IJST (2014) 38A3 (Special issue-Mathematics): 321-327
Iranian Journal of Science & Technology

http://ijsts.shirazu.ac.ir

Hidden state estimation in the state space model with first-order autoregressive process noise

R. Farnoosh* and A. Hajrajabi

Department of Applied Mathematics, Faculty of Mathematics, Iran University of Science and Technology, Narmak, Tehran 16844, Iran
E-mail: rfarnoosh@iust.ac.ir

Abstract

In this article, the discrete time state space model with first-order autoregressive dependent process noise is considered and the recursive method for filtering, prediction and smoothing of the hidden state from the noisy observation is designed. The explicit solution is obtained for the hidden state estimation problem. Finally, in a simulation study, the performance of the designed method for discrete time state space model with dependent process noise is verified.

Keywords*:* State space model; dependent process noise; estimation of the hidden state; estimation of the error covariance

1. Introduction

Some phenomena with time varying systems can be modeled as the state space models that are very common in engineering and physics applications for example, in modeling GPS and inertial navigation, target tracking, telecommunications, stochastic optimal control and control engineering (Grewal et al., 2001; Bar-Shalom et al., 2001; Kiaipio and Kirubarajan, 2005; Julier and Uhlmann, 2004; Sarkka, 2007; Farnoosh and Nabati, 2013). Some applications of non-linear and non-Gaussian state space modelling in earthquake counts, polio counts, rainfall occurrence data, glacial varve data and daily returns on a share by means of hidden Markov models are presented in (Langrock, 2011). State estimation of a system that changes over time using a sequence of noisy measurements made on the system is the main purpose in these models because the state vector contains all relevant information required to describe the system under investigation. The Kalman filter that is well described in the literature (Kalman, 1960) considers estimation of the dynamic state from noisy measurements in the class of estimation problems in linear Gaussian state space models. Its modified methods like the Extended Kalman filter (EKF), the Unscented Kalman filter (UKF) and Gaussian filter have been applied in non-liner Gaussian state space models (Julier and Uhlmann, 2004; Sarkka, 2007; Wu et al., 2006). Sequential Monte Carlo (SMC) methods, which provide very good approximations to the optimal filter under weak assumptions for non-linear non-Gaussian state space models, have been studied in many recent researches (Kantas et al., 2009; Andrieu et al., 2010; Doucet et al., 2000). Optimal smoothing and numerous approximation methods that are closely related to the optimal filtering, for computing estimates of current and future states of the system have been studied in works of (Godsill et al., 2004; Sarkka, 2008). Many types of state space models based on their applications have been studied in recent years. Structured autoregressive state space models and hidden state filtering based on SMC in these models have been presented in (Prado and Lopes, 2013). State space modeling of linear stochastic time-delay systems and optimal state estimate based on mean-square filtering problem is studied in (Basin et al., 2011). The process and measurement noise in these models are both assumed independent over time but the case of dependent noise processes might be more common in practice (Kailath et al., 2000). Analyzing of these models based on estimation of the hidden state with the particle approach by considering two types of dependence noise between process and measurement is studied in (Saha, 2012). In many applied areas including econometrics and environmental signal processing, independence of process noise is an unrealistic assumption. In this note, dependence of process noise with first-order autoregressive is considered and the optimal filtering, prediction and smoothing

*Corresponding author
Received: 24 March 2013 / Accepted: 20 May 2014

problem for linear systems with dependent process noise over observations is treated to obtain the optimal estimate and the error covariance. As a result, the optimal estimate equations are derived and the performance of the designed optimal estimate for this system with dependent process noise is verified. To the best of knowledge, the related optimal filtering, prediction and smoothing problem for the state space model with dependent process noise has not been investigated.

The outline of this paper is as follows. In Section 2, the discrete state space model with first-order autoregressive as the process noise is described. The filtering and prediction problem of the hidden state in view of the optimal estimate and the error variance are derived in section 3, respectively.

