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On 1-Manifolds and 2-Manifolds

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Abstract

In this work, different types of chaotic 1-manifolds which lie on the chaotic spheres or on a torus are introduced. Some types of retractions of the chaotic spheres affect on the 1-chaotic systems, and other types of retractions occur to the geometric manifold but make the 1-chaotic manifold invariant. The existed retractions are discussed through new proved theorems. Also we construct different types of folding of 1-chaotic manifolds which are homeomorphic to $S^1$and their indicatrixes.

Keywords*:* Chaotic; manifolds; folding; retraction; geodesics

1. Introduction

In this section, we present some definitions and background about chaotic manifolds and some geometric transformations.

Definition 1.1. (El-Ghoul, 2001 & Kostelich 1996). The chaotic manifold is a manifold changed by time into homeomorphic manifolds either with fixed points pi, i=1, 2,..., n or with no-fixed points .

Definition 1.2. (Thurston, 1977). An n-dimensional manifold is a Hausdorff topological space such that each point has an open neighborhood homeomorphic to the open n-dimensional disc $Un = \{x \epsilon R^n : |x| < 1\}$.

Definition 1.3. (Docarmo, 1976 & El-Abed, 2007). A geodesic in Riemannian manifold M is a parametrized curve γ such that the velocity vector $(\gamma *)$ is parallel along γ.

Definition 1.4. (El-Ghoul, El-Ahmady & A. El-Abed 2007). Canonical geodesic equations is a non-linear system of differential equations such that $\frac{d}{ds}\left(\frac{\partial T}{\partial \theta'_i}\right) - \left(\frac{\partial T}{\partial \theta'_i}\right) = 0, i = \overline{1, n}$
Where $T = 1/2\overline{ds^2}$and $\overline{ds^2} = \overline{dx_1}^2 + \overline{dx_2}^2 + \ldots + \overline{dx_n^2}$.

Definition 1.5. (El-Kholy, 1981 & Robertson, 1977). For Riemannian manifolds M and N (not necessarily of the same dimension), a map $f: M \to N$ is said to be a topological folding of M into N if, for each piecewise geodesic path $\gamma: I \to M$ $(I = [0,1] \subseteq R)$, the induced path $f \circ \gamma: I \to N$ is piecewise geodesic. If, in addition, $f: M \to N$ preserves lengths of paths, we call f an isometric folding of M into N. Thus an isometric folding is necessarily a topological folding. Many types of foldings are discussed and some applications are introduced in (Difrancesco, 2000 & El-Abed, 2013).

Definition 1.6. (Franchetti, 2003 & Goodykoontz, 1985). A subset A of a topological space X is called a retract of X if there exists a continuous map $r: X \to A$ (called a retraction) such that $r(a) = a$ for any $a \in A$.Many types of retractions are also presented in (Michael, 2002 & Pellicer, 2004).

2. Main results

In what follows we introduce different types of folding of 1-chaotic manifolds which are homeomorphic to $S^1$and their indicatrixes. Consider a system of 1-chaotic manifolds homeomorphic to S^1 with $\tau_i = 0$ and indecatrix representation $t(s) = (x'_i, y'_i; \overline{\mu_1}, \overline{\mu_2} \ldots; \underline{\mu_1}, \underline{\mu_2}, \ldots)$ as in Fig. 1.

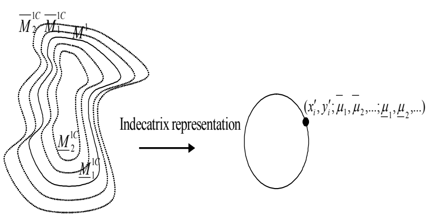

Fig. 1. indicatrix representation of chaotic 1-manifolds such that $x'_i = \cos s$, $y'_i = \sin s, \overline{\mu_1}, \overline{\mu_2}, \ldots$ are the

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physical characters of the outside system, $\underline{\mu_1}, \underline{\mu_2}, \ldots$ are the physical characters of the inside chaotic manifolds of S^1. The indicatrix representation of the geometric one is $(x'_i, y'_i; 0,0,\ldots; 0,0,\ldots)$ The representation of the interior system is $(x'_i, y'_i; 0,0\ldots; \underline{\mu_1}, \underline{\mu_2}, \ldots)$ for the outside system will be $(x'_i, y'_i; \overline{\mu_1}, \overline{\mu_2}\ldots; 0,0,\ldots)$.

