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"Research Note"

ON R-QUADRATIC FINSLER METRICS*

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Abstract – We prove that every R-quadratic metric of scalar flag curvature with a dimension greater than two is of constant flag curvature. Then we show that generalized Douglas-Weyl metrics contain R-quadratic metrics as a special case, but the class of R-quadratic metric is not closed under projective transformations.

Keywords – R-quadratic metric, Landsberg metric, generalized Douglas-Weyl metric

1. INTRODUCTION

In this paper, we prove that every R-quadratic Finsler metric of scalar flag curvature with a dimension greater than two is of constant flag curvature. Given a manifold M, the class of generalized Douglas-Weyl metrics is denoted by $GDW(M)$.

It is well-known that this class of Finsler metrics is closed under projective transformation [1]. More precisely, let F be projectively related to a Finsler metric in $GDW(M)$, then $F \in GDW(M)$. Here, we show that GDW-metrics contain R-quadratic metrics. Then, we give an example indicating the class of R-quadratic metric is not closed under projective transformations. Finally, we study the problem of reducing R-quadratic Finsler metrics to Landsberg and Berwald metrics. Here, we prove that every R-quadratic metric with *constant* isotropic Landsberg curvature (resp. constant isotropic Berwald curvature) is a Landsberg metric (resp. Berwald metric).

Throughout this paper we make use of *Einstein* convention. We also set the *Berwald connection* on Finsler manifolds. The h - and v - covariant derivatives of a Finsler tensor field are denoted by "$|$" and " , " respectively.

2. PRELIMINARIES

Let M be an n-dimensional C^∞ manifold. Denote T_xM, the tangent space at $x \in M$, and by $TM = \bigcup_{x\in M} T_xM$ the tangent bundle of M. Each element of TM has the form (x,y), where $x \in M$ and $y \in T_xM$. The natural projection $\pi : TM \to M$ is given by $\pi(x,y) = x$. Let $TM_0 = TM \setminus \{0\}$. The *pull-back tangent bundle* π^*TM is a vector bundle over TM_0 whose fiber π^*_vTM at $v \in TM_0$ is just T_xM, where $\pi(v) = x$. Then $\pi^*TM = \{(x,y,v) \mid \ y \in T_xM_0, \ v \in T_xM\}$.

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A *Finsler metric* on a manifold M is a function $F:TM \to [0,\infty)$ having the following properties: (i) F is C^∞ on TM_0; (ii) $F(x,\lambda y)=\lambda F(x,y)$, $\lambda>0$; (iii) the Hessian of F^2 with elements $2g_{ij}(x,y)=[F^2]_{y^i y^j}$ is positively defined on TM_0.

Let $\partial_i = \frac{\partial}{\partial x^i}$, $\partial^\bullet_i = \frac{\partial}{\partial y^i}$. The *Cartan* tensor **C** is defined by $C(U,V,W)=C_{ijk}(y)U^iV^jW^k$, where $U=U^i\partial_i$, $V=V^i\partial_i$, $W=W^i\partial_i$, and $4C_{ijk}=[F^2]_{y^iy^jy^k}(y)$. The tensor **L** on π^*TM is defined by $L(U,V,W)=L_{ijk}(y)U^iV^jW^k$, where $L_{ijk}=C_{ijk|s}y^s$. We call **L** the *Landsberg tensor*. A Finsler metric is called a *Landsberg metric* if **L**=0. A Finsler metric *F* is said to be *isotropic Landsberg metric* if L+cFC=0 for some scalar function *c* on M. For more details see [2].

Given a Finsler manifold (M, F), then a global vector field G is induced by F on TM_0, which, in a standard coordinate, (x^i,y^i) for TM_0 is given by $G=y^i\partial_i - 2G^i(x,y)\partial^\bullet_i$, where $G^i(x,\lambda y)=\lambda^2 G^i(x,y)$ $\lambda>0$. G is called the associated *spray* to F. The projection of an integral curve of G is called a *geodesic* in M.

Set $B^i{}_{jkl} := (G^i(y))_{y^j y^k y^l}$. For $y\in T_xM_0$, define $B_y: T_xM\otimes T_xM\otimes T_xM \to T_xM$ by $B_y(u,v,w):=B^i{}_{jkl}(y)u^jv^kw^l\ \partial_i|_x$, and $B_y(u,v,w)$ is symmetric in u, v, w. *B* is called the *Berwald curvature*. A Finsler metric with vanishing Berwald curvature is said to be *Berwald metric*. *F* is said to be *isotropic Berwald metric* if its Berwald curvature satisfies the following $B^i{}_{jkl}=c(x)\{F_{y^jy^k}\delta^i{}_l + F_{y^ky^l}\delta^i{}_j + F_{y^ly^j}\delta^i{}_k + F_{y^jy^ky^l}y^i\}$, where *c* is scalar function on M [3].