Section 4 represents the smoothing problem of the hidden state and in sections 4.1 and 4.2 the explicit solution for the lag-one and the lag-two covariance is derived. In section 5, Some numerical simulation examples performed by R programming software are conducted to verify the accuracy of the proposed method. Some conclusions are given in section 6.

2. Problem formulation

The discrete time linear state space model considered in this paper is

$$\begin{aligned} x_k &= A_k x_{k-1} + q_k \\ y_k &= H_k x_k + r_k, \end{aligned} \tag{1}$$

where q_k is the process noise, $r_k \sim N(0,R)$ is the measuremet noise with covariance matrix R. The measurement, y_k, is a q-dimensional vector and the state, x_k, is a p-dimensional vector. Time is indexed by k that runs from 0 to n and the matrices A_k, H_k and R are assumed to be known. The dependent process noise is defined as the first- order autoregressive vector, AR(1)

$$q_k = \rho q_{k-1} + u_k, \tag{2}$$

where $u_k \sim N(0,Q)$ is independent and identically distributed (i.i.d) with zero mean and covariance matrix Q. The model (1) can be written as

$$\begin{aligned} x_k &= (A_k+\rho)x_{k-1} - (\rho A_{k-1})x_{k-2} + u_k \\ &= Ax_{k-1} - Bx_{k-2} + u_k \\ y_k &= H_k x_k + r_k, \end{aligned} \tag{3}$$

where $A = (A_k + \rho)$ and $B = (\rho A_{k-1})$. u_k and x_k are independent for each k. Also at time 0 and 1 there are no measurements, only the prior joint distribution. The system state dynamics depend on a delayed state x_{k-1} and x_{k-2} and also u_k and r_k are assumed independent in the model (3).
The distribution of $x = (x'_0, x'_1)'$ is as follows:

$$f(x) = \frac{1}{(2\pi)^{\frac{p}{2}} |\Sigma|^{\frac{1}{2}}} exp(-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)), \tag{4}$$

where $x = (x'_0, x'_1)'$ is a 2p- dimensional vector with the mean $\mu = (\mu'_0, \mu'_1)'$ and Σ is the covariance matrix between x_0 and x_1 with $2p \times 2p$ dimensional

$$\Sigma = \begin{bmatrix} \Sigma_{00} & \Sigma_{01} \\ \Sigma_{10} & \Sigma_{11} \end{bmatrix} \tag{5}$$

The estimation problem is to find the best estimate of the unobserved state x_k, given the measurements $Y_s = \{y_2, \ldots, y_s\}$, to time s by noting that Y_0 and Y_1 are empty. When $s < k$, $s = k$ and $s > k$ the problem is called a prediction, filtering and smoothing, respectively. The following definitions will be used for solving these estimation problems.

$$m_k^s = E(x_k \mid Y_s), \tag{6}$$

$$p_{k_1,k_2}^s = E\{(x_{k_1} - m_{k_1}^s)(x_{k_1} - m_{k_1}^s)'\}. \tag{7}$$

When $k_1 = k_2 = k$, in (7), p_k^s will be written for convenience.

3. Filtering and prediction problem of the hidden state

The stated filtering problem is solved by the followng theorem.

Theorem 1. The optimal filtering equations for the model (3) can be evaluated in closed form with initial conditions $m_0^0 = m_0$, $m_1^1 = m_1$, $p_0^0 = \Sigma_{00}$, $p_1^1 = \Sigma_{11}$ *and* $p_{1,0}^1 = \Sigma_{10}$ *for* $k = 2, \ldots, n$

1. The prediction step for the mean and covariance

is as follows:

$$\begin{cases} m_k^{k-1} = (A_k+\rho)m_{k-1}^{k-1} - (\rho A_{k-1})m_{k-2}^{k-2}, \\ p_k^{k-1} = (A_k+\rho)p_{k-1}^{k-1}(A_k+\rho)' - (A_k+\rho)p_{k-1,k-2}^{k-1}(\rho A_{k-1})' \\ -(\rho A_{k-1})p_{k-2,k-1}^{k-1}(A_k+\rho)' + (\rho A_{k-1})p_{k-2}^{k-2}(\rho A_{k-1})' + Q \end{cases} \tag{8}$$