(a)Folding without singularity: Let $f_1 \colon M^1 \to M^2$, be a folding of the pure geometric into itself without singularity, then this folding induces two systems of foldings

$\bar{f}_{1i} \colon \bar{M}_i^{1c} \to \bar{M}_i^{2c}, \underline{f_{1i}} \colon \underline{M_i^{1c}} \to \underline{M_i^{2c}}$

such that M^2 is homeomorphic to $\bar{M}_i^{2c}, \underline{M_i^{2c}}$, the indecatrices representation is invariant. See Fig. 2.

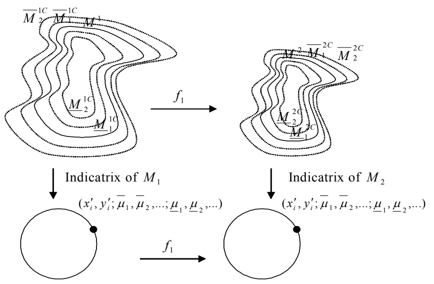

Fig. 2. Nonsingularity folding with invariant

So we obtain the following:

Theorem 2.1. Any folding of 1-chaotic manifold without singularity to itself preserves the indecatrices of the chaotic manifold invariant.

Corollary 2.1. The folding of pure chaotic 1-manifold without singularity to another chaotic changes in its indicatrix.

Proof: Define a folding $\overline{f^*}_{11} \colon \bar{M}_1^{1c} \to \bar{M}_2^{2^*c}$, of pure chaotic 1- manifold $\bar{M}_1^{1c}$ on another pure chaotic $\bar{M}_2^{1c}$such that $\overline{f^*}_{11}(\bar{M}_1^{1c}) = \bar{M}_2^{2^*c}$then its geometric and the other chaotic manifolds not change but the indecatrix representation will be $(x'_i, y'_i; 0; \overline{f^*}_{11}(\overline{\mu_2}), \overline{\mu_3}\ldots; \underline{\mu_1}, \underline{\mu_2}, \ldots)$ instead of $(x'_i, y'_i; \overline{\mu_1}, \overline{\mu_2}\ldots; \underline{\mu_1}, \underline{\mu_2}, \ldots)$. See Fig. 3.

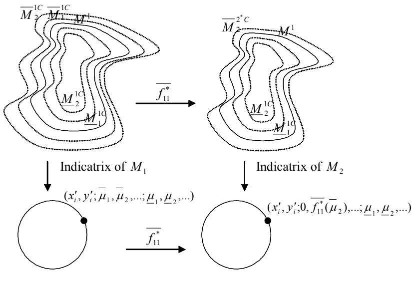

Fig. 3. Nunsingularity folding with variant indecatrix

(b) Folding with singularity: Consider the folding $f_2 \colon M^1 \to M^2, f_2$ restricted on the geometric one, then there are induced systems of foldings

$\bar{f}_{2i} \colon \bar{M}_i^{1c} \to \bar{M}_i^{2c}, \underline{f_{2i}} \colon \underline{M_i^{1c}} \to \underline{M_i^{2c}}$,

which induce a folding of the indicatrix as shown in Fig. 4.

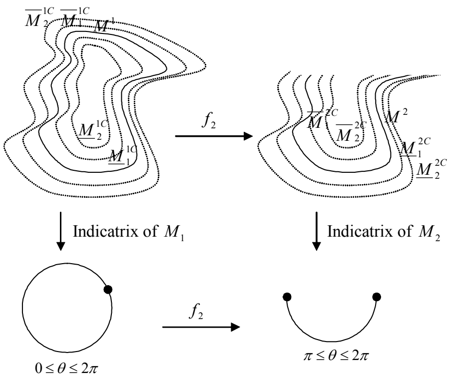

Fig. 4. Singularity folding and indicatrix

Hence we can formulate the following

Theorem 2.2. The folding of 1-chaotic manifold with singularity induces a folding for its indecatrix.

Now we will discuss the effect of retraction on the indecatrix. Let $r_1 \colon \bar{M}_i^{1c} \to \bar{M}_i^{1c}$,,such that $r_1(\bar{\mu}_j) = \frac{\bar{\mu}_j}{n_1}$, then the retraction will reduce the density function. The indecatrix will be in the form $(x'_i, y'_i; \overline{\mu^1}, \ldots \frac{\bar{\mu}_j}{n_1}, \ldots; \underline{\mu_1}, \underline{\mu_2}, \ldots)$. see Fig. 5.