Let $2E_{jk}(y):=B^m{}_{jkm}(y)$. This set of local functions give rise to a tensor on TM_0. Define $E_y: T_xM\otimes T_xM \to R$ by $E_y(u,v):=E_{jk}(y)u^jv^k$, and $E_y(u,v)$ is symmetric in u and v. *E* is called the *mean Berwald curvature* [4].

Theorem 1. [5] Let (M, F) be an n-dimensional (n>2) Finsler manifold of scalar curvature. Then *F* is of constant flag curvature if and only if $E_{ij|s}y^s=0$.

J. Douglas introduced a new quantity $D_y: T_xM\otimes T_xM\otimes T_xM \to T_xM$, which is trilinear form $D_y(u,v,w):=D^i{}_{jkl}(y)u^jv^kw^l\ \partial_i|_x$, defined by

$$D^i{}_{jkl} := B^i{}_{jkl} - \frac{2}{n+1}\{E_{jk}\delta^i{}_l + E_{jl}\delta^i{}_k + E_{kl}\delta^i{}_j + E_{jk,l}y^i\}.$$

We call $D:=\{D_y\}_{y\in TM_0}$ the *Douglas curvature*. A Finsler metric with vanishing Douglas curvature is said to be *Douglas metric*. A Finsler metric *F* is said to be a *generalized Douglas-Weyl metric* or briefly *GDW*-metric if its Douglas curvature satisfies the following $h^i{}_\alpha D^\alpha{}_{jkl|m}y^m=0$ [6].

Riemann curvature $R_y: T_xM \to T_xM$ is defined by $R_y(u):=R^i{}_k(y)u^k\ \partial_i$, where

$$R^i{}_k(y)=2(G^i)_{x^k} - (G^i)_{x^jy^k}y^j + 2G^j(G^i)_{y^jy^k} - (G^i)_{y^j}(G^j)_{y^k}.$$

The family $R:=\{R_y\}_{y\in TM_0}$ is called the Riemann curvature [4]. A Finsler metric *F* is said to be *R-quadratic* if R_y is quadratic in $y\in T_xM$ at each point $x\in M$ [4].

Theorem 2. [4] Every compact R-quadratic Finsler metric is a Landsberg metric.

Theorem 3. Let (M, F) be an n-dimensional (n>2) R-quadratic Finsler manifold. Suppose that *F* is of scalar flag curvature. Then F is of constant flag curvature.

Proof: The curvature form of the Berwald connection is:

$$\Omega^i{}_j = d\,\omega^i{}_j - \omega^k{}_j \wedge \omega^i{}_k = \frac{1}{2} R^i{}_{jkl}\,\omega^k \wedge \omega^l - B^i{}_{jkl}\,\omega^k \wedge \omega^{n+l}. \tag{1}$$

For the Berwald connection, we have the following structure equation:

$$dg_{ij} - g_{jk}\Omega^k{}_i - g_{ik}\Omega^k{}_j = -2L_{ijk}\,\omega^k + 2C_{ijk}\,\omega^{n+k}. \tag{2}$$

Differentiating (2) yields the following Ricci identity:

$$g_{pj}\,\omega^p{}_i - g_{pi}\,\omega^p{}_j = -2L_{ijk|l}\,\omega^k \wedge \omega^l - 2L_{ijk,l}\,\omega^k \wedge \omega^{n+l} - 2C_{ijl|k}\,\omega^k \wedge \omega^{n+l}$$
$$-2C_{ijl,k}\,\omega^{n+k} \wedge \omega^{n+l} - 2C_{ijp}\Omega^p{}_l\, y^l. \tag{3}$$

It follows from (3) that:

$$2C_{ijl|k} + 2L_{ijk,l} = g_{pj}\, B^p{}_{ikl} + g_{ip}\, B^p{}_{jkl}. \tag{4}$$

Differentiating of (1) yields:

$$d\,\Omega^j{}_i - \omega^k{}_i \wedge \Omega^j{}_k + \omega^j{}_k \wedge \Omega^k{}_i = 0 \tag{5}$$

Define $B^i{}_{jkl|m}$, $B^i{}_{jkl,m}$, $R^i{}_{jkl|m}$ and $R^i{}_{jkl,m}$ by:

$$dB^i{}_{jkl} - B^i{}_{mkl}\,\omega^m{}_i - B^i{}_{jml}\,\omega^m{}_k - B^i{}_{jkm}\,\omega^m{}_l - B^i{}_{jkl}\,\omega^i{}_m := B^i{}_{jkl|m}\,\omega^m + B^i{}_{jkl,m}\,\omega^{n+m}. \tag{6}$$

$$dR^i{}_{jkl} - R^i{}_{mkl}\,\omega^m{}_i - R^i{}_{jml}\,\omega^m{}_k - R^i{}_{jkm}\,\omega^m{}_l - R^i{}_{jkl}\,\omega^i{}_m := R^i{}_{jkl|m}\,\omega^m + R^i{}_{jkl,m}\,\omega^{n+m}. \tag{7}$$

From (5), (6) and (7) one obtains the following Bianchi identity:

$$R^i{}_{jkl|m} + R^i{}_{jlm|k} + R^i{}_{jmk|l} = 0, \tag{8}$$

$$B^i{}_{jkl|m} - B^i{}_{jmk|l} = R^i{}_{jml,k}, \tag{9}$$

From (9) we get $E_{jk|m}\, y^m = 0$. By Theorem 1, F is of constant flag curvature.