2. The update step for the mean and covariance is as follows:

$$\begin{cases} \varepsilon_k = y_k - H_k m_k^{k-1}, \\ S_k = H_k p_k^{k-1} H'_k + R, \\ k_k = p_k^{k-1} H'_k S_k^{-1}, \\ m_k^k = m_k^{k-1} + k_k \varepsilon_k, \\ p_k^k = p_k^{k-1} - p_k^{k-1} H'_k S_k^{-1} H_k p_k^{k-1}. \end{cases} \tag{9}$$

Proof (i): The derivations of (8) and (9) follow from straight forward calculations,

$$\begin{aligned} &= Am_{k-1}^{k-1} - Bm_{k-2}^{k-2} \\ &= (A_k+\rho)m_{k-1}^{k-1} - (\rho A_{k-1})m_{k-2}^{k-2}. \end{aligned} \tag{10}$$

Note that the conditional expectation equality $E(x_{k-h} | Y_k) = E(x_{k-h} | Y_{k-h}), h = 1,2$ is valid (Basin et al., 2005; Pugachev and Sinitsyn, 2001). The prediction covariance matrix can be written as

$$\begin{aligned} p_k^{k-1} &= E\{(x_k - m_k^{k-1})(x_k - m_k^{k-1})'\} \\ &= E\{[A(x_{k-1} - m_{k-1}^{k-1}) - B(x_{k-2} - m_{k-2}^{k-2}) + u_k] \\ &[A(x_{k-1} - m_{k-1}^{k-1}) - B(x_{k-2} - m_{k-2}^{k-2}) + u_k]'\} \\ &= Ap_{k-1}^{k-1}A' - Ap_{k-1,k-2}^{k-1}B' - Bp_{k-2,k-1}^{k-1}A' + Bp_{k-2}^{k-2}B' + Q \\ &= (A_k+\rho)p_{k-1}^{k-1}(A_k+\rho)' - (A_k+\rho)p_{k-1,k-2}^{k-1}(\rho A_{k-1})' \\ &-(\rho A_{k-1})p_{k-2,k-1}^{k-1}(A_k+\rho)' + (\rho A_{k-1})p_{k-2}^{k-2}(\rho A_{k-1})' + Q. \end{aligned} \tag{11}$$

Proof (ii): First, the innovations are defined for $k = 2,\dots,n$ as

$$\begin{aligned} \varepsilon_k &= y_k - E(y_k | Y_{k-1}) \\ &= y_k - E(H_k x_k + r_k | Y_{k-1}) = y_k - H_k m_k^{k-1}. \end{aligned}$$
(12)

Not that $E(\varepsilon_k) = 0$ and

$$\begin{aligned} S_k &= var(\varepsilon_k) = var[y_k - H_k m_k^{k-1}] \\ &= var[H_k(x_k - m_k^{k-1}) + r_k] \\ &= H_k p_k^{k-1} H'_k + R. \end{aligned} \tag{13}$$

Furthermore, the conditional covariance between x_k and ε_k given Y_{k-1} is

$$\begin{aligned} cov(x_k, \varepsilon_k | Y_{k-1}) &= cov[x_k, H_k(x_k - m_k^{k-1}) + r_k | Y_{k-1}] \\ &= p_k^{k-1} H'_k. \end{aligned} \tag{14}$$

Using these results, the joint conditional distribution of x_k and ε_k given Y_{k-1} is

$$\begin{bmatrix} x_k \\ \varepsilon_k \end{bmatrix} | Y_{k-1} \sim N(\begin{bmatrix} m_k^{k-1} \\ 0 \end{bmatrix}, \begin{bmatrix} p_k^{k-1} & p_k^{k-1}H'_k \\ H_k p_k^{k-1} & S_k \end{bmatrix}). \tag{15}$$

