}\\ &+\begin{cases}5, & n=3,\\ 0, & n\neq 3.\end{cases}\end{aligned} \tag{76}$$

Taking the Laplace transform from both sides of (76) we get

$$L[(B_0)_n(t)] = 0,\ \ n\neq 3,5 \tag{77}$$

and

$$L\left[(B_0)_3(t)\right]$$

$$\begin{aligned}&= L\left[\int_0^t\tau^{2\alpha-1}E_{(\alpha,2\alpha}(-a\tau^\alpha,-K(3\pi)^2\tau^{2\alpha})\right.\\ &\left.H(t-\tau)d\tau\right]+L[5]\\ &= L\left[E_{(\alpha,2\alpha),2\alpha}(-at^\alpha,-K(3\pi)^2t^{2\alpha})\right]\\ &\times L[H(t)]+\frac{5}{s}\\ &= L\left[\sum_{k=0}^{\infty}\sum_{l_1=0}^{k}\binom{k}{l_1}\frac{(-at^\alpha)^{l_1}(-K(3\pi)^2t^{2\alpha})^{k-l_1}}{\Gamma(2\alpha+l_1\alpha+(k-l_1)2\alpha)}\right]\\ &\times L[H(t)]+\frac{5}{s}\\ &= \left(\sum_{k=0}^{\infty}\sum_{l_1=0}^{k}\binom{k}{l_1}\frac{(-a)^{l_1}(-K(3\pi)^{2^{k-l_1}}}{s^{2\alpha+l_1\alpha+(k-l_1)2\alpha}}\right)\\ &\times L[H(t)]+\frac{5}{s}\\ &= \left(\sum_{k=0}^{\infty}\frac{(-K(3\pi)^2)^k}{s^{2\alpha+2\alpha k}}\left(\frac{as^\alpha}{K(3\pi)^2}+1\right)^k\right.\\ &\times L[H(t)]+\frac{5}{s}\\ &= \left(\frac{1}{s^{2\alpha}}\sum_{k=0}^{\infty}\left(\frac{(-K(3\pi)^2)}{s^{2\alpha}}\left(\frac{as^\alpha}{K(3\pi)^2}+1\right)\right)^k\right.\\ &\times L[H(t)]+\frac{5}{s}\\ &= \frac{1}{s^{2\alpha}}\times\frac{1}{1+\frac{K(3\pi)^2}{s^{2\alpha}}\left(\frac{as^\alpha}{K(3\pi)^2}+1\right)}\times\left(\frac{\Gamma(\beta+1)}{s^{\beta-2\alpha+1}}\right.\\ &\left.+a\frac{\Gamma(\beta+1)}{s^{\beta-\alpha+1}}+K(3\pi)^2\left[\frac{\Gamma(\beta+1)}{s^{\beta+1}}+\frac{5}{s}\right]\right)+\frac{5}{s}\\ &= \frac{\Gamma(\beta+1)}{s^{\beta+1}}+\frac{5}{s}\end{aligned} \tag{78}$$

From (77) and (78), we get

$$(B_0)_n(t)=\begin{cases} t^{\beta}+5, & n=3,\\ t^{\gamma}, & n=5\\ 0, & n\neq 3,5.\end{cases}$$

Therefore, the solution of (11) given the Neumann boundary conditions is in the form:

$$\widetilde{W}_0(x,t)=(t^{\beta}+5)\cos(3\pi x)+t^{\gamma}\cos(5\pi x).$$

Again with similar progress in example 1, and by some computations, we obtain

$$(\widetilde{W}_0)_i(x,t)=0,\ \ i=1,2,\ldots$$

Then the exact solution of the fractional nonlinear telegraph equation (66) with given conditions in (68) is

$$u(x,t)=(t^{\beta}+5)\cos(3\pi x)+t^{\gamma}\cos(5\pi x)+tx^2+x.$$

6. Conclusions

In this paper, we considered the fractional nonlinear telegraph equation with Dirichlet and Neumann boundary conditions. Analytical solutions were obtained in both cases by using MHPM and separation method. We derived the exact solutions in the closed form for the two problems. We assume is possible to applying the proposed method for other boundary conditions such as Robin boundary condition.

References

Chakraverty, S., & Behera, D. (2013). Dynamic responses of fractionally damped mechanical system using homotopy perturbation method. *Alexandria Engineering Journal*, 52(3), 557–562.

Chalco-Cano, Y., & et al. (2013). Solution set for fractional differential equations with Riemann-Liouville derivative. *Fractional Calculus and Applied Analysis***,** 16(3), 682–694.

Chen, J., et al. (2008). Analytical solution for the time-fractional telegraph equation by the method of separating variables. *Journal of Mathematical Analysis and Applications*, 338(2), 1364–1377.

Chowdhury, M., & et al. (2013). Accurate analytic solution for differential and integral equations by modified homotopy perturbation method. *Middle-East Journal of Scientific Research,* 13, 50–58.

Demir, A., et al. (2013). Analysis of the new homotopy perturbation method for linear and nonlinear problems. *Boundary Value Problems,* 2013(1), 1–11.

Dimovski, I. H. (1990). *Convolutional calculus*, Springer.

Fino, A. Z., & Ibrahim H. (2013). Analytical solution for a generalized space-time fractional telegraph equation. *Mathematical Methods in the Applied Sciences*, 36(14), 1813–1824.

Ford, N. J., et al. (2013). Numerical analysis of a two-parameter fractional telegraph equation. *Journal of Computational and Applied Mathematics*, 249, 95–106.

Garg, M., et al. (2013). Solution of generalized space-time fractional telegraph equation with composite and Riesz-Feller fractional derivatives. *International Journal of Pure and Applied Mathematics*, 83(5), 685–691.

He, J.-H. (1999). Homotopy perturbation technique. *Computer Methods in Applied Mechanics and Engineering*, 178(3), 257–262.

Irandoust-Pakchin, S., et al. (2013). Efficient computational algorithms for solving one class of fractional boundary value problems. *Computational Mathematics and Mathematical Physics,* 53(7): 920–932.

Jiang, W., & Lin, Y. (2011). Representation of exact solution for the time-fractional telegraph equation in the reproducing kernel space. *Communications in Nonlinear Science and Numerical Simulation*, 16(9), 3639–3645.

Luchko, Y., & Gorenflo, R. (1999). An operational method for solving fractional differential equations with the Caputo derivatives. *Acta Mathematica Vietnamica,* 24(2), 207–233.

Orsingher, E., & Beghin, L. (2004). Time-fractional telegraph equations and telegraph processes with Brownian time. *Probability Theory and Related Fields*, 128(1), 141–160.

Orsingher, E., & Zhao, X. (2003). The space-fractional telegraph equation and the related fractional telegraph process. *Chinese Annals of Mathematics*, 24(01), 45–56.

Ouloin, M., et al. (2013). Laplace-Transform Based Inversion Method for Fractional Dispersion. *Transport in Porous Media*, 98(1), 1–14.

Podlubny, I. (1998). *Fractional differential equations: an introduction to fractional derivatives*, fractional differential equations, to methods of their solution and some of their applications, Academic press.

Samko, S. G., et al. (1993). *Fractional integrals and derivatives*. Theory and Applications, Gordon and Breach, Yverdon 1993.

Yang, X.-J., et al. (2013). Cantor-type cylindrical-coordinate method for differential equations with local fractional derivatives. *Physics Letters A*, 377(28), 1696–1700.

Zhao, L., & Deng. W. (2014). Jacobian-predictor-corrector approach for fractional differential equations. *Advances in Computational Mathematics* 40(1), 137–165.