Proposition 1. Every R-quadratic Finsler metric of non-zero scalar flag curvature $K(x)$, depending on position alone, is Riemannian.

Proof: F is of scalar flag curvature $K(x)$, $R^i{}_k(x,y) = K(x)(F^2\delta^i{}_k - y_k y^i)$ and R-quadratic $R^i{}_k(x,y) = R^i{}_{jkl}(x)y^j y^l$. Then $(n-1)K(x)F^2 = R^m{}_{jml}(x)y^j y^l$. This means that F^2 is quadratic. Then F is Riemannian.

Theorem 4. Every R-quadratic Finsler metric is a GDW metric.

Proof:

$$D^i{}_{jkl} := B^i{}_{jkl} - \frac{2}{n+1}\{E_{jk}\delta^i{}_l + E_{jl}\delta^i{}_k + E_{kl}\delta^i{}_j + E_{jk,l}\, y^i\}. \tag{10}$$

Then

$$D^i{}_{jkl|m}\, y^m = B^i{}_{jkl|m}\, y^m - \frac{2}{n+1}\{E_{jk|m}\, y^m \delta^i{}_l + E_{jl|m}\, y^m \delta^i{}_k + E_{kl|m}\, y^m \delta^i{}_j + E_{jk,l|m}\, y^m y^i\}. \tag{11}$$

It follows from (9) that $E_{jk|m}\,y^m = R^p{}_{jmp,k}\,y^m$. We obtain:

$$D^\alpha{}_{jkl|m}y^m := R^\alpha{}_{jml,k}y^m - \frac{2}{n+1}\{R^p{}_{jmp,k}y^m\delta^\alpha{}_l + R^p{}_{lmp,j}y^m\delta^\alpha{}_k + R^p{}_{kmp,l}y^m\delta^\alpha{}_j\}\,. \tag{12}$$

F is R-quadratic, then $D^\alpha{}_{jkl|m}y^m = 0$. Then F is a GDW-metric.
The following example shows that the class of R-quadratic metrics is a proper subclass of the class of GDW-metrics on a manifold.

Example 1. [4] Let $X = (x, y, z) \in B^3(1) \subset R^3$ and $Y = (u, v, w) \in T_x B^3(1)$. Let **A:=** $(x^2+y^2+z^2)u - 2x(xu+yv+zw)$, **B:=** $1-(x^2+y^2+z^2)^2$, **C:=** $(u^2+v^2+w^2)$.
Define $F = F(x, y)$ by $F := \alpha + \beta = (\sqrt{A^2+BC} + A)B^{(-1)}$. The flag curvature of F is given by $K = -3F^{-1}u + x^2 - 2y^2 - 2z^2$. F is of scalar flag curvature, therefore F is a GDW- metric on $B^3(1)$. But F is not R-quadratic metric by using Theorem 3.

The class of GDW-metrics is closed under projective change. The following example shows that the class of R-quadratic metrics is not closed under projective change.

Example 2. Let $F := |y| + (\sqrt{1+|x|^2})^{-1} < x, y >$, $y \in T_x R^n = R^n$ where $|.|$ and $< , >$ denote the Euclidian norm and inner product on R^n respectively. F is of scalar flag curvature. Then by Theorem 3, this Randers metric is not R-quadratic, however it is projectively related to an R-quadratic metric, i.e., Euclidean metric [4].

3. REDUCTION TO LANDSBERG AND BERWALD METRICS

In this section, we give some conditions that, under them a R-quadratic metric reduces to a Landsberg or Berwald metric.

Lemma 1. Let (M, F) be a Finsler manifold. If F is R-quadratic metric, then the Landsberg curvature satisfies the following equation $L_{ijk|s}y^s = 0$.

Theorem 5. Let (M, F) be a constant isotropic Landsberg manifold. Suppose that F is an R-quadratic Finsler metric. Then F is a Landsberg metric.

Proof: Since F is constant isotropic Landsberg, $L_{jkl} + cFC_{jkl} = 0$, where c is a real number. Then we have: $L_{jkl|m}\,y^m + cFL_{jkl} = 0$. Then *F* is Landsberg metric.

Lemma 2. [6] Every isotropic Berwald manifold is isotropic Landsberg manifold.

Corollary 1. Let (M, F) be a constant isotropic Berwald manifold. Suppose that *F* is an R-quadratic metric. Then *F* is a Berwald metric.

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