Using the properties of the conditional expectation of normal distribution, the mean and variance of the filtering distribution can be written as

$$\begin{aligned} m_k^k &= E(x_k | Y_{k-1}, y_k) = E(x_k | Y_{k-1}, \varepsilon_k) \\ &= m_k^{k-1} + k_k \varepsilon_k, \end{aligned} \tag{16}$$

and

$$\begin{aligned} p_k^k &= E\{(x_k - m_k^k)(x_k - m_k^k)'\} = var(x_k | Y_{k-1}, \varepsilon_k) \\ &= p_k^{k-1} - p_k^{k-1} H'_k S_k^{-1} H_k p_k^{k-1}. \end{aligned} \tag{17}$$

4. Smoothing problem of the hidden state

The purpose of the optimal smoothing is to compute the marginal posterior distribution of the state x_k at the time step k after receiving the measurements up to a time step n.

Theorem 2. The optimal smoothing equations for the model (3) can be evaluated in closed form with initial conditions m_n^n and p_n^n *for* $k = n,\dots,2$.
The smoothing step for the mean and covariance is as follows:

$$\begin{cases} J_{k-1} = [p_{k-1}^{k-1}(A_k+\rho)' - p_{k-1,k-2}^{k-1}(\rho A_{k-1})'](p_k^{k-1})^{-1}, \\ m_{k-1}^n = m_{k-1}^{k-1} + J_{k-1}(m_k^n - m_k^{k-1}), \\ p_{k-1}^n = p_{k-1}^{k-1} + J_{k-1}(p_k^n - p_k^{k-1})J'_{k-1}. \end{cases} \tag{18}$$

Proof: The proposed solution to this smoothing problem is based on (Shumway, 2006). First we define $\tau_k = \{r_k,\dots,r_n,u_{k+1},\dots,u_n\}$ and $\gamma_{k-1} = E(x_{k-1} | Y_{k-1}, x_k - m_k^{k-1}, \tau_k)$ for $k = 2,\dots,n$. Because Y_{k-1}, $x_k - m_k^{k-1}$ and τ_k are mutually independent and generate Y_n, and also x_{k-1} and τ_k are independent, γ_{k-1} can be written as

$$\gamma_{k-1} = m_{k-1}^{k-1} + J_{k-1}(x_k - m_k^{k-1}), \tag{19}$$

where

$$J_{k-1} = cov(x_{k-1}, x_k - m_k^{k-1})(p_k^{k-1})^{-1}$$
$$= cov(x_{k-1}, A(x_{k-1} - m_{k-1}^{k-1}) - B(x_{k-2} - m_{k-2}^{k-2}) + u_k)(p_k^{k-1})^{-1}$$
$$= [p_{k-1}^{k-1}A' - p_{k-1,k-2}^{k-1}B'](p_k^{k-1})^{-1}$$
$$= [p_{k-1}^{k-1}(A_k + \rho)' - p_{k-1,k-2}^{k-1}(\rho A_{k-1})'](p_k^{k-1})^{-1}. \tag{20}$$

Because Y_{k-1}, $x_k - m_k^{k-1}$ and τ_k generate Y_n, the equation (18) is obtained

$$m_{k-1}^n = E(\gamma_{k-1} \mid Y_n)$$
$$= m_{k-1}^{k-1} + J_{k-1}(m_k^n - m_k^{k-1}). \tag{21}$$

The smoothing covariance, p_{k-1}^n, is obtained by straight-forward calculations. By using the smoothing mean in the equation (18),

$$x_{k-1} - m_{k-1}^n = (x_{k-1} - m_{k-1}^{k-1}) - J_{k-1}(m_k - m_k^{k-1}), \tag{22}$$

Multiplying each side of (22) by the transpose of itself and making an assumption, smoothing covariance is obtained.

