1-TYPE AND BIHARMONIC FRENET CURVES IN LORENTZIAN 3-SPACE[*]

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Abstract – 1-type and biharmonic curves by using Laplace operator in Lorentzian 3-space are studied and some theorems and characterizations are given for these curves.

1. INTRODUCTION

Chen and Ishikawa [1] classified biharmonic curves in semi-Euclidean space E_v^n. They showed that every biharmonic curve lies in a 3-dimensional totally geodesic subspace, thus, it suffices to classify biharmonic curves in a semi-Euclidean 3-space. Inoguchi [2] pointed out that every biharmonic Frenet curve in Monkowski 3-space E_1^3 is a helix whose curvature κ and torsion τ satisfy $\kappa^2 = \tau^2$.

In this paper we shall give the characterizations of 1-type and biharmonic curves in semi-Euclidean 3-space in terms of curvature and torsion from a different point. Firstly, we point out the general differential equation characterizing Frenet curves (both non-null and null) in a Lorentz 3-space. Moreover, we classify the special curves from the differential equations. In the final section, we give some theorems, corollaries and propositions.

2. PRELIMINARIES

Let (M^3, g) be a time-oriented Lorentz 3-manifold. Let $\gamma : I \to M$ be a unit speed curve. Namely the velocity vector field γ' satisfies $g(\gamma', \gamma') = \varepsilon_1 = \mp 1$. The constant ε_1 is called the *causal character* of γ. A unit speed curve is said to be *spacelike* or *timelike* if its causal character is 1 or -1, respectively, a unit speed curve γ is said to be a *geodesic* if $\nabla_{\gamma'}\gamma' = 0$, where ∇ is the Levi-Civita connection of (M, g). A unit speed curve γ is said to be a *Frenet curve* if $g(\gamma'', \gamma'') \neq 0$. Like Euclidean geometry, every Frenet curve γ on (M, g) admits a Frenet frame field along γ. Here, a Frenet frame filed $\{V_1, V_2, V_3\}$ is an orthonormal frame filed along γ such that $V_1 = \gamma'(s)$ and $\{V_1, V_2, V_3\}$ satisfies the following *Frenet-Serret formula* [3, 4, 5].

$$\begin{bmatrix} \nabla_{\gamma'}V_1 \\ \nabla_{\gamma'}V_2 \\ \nabla_{\gamma'}V_3 \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon_2\kappa & 0 \\ -\varepsilon_1\kappa & 0 & -\varepsilon_3\tau \\ 0 & \varepsilon_2\tau & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}. \tag{1}$$

The functions $\kappa \geq 0$ and τ are called the *curvature* and *torsion,* respectively, the vector fields V_1, V_2, V_3 are called *tangent vector field, principle normal vector field* and *binormal vector field* of γ, respectively. The constants ε_2 and ε_3 defined by

$$\varepsilon_i = g(V_i, V_i), \quad i = 2, 3 \tag{2}$$

are called *second causal character* and *third causal character* of γ, respectively. Note that

$$\varepsilon_3 = -\varepsilon_1 \varepsilon_2. \tag{3}$$

As in the case of Riemannian geometry, a Frenet curve γ is a geodesic if and only if $\kappa = 0$. A Frenet curve with constant curvature and zero torsion is called a *pseudo circle.* A circular helix is a Frenet curve whose curvature and torsion are constants. Pseudo circles are regarded as degenerate helices. Helices, which are not circles, are frequently called *proper helices.*

The mean curvature vector field H of a unit speed curve γ is $H = \varepsilon_1 \nabla_{\gamma'} \gamma'$. If γ is a Frenet curve, then H is given by

$$H = -\varepsilon_3 \kappa V_2. \tag{4}$$

To close this section, we recall the notion of biharmonicity for unit speed curves.

Let $\gamma = \gamma(s)$ be a unit speed curve on the Lorentz 3-manifold (M, g) defined on an interval I. Denote by $\gamma * TM$ the vector bundle over I obtained by pulling back the tangent bundle TM :

$$\gamma * TM := \bigcup_{s \in I} T_{\gamma(s)} M . \tag{5}$$

The *Laplace operator* action on the space $\Gamma(\gamma * TM)$ of all smooth sections of $\gamma * TM$ is given explicitly by [2].

$$\Delta = -\varepsilon_1 \nabla_{\gamma'} \nabla_{\gamma'}. \tag{6}$$

Definition 2.1. [2]. A unit speed curve $\gamma : I \to M$ on a Lorentz 3-manifold M is said to be biharmonic if $\Delta H = 0$.

If M is the semi-Euclidean 3-space, then γ is biharmonic if and only if $\Delta \Delta \gamma = 0$

Definition 2.2. [6]. A unit speed curve $\gamma : I \to M$ on a Lorentz 3-manifold M is said to be 1-type if $\Delta H = \lambda H$.

Definition 2.3. [7, 8]. A unit speed curve $\gamma : I \to M$ on a Lorentz 3-manifold M is said to be a general helix, which means that its $\dfrac{\kappa}{\tau}$ is constant but κ and τ are not.