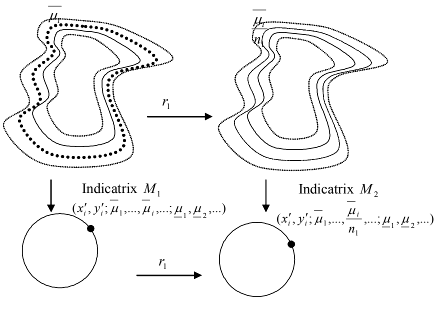

Fig. 5. Retraction of the density function

$if\, r_2 \colon \bar{M}_i^{1c} \to \bar{M}_i^{1c}$, $r_2(\bar{\mu}_j) = \frac{\bar{\mu}_j}{n_2}$, $n_2 > n_1$, similarly if $r_3(\bar{\mu}_j) = \frac{\bar{\mu}_j}{n_3}$,$n_3 > n_2 > n_1$, and so on we get $\lim_{j\to\infty} r_j\left(\frac{\bar{\mu}_j}{n_j}\right) = 0$, in this case the indecatrix is $(x'_i, y'_i; \overline{\mu_1}, \ldots \bar{\mu}_{j-1}, 0, \ldots, \bar{\mu}_{j+1} \ldots; \underline{\mu_1}, \underline{\mu_2}, \ldots)$

So, the limit of all retractions for all physical characters of chaotic 1- manifolds will have the indecatrix $(x'_i, y'_i; 0,0,\ldots,0,0,\ldots)$

Corollary 2.2. Any variation of the curvature of chaotic 1- manifolds does not induce any variation on indecatrix representation curvature.

Proof: Since the indecatrix representation of chaotic 1-manifolds is $(s) = (x'_i, y'_i; \overline{\mu_1}, \overline{\mu_2} \dots ; \underline{\mu_1}, \underline{\mu_2}, \dots)$, then $t'(s) = (x'', y'', 0)$ for the geometric manifold and $\overline{t}'_i(s) = (x'', y'', \bar{\mu}_i)$, $\underline{t}'_i(s) = (x'', y'', \underline{\mu}_i)$ for chaotic systems, which have a unique curvature $k(s) = |t'(s)| = 1$.

In this section we will present some types of chaotic 1- manifolds which lie on the chaotic spheres, some kinds of retractions of the chaotic spheres that affect on the 1-chaotic systems, also other types of retractions that occur onto the geometric manifold but make the 1-chaotic manifold invariant without any change.

Consider a system of chaotic 1- manifolds $\cup\, s_i^{1c}$ lie on the chaotic spheres as in Fig. 6.

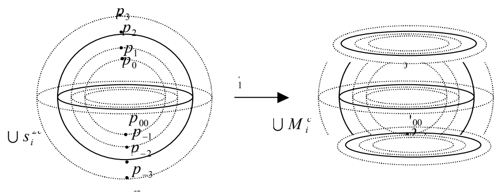

Fig. 6. Retraction of chaotic 1-manifolds lie on chaotic spheres

If $r_1 : \{\cup\, s_i^{2c} - p_i\} \to \cup\, M_i^c$ where $\cup\, s_i^{2c}$ are the chaotic spheres and $\cup\, M_i^c$ are the retracted chaotic manifolds and p_i, p_{-i} are any two antipodal points of the sphere s_i^{2c} then we find $r_{1i}(\{\cup\, s_i^{2c} - p_i\})$,$i = 1,2,\dots$. have two dimension and the 1-chaotic manifolds are still invariant. The limit of this type r_{1i} will be these chaotic 1- manifolds i.e.,

$$\lim_{j\to\infty} r_{1i}(\{\cup\, s_i^{2c} - p_i\}) = \cup\, s_i^{1c}$$

see Fig. 7.

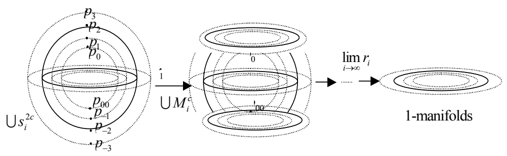

Fig. 7. The limit of retraction r1i

Let $r_2 : \{\cup\, s_i^{2c} - q_i\} \to \cup\, H_i^c$, such that $\cup\, H_i^c$ are 2-chaotic manifolds that intersect with chaotic 1-manifolds $\cup\, s_i^{1c}$, in this case the chaotic manifolds $\cup\, s_i^{1c}$ will be variant under this retraction and, in the end, we obtain $\lim_{j\to\infty} r_{1i}(\{\cup\, s_i^{2c} - p_i\}) = \cup\, d_i^{1c}$ where $\cup\, d_i^{1c}$ are chaotic 1- manifolds, while $\lim_{j\to\infty} r_{1i}(\{\cup\, s_i^{2c} - q_i\}) = 0$- chaotic manifolds. See Fig. 8.