$$p_{k-1}^n = p_{k-1}^{k-1} + J_{k-1}E\{(m_k^n - m_k^{k-1})(m_k^n - m_k^{k-1})'\}J'_{k-1}$$
$$= p_{k-1}^{k-1} + J_{k-1}(p_k^n - p_k^{k-1})J'_{k-1}, \tag{23}$$

since the cross-product terms are zero and

$$E(m_k^{k-1}m'^{k-1}_k) = E(x_k x'_k) - p_k^{k-1}. \tag{24}$$

$$E(m_k^n m'^n_k) = E(x_k x'_k) - p_k^n. \tag{25}$$

4.1. The lag-one covariance smoother

For the state space model specified in the model (3), with k_k, J_k, ($k = 2,\ldots,n$), obtained from Theorem (1) and (2), the lag-one covariance can be written for $k = n, n-1,\ldots,3$,

$$p_{k-1,k-2}^n = p_{k-1,k-2}^{k-1} - J_{k-1}((A_k + \rho)p_{k-1}^{k-1}$$
$$-(\rho A_{k-1})p_{k-2,k-1}^{k-1} - p_{k,k-1}^n)J'_{k-2}, \tag{26}$$

by considering the initial condition

$$p_{n,n-1}^n = (I - k_n H_n)((A_k + \rho)p_{n-1}^{n-1}$$
$$-(\rho A_{k-1})p_{n-2,n-1}^{n-1}). \tag{27}$$

Proof: We first calculate the lag-one covariance $p_{k,k-1}^k$ using the Theorem (1) and (2)

$$p_{k,k-1}^k = E\{(x_k - m_k^k)(x_{k-1} - m_{k-1}^k)'\}$$
$$= E\{[(x_k - m_k^{k-1}) - k_k(y_k - H_k m_k^{k-1})]$$
$$[(x_{k-1} - m_{k-1}^{k-1}) - J_{k-1}(k_k(y_k - H_k m_k^{k-1}))]'\}$$
$$= E\{[(x_k - m_k^{k-1}) - k_k(H_k(x_k - m_k^{k-1}) + r_k)]$$
$$[(x_{k-1} - m_{k-1}^{k-1}) - J_{k-1}(k_k(H_k(x_k - m_k^{k-1}) + r_k))]'\}, \tag{28}$$

expanding terms and taking expectation, we can write

$$p_{k,k-1}^k = p_{k,k-1}^{k-1} - p_k^{k-1}H'_k k'_k J'_{k-1} - k_k H_k p_{k,k-1}^{k-1}$$
$$+ k_k(H_k p_k^{k-1}H'_k + R)k'_k J'_{k-1}. \tag{29}$$

using the following equations

$$p_{k,k-1}^{k-1} = E\{(x_k - m_k^{k-1})(x_{k-1} - m_{k-1}^{k-1})'\}$$
$$= E\{[A(x_{k-1} - m_{k-1}^{k-1}) - B(x_{k-2} - m_{k-2}^{k-2}) + u_k][x_{k-1} - m_{k-1}^{k-1}]'\}$$
$$= Ap_{k-1}^{k-1} - Bp_{k-2,k-1}^{k-1}, \tag{30}$$

and

$$p_k^{k-1}H'_k = k_k(H_k p_k^{k-1}H'_k + R), \tag{31}$$

the equation (29) is obtained as follows:

$$p_{k,k-1}^k = (I - k_k H_k)p_{k,k-1}^{k-1}$$
$$= (I - k_k H_k)(Ap_{k-1}^{k-1} - Bp_{k-2,k-1}^{k-1})$$
$$= (I - k_k H_k)((A_k + \rho)p_{k-1}^{k-1} - (\rho A_{k-1})p_{k-2,k-1}^{k-1}). \tag{32}$$

By substituting $k = n$ in the equation (32), the equation (27) is obtained.

The basic step in the derivation of (26) is to use the Theorem (2)

$$x_{k-1} - m_{k-1}^n = (x_{k-1} - m_{k-1}^{k-1}) + J_{k-1}(m_k^n - m_k^{k-1}), \tag{33}$$

and

$$x_{k-2} - m_{k-2}^n = (x_{k-2} - m_{k-2}^{k-2}) + J_{k-2}(m_{k-1}^n - m_k^{k-2}), \tag{34}$$

Next, multiply the left-hand side of (33) by the transpose of the left-hand side of (34), and equate that to the corresponding result of the right-hand sides of (33) and (34). Then, making an assumption of both sides, the result reduces to the equation (26).

$$p^{n}_{k-1,k-2} = p^{k-1}_{k-1,k-2} - J_{k-1}E[(m^{n}_{k} - m^{k-1}_{k})(m^{n}_{k-1} - m^{k-2}_{k-1})']J'_{k-2}$$
$$= p^{k-1}_{k-1,k-2} - J_{k-1}(p^{k-1}_{k,k-1} - p^{n}_{k,k-1})J'_{k-2}$$
$$= p^{k-1}_{k-1,k-2} - J_{k-1}(Ap^{k-1}_{k-1} - Bp^{k-1}_{k-2,k-1} - p^{n}_{k,k-1})J'_{k-2}$$
$$= p^{k-1}_{k-1,k-2} - J_{k-1}((A_{k} + \rho)p^{k-1}_{k-1}$$
$$-(\rho A_{k-1})p^{k-1}_{k-2,k-1} - p^{n}_{k,k-1})J'_{k-2}. \quad (35)$$

4.2. The lag-two covariance smoother

For the state-space model specified in the model (3), with k_k, J_k, $k = 2,\dots,n$, obtained from Theorem (1) and (2), the lag-two covariance can be written for $k = n, n-1,\dots,3$,

$$p^{n}_{k-1,k-3} = p^{k-1}_{k-1,k-3} - J_{k-1}(p^{k}_{k,k-2} - p^{n}_{k,k-2})J'_{k-3}, \quad (36)$$

with initial condition

$$p^{n}_{n,n-2} = (I - k_n H_n)((A_n + \rho)p^{n-1}_{n-1,n-2} - (\rho A_{n-1})p^{n-2}_{n-2})$$
$$+(k_n H_n - I)((A_n + \rho)p^{n-2}_{n-1}$$
$$-(\rho A_{n-1})p^{n-2}_{n-2,n-1})H'_{n-1}k'_{n-1}J'_{n-2}. \quad (37)$$

Proof: Similar to the process of proving the lag-one covariance smoother, we first calculate the lag-two covariance $p^{k}_{k,k-2}$ using the Theorem (1) and (2)

$$p^{k}_{k,k-2} = E\{[x_k - m^{k}_{k}][x_{k-2} - m^{k}_{k-2}]'\}$$
$$= E\{[(x_k - m^{k-1}_{k}) - k_k(y_k - H_k m^{k-1}_{k})]$$
$$[(x_{k-2} - m^{k-2}_{k-2}) - J_{k-2}(m^{k}_{k-1} - m^{k-2}_{k-1})]'\}$$
$$= E\{[(x_k - m^{k-1}_{k}) - k_k(H_k(x_k - m^{k-1}_{k}) + r_k)]$$
$$[(x_{k-2} - m^{k-2}_{k-2}) - J_{k-2}(k_{k-1}(H_{k-1}(x_{k-1} - m^{k-2}_{k-1}) + r_{k-1}))]'\}$$
$$= (I - k_k H_k)(Ap^{k-1}_{k-1,k-2} - Bp^{k-2}_{k-2})$$
$$+ (k_k H_k - I)(Ap^{k-2}_{k-1} - Bp^{k-2}_{k-2,k-1})H'_{k-1}k'_{k-1}J'_{k-2}. \quad (38)$$

By substituting $k = n$ in the equation (38), the equation (37) is obtained. The basic step in the derivation of (36) is to use the mean of smoothing in the Theorem (2)

$$x_{k-1} - m^{n}_{k-1} = (x_{k-1} - m^{k-1}_{k-1}) + J_{k-1}(m^{n}_{k} - m^{k-1}_{k}), \quad (39)$$

and

$$x_{k-3} - m^{n}_{k-3} = (x_{k-3} - m^{k-3}_{k-3}) + J_{k-3}(m^{n}_{k-2} - m^{k-3}_{k-2}), \quad (40)$$

Next, multiply the left-hand side of (39) by the transpose of the left-hand side of (40), and equate that to the corresponding result of the right-hand sides of (39) and (40). Then, taking expectation of both sides, the result reduces to the equation (36).