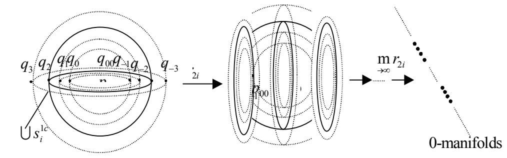

Fig. 8. Limit of retration r2i

Hence we obtain the following.

Theorem 2.3. The limit of retractions of 2-chaotic manifolds $\cup\, s_i^{2c}$ to 1-chaotic manifolds $\cup\, s_i^{1c}$ which lie on them is not followed by a variant in the 1-chaotic manifolds $\cup\, s_i^{1c}$

Theorem 2.4. The limit of retractions of 2-chaotic manifolds, $\cup\, s_i^{2c}$ chaotic manifolds $\cup\, d_i^{1c}$ intersect perpendicular with the 1-chaotic manifolds $\cup\, s_i^{1c}$change $\cup\, s_i^{1c}$ to 0-chaotic manifolds.

Now, we will discuss the effect of the geodesic retraction of the chaotic spheres on the chaotic stereographic projections

Let us consider a sphere s_1^2 of radius unity, with the differential equation $X\dot{X} + Y\dot{Y} + Z\dot{Z} = 0$, then its chaotic equations will be $\bar{X}\,\dot{\bar{X}} + \bar{Y}\dot{\bar{Y}} + \bar{Z}\,\dot{\bar{Z}} = 0$, and $\underline{X\dot{X}} + \underline{Y\dot{Y}} + \underline{Z\dot{Z}} = 0$. see Fig. 9

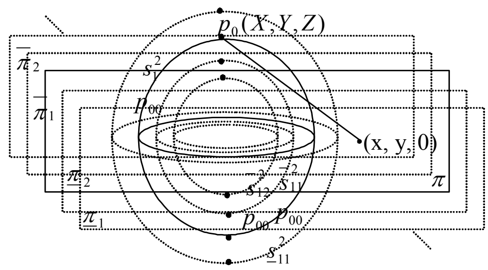

Fig. 9. Stereographic projection

The relation between the set of points on the geometric sphere $s_1^2\ (X, Y, Z)$, and any point on the geometric plane $\pi\ (x, y, 0)$ is

$$X = \frac{2X}{X^2+Y^2+1}, Y = \frac{2Y}{X^2+Y^2+1}, Z = \frac{X^2+Y^2-1}{X^2+Y^2+1}$$

Then there are two induced systems

$$(\bar{X}_i, \bar{Y}_i, \bar{Z}_i) \in \bar{s}_{1i}^2, (\underline{X_i}, \underline{Y_i}, \underline{Z_i}) \in \underline{S_{1i}^2}$$

and two chaotic systems of pure chaotic planes

$(\bar{x}_i, \bar{y}_i, 0) \in \bar{\pi}_i$, $(\underline{x_i}, \underline{y_i}, 0) \in \underline{\pi_i}$

we can obtain chaotic 1-manifolds by the following retraction of $(s_1^2 - \{p_i\})$

$r : (s_1^2 - \{p_i\}) \to s_1^1$ which induce two types of geodesic retractions of the chaotic spheres $\underline{\bar{s}}_{1i}^2$

$\bar{r}_i : (\bar{s}_1^2 - \{\bar{p}_i\}) \to \bar{s}_1^1$ and $\underline{r_i} : (\underline{s_1^2} - \{\underline{p_i}\}) \to \underline{s_1^1}$, which must be followed by stereographic projections retractions

$r^* : \pi \to R^1$, which induce,

$\bar{r}_i^* : \bar{\pi}_i \to \bar{R}^1$, $\underline{r_i}^* : \underline{\pi_i} \to \underline{R}^1$, see Fig. 10.