$$p^{n}_{k-1,k-2} = p^{k-1}_{k-1,k-3} - J_{k-1}E[(m^{n}_{k} - m^{k-1}_{k})(m^{n}_{k-2} - m^{k-3}_{k-2})']J'_{k-3}$$
$$= p^{k-1}_{k-1,k-3} - J_{k-1}(p^{k}_{k,k-2} - p^{n}_{k,k-2})J'_{k-3}. \quad (41)$$

5. Simulation study

This section presents an example of designing the optimal estimate of the hidden state for a state space model (3) with dependent process noise . For this purpose, the simulation is carried out for this model with $A_k = A_{k-1} = 1$, $\rho = 0.5$, the measurement noise variance $R = 1.2$ and the process noise variance $Q = 1.5$. The following initial values are assigned: $m_0 = 0$, $m_1 = 1$, $\Sigma_0 = 0.7$, $\Sigma_1 = 1$ and $\Sigma_{10} = 0.2$. The simulated hidden state of the system, x_k, and the observations, y_k, are shown in Fig. 1.

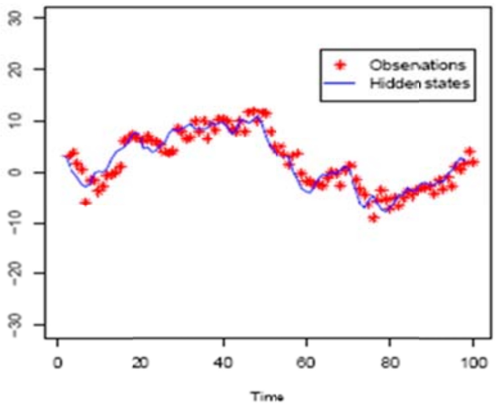

Fig. 1. The simulated state x_t (solid line) and the observations y_t (star)

The purpose is to estimate the hidden state using the optimal estimation method based on a discrete state space form of the model. $n = 10$ values of the corrupted states y_k, the states x_k, the mean and variance of the prediction m^{k-1}_{k}, p^{k-1}_{k}, the filtering m^{k}_{k}, p^{k}_{k} and the smoothing m^{n}_{k}, p^{n}_{k} distribution for estimation of the state in the model are obtained and are illustrated in Table 1. As we see, $p^{k-1}_{k} > p^{k}_{k} > p^{n}_{k}$.

In Fig. 2, the mean value of the prediction, the filtering and the smoothing distribution for $k = 1,\dots, n = 10$ are shown as lines and their confidence intervals which are computed by

$m_k^{k-1} \pm 1.96\sqrt{p_k^{k-1}}$, $m_k^k \pm 1.96\sqrt{p_k^k}$ and $m_k^n \pm 1.96\sqrt{p_k^n}$ as dashed lines. The simulation results obtained from Table 1 and Fig. 2 show that the prediction of the current is more uncertain than the corresponding filtered value, which in turn, is more uncertain than the corresponding smoother value. Consequently, the confidence interval for smoother is shorter than the filter and the filter is shorter than the predicted. Figure 3 shows the mean values with confidence interval of the prediction, the filtering and the smoothing of the hidden state, $x_k = (x_{1k}, x_{2k})', k = 1,\ldots,25$, that is simulated from the model with $A_k = A_{k-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\rho = \begin{pmatrix} 0.7 & 0.2 \\ 0.2 & 0.7 \end{pmatrix}$, the measurement noise variance $R = \begin{pmatrix} .9 & 0 \\ 0 & 0.9 \end{pmatrix}$ and the process noise variance $Q = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. The following initial values are assigned: $m_0 = (0.5, 0.75)'$, $m_1 = (2, 1.1)'$, $\Sigma_0 = \begin{pmatrix} .1 & 0 \\ 0 & 1 \end{pmatrix}$, $\Sigma_1 = \begin{pmatrix} .002 & 0 \\ 0 & 0.2 \end{pmatrix}$ and $\Sigma_{10} = \begin{pmatrix} 0.01 & 0.01 \\ 0.01 & 0.01 \end{pmatrix}$. The simulation results show a definite advantage of the designed optimal estimate with regard to proximity of the estimate to the real state value.