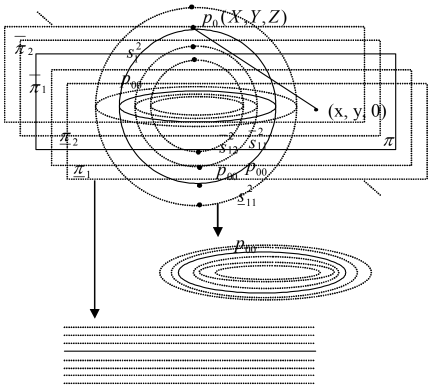

Fig. 10. Geodesic retraction of the chaotic spheres

So we can state the following theorem:

Theorem 2.5. The geodesic retraction of chaotic spheres induce two chaotic retractions of the stereographic projections which reduce the dimension of chaotic stereographic planes $\overline{\pi}_i$

Let the tours $T = {S_1}^1 \times {S_2}^1$ be a 2-manifold which have chaotic 1-manifolds $\cup\, s_3^{1c}$ and let us introduce the following retractions on T,

1. $r_1: T - S_2^1 \rightarrow \cup\, s_3^{1c}$,
2. $r_2: T - S_2^1 \rightarrow s_h^1, where\ s_h^1 \in T$
3. $r_3: T - S_1^1 \rightarrow annulus$
4. $r_4: T - S_1^1 \rightarrow s_b^1\ , such\ that\ s_b^1 \in T,$ as shown in Fig. 11.

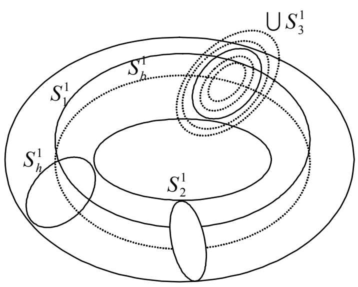

Fig. 11. Retraction preserve the 1-chaotic manifolds

Then we obtain the following theorem:

Theorem 2.6. The retraction $r_1(T - S_2^1)$ of a geometric manifold does not change the 1-chaotic manifolds $\cup\, s_3^{1c}$ while the retraction $r_2(T - S_2^1)$ of a geometric manifold removes the 1-chaotic manifolds.

Theorem 2.7. The retraction $r_3(T - S_1^1)$ of a geometric manifold preserves the dimension of chaotic manifolds $\cup\, s_3^{1c}$ but the retraction $r_4(T - S_1^1)$ induces 0-chaotic manifold.

Proof: $\cup\, s_3^{1c}$ represent 1-chaotic manifolds that lie on the torus $T = {S_1}^1 \times {S_2}^1$ (2-manifold), since $r_1(T - S_2^1) = \cup\, s_3^{1c}$ so this type of retraction gives the same 1-chaotic manifolds $\cup\, s_3^{1c}$while the retracion $r_2(T - S_2^1)$ leads to the circle $s_h^1 \in T$ as shown in Fig. 12. Since $s_h^1 \notin \cup\, s_3^{1c}$then the 1-chaotic manifolds $\cup\, s_3^{1c}$vanish.

In retracion $r_3(T - S_1^1)$of a geometric manifold the torus T becomes annulus of dimension 2 but has different geometric characteristics, so $\cup\, s_3^{1c}$also preserve its dimension. While the retraction of $r_4(T - S_1^1)$ decreases the dimension of the torus T to 1 which is the dimension of circless_b^1 . Then we get a decreasing in dimension of 1-chaotic manifolds $\cup\, s_3^{1c}$ to 0-chaotic manifolds.

3. Application

In this section we introduce an application in the medical field.

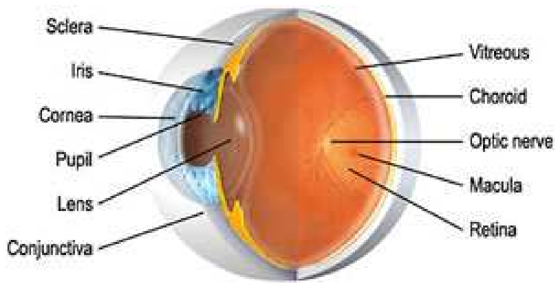

Fig. 12. Eyeball instruction

Let the eyeball represent 2- manifold (geometric) and its 2-chaotic manifolds can be sclera, choroid, retina, lens, cornea, vitreous body and lacrimal secretion carry 1-chaotic manifolds as central retinal artery, vein and nerve.

Small size malignancy behind the eye can affect the eyeball (eye deformity) with its layers without effecting blood vessels and nerves (1- manifold) while

huge size malignancy can affect eyeball layers and also affect blood vessels and nerves.

External trauma can change the size of eyeball leading to rupture of eye, lens dislocation and retinal detachment associated with blood vessels lesions.

Glaucoma increase the size of eye without change in artery, vein and nerve

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