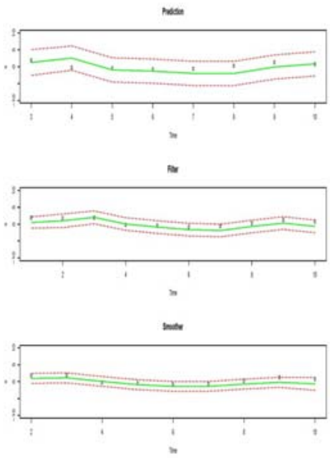

Fig. 2. The simulated values of x_k, for $k = 1,..,10$ are shown as points. In top, middle and bottom panel, the mean of prediction m_k^{k-1}, the mean of filter m_k^k and the mean of smoother m_k^n with their confidence intervals are shown as a line and dashed lines, respectively

6. Conclusion

The discrete time state space model is considered and an optimal filtering, prediction and smoothing method is proposed to estimate the hidden state from the noisy observation. In the presented model, dependent process noise is defined as first-order autoregressive AR(1). The closed form is obtained for the optimal estimate and the error variance. The simulation results show that the values of the estimated calculated close to the real values of the simulated hidden state. Furthermore, the confidence intervals are found for trajectories of the solution. We observed that the smoothers, the filters and the predictions for the state having the best confidence intervals, respectively.

Table 1. Predictions, filters and smoothers

t	y_k	x_k	m_k^{k-1}	p_k^{k-1}	m_k^k	p_k^k	m_k^n	p_k^n
0	-	1.915	-	-	0.500	0.700	-	-
1	-	1.819	-	-	1.0008	1.000	0.954	0.540
2	2.283	1.890	1.250	3.625	2.026	0.901	1.131	0.544
3	-0.870	-0.238	2.539	3.256	0.047	0.876	0.065	0.529
4	-0.830	-0.474	-0.941	3.222	-0.860	0.874	-0.893	0.527
5	-1.721	-0.827	-1.315	3.215	-1.610	0.873	-1.405	0.527
6	-1.829	-0.632	-1.985	3.214	-1.871	0.873	-1.413	0.527
7	-0.193	0.253	-2.000	3.213	-0.685	0.873	-0.726	0.527
8	0.488	1.271	-0.092	3.213	0.330	0.873	-0.206	0.527
9	-1.216	0.723	0.838	3.213	-0.657	0.873	-0.657	0.527

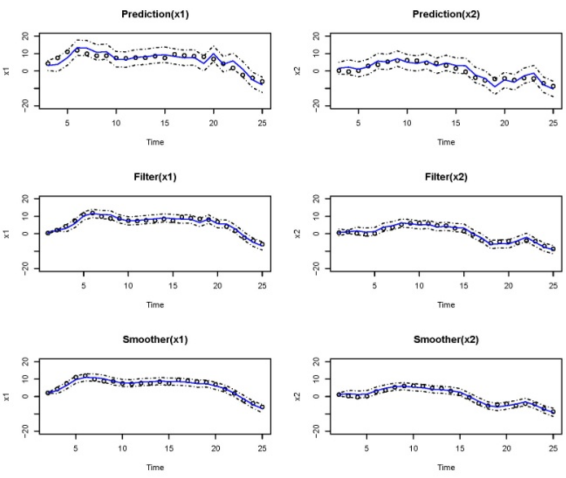

Fig. 3. The simulated values of $x_k = (x_{1k}, x_{2k})'$, for $k = 1,..,25$ are shown as points. In top, middle and bottom panel, the mean of prediction $m_k^{k-1} = (m_{1k}^{k-1}, m_{2k}^{k-1})'$, the mean of filter $m_k^k = (m_{1k}^k, m_{2k}^k)'$ and the mean of smoother $m_k^n = (m_{1k}^n, m_{2k}^n)'$ with their confidence intervals are shown as a line and dashed lines, respectively.

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