3. 1-TYPE AND BIHARMONIC CURVES IN LORENTZIAN 3-MANIFOLDS

Chen and Ishikawa classified biharmonic curves in a semi-Euclidean 3-space. In particular, they showed that in a Euclidean 3-space, there are no proper biharmonic curves (i.e., biharmonic curves which are not harmonic). On the other hand, in an indefinite semi-Euclidean 3-space, proper biharmonic curves exist. Inoguchi pointed out that every biharmonic Frenet curve in Minkowski 3-space E_1^3 is a helix whose curvature κ and torsion τ satisfy $\kappa^2 = \tau^2$. Here, we recall their classification theorems.

Theorem 3.1. [1]. Let γ be a spacelike curve in indefinite semi-Euclidean 3-space E_v^3. Then γ is biharmonic if and only if γ is congruent to one of the following:

(1) a spacelike line;

(2) a spacelike curve $\gamma(s) = (as^3 + bs^2, as^3 + bs^2, s)$ in E_1^3, where a and b are constants such that $a^2 + b^2 \neq 0$;

(3) a spacelike curve $\gamma(s) = (as^3/6, as^2/2, -a^2s^3/6 + s)$ in E_1^3, where a is a nonzero constant;

(4) a spacelike curve $\gamma(s) = (as^3/6, as^2/2, a^2s^3/6 + s)$ in E_2^3, where a is a nonzero constant.

Theorem 3.2. [2]. Let $\gamma : I \to M$ be a Frenet curve on a Lorentz 3-manifold (M, g). Denote by Δ the Laplace operator acting on $\Gamma(\gamma^* TM)$. Then γ satisfies $\Delta H = \lambda H$ if and only if γ is a helix (including a geodesic). In this case the eigenvalue λ is

$$\lambda = -\varepsilon_3(\varepsilon_1 \kappa^2 + \varepsilon_3 \tau^2).$$

Corollary 3.3. [2]. Let γ be a Frenet curve on a Lorentz 3-manifold (M, g). Then γ is a non-geodesic biharmonic curve if and only if it is one of the following:

(1) γ is a spacelike helix with a spacelike principal normal such that $\kappa = \mp\tau$;

(2) γ is a timelike helix such that $\kappa = \mp\tau$.

Note that no biharmonic spacelike curve on M exists with spacelike principal normals.

Corollary 3.4. [2]. Let γ be a Frenet curve on (M, g). Then γ is a helix if and only if

$$\nabla_{\gamma'}\nabla_{\gamma'}\nabla_{\gamma'}\gamma' - \lambda\nabla_{\gamma'}\gamma' = 0 \tag{7}$$

for some constant λ. In this case λ equals $-\varepsilon_3(\varepsilon_1\kappa^2 + \varepsilon_3\tau^2)$.

Note that Ikawa obtained Corollary 3.4 for timelike curves [9]. Thus, here Inoguchi gave an analytic meaning of (7). Since he treated both spacelike and timelike curves in corollary 3.4, he got a generalization of [9].

In the case where M is the Minkowski 3-space E_1^3, it is known that helices with $\tau = \pm\kappa \neq 0$ are cubic curves, and one can explicitly give the formula of such helices [10]. Moreover, it is easy to check that such spacelike helices are congruent to the curves given in Theorem 3.1.

Now, we rephrase the classification due to Chen and Ishikawa. Since case (4) in Theorem 3.1 is the image of a timelike helix satisfying $\kappa^2 = \tau^2 = a^2$ under the following anti-isometry from E_1^3 onto E_2^3:

$$E_1^3 \ni (u, v, w) \to (w, u, v), \tag{8}$$

we may restrict our attention to curves in Minkowski 3-space E_1^3.

Proposition 3.5. [1]. Let γ be a unit speed curve in Minkowski 3-space E_1^3. Then γ is biharmonic if and only if γ is congruent to one of the following:

(1) a spacelike or timelike line;

(2) a spacelike curve such that $g(\gamma'', \gamma'') = 0$ given by

$$\gamma(s) = (as^3 + bs^2, as^3 + bs^2, s),$$

where a and b are constants such that $a^2 + b^2 \neq 0$;

(3) a spacelike helix with a spacelike principal normal vector field satisfying $\kappa^2 = \tau^2 = a^2$,

$$\gamma(s) = \left(\frac{a^2 s^3}{6}, \frac{as^2}{2}, -\frac{a^2 s^3}{6+s} \right),$$

(4) a timelike helix satisfying $\kappa^2 = \tau^2 = a^2$.

$$\gamma(s) = \left(\frac{a^2 s^3}{6+s}, \frac{as^2}{2}, \frac{a^2 s^3}{6} \right).$$

4. A GENERAL CHARACTERIZATION FOR A NON-NULL FRENET CURVE AND SOME RESULTS

Theorem 4.1. Let γ be a unit speed non-null Frenet curve in a Lorentz 3-space. Then γ satisfies the following differential equation

$$\nabla_\gamma^3 V_1 + \lambda_1 \nabla_\gamma^2 V_1 + \lambda_2 \nabla_\gamma V_1 + \lambda_3 V_1 = 0, \tag{9}$$

where

$$\lambda_1 = -\left(2\frac{\kappa'}{\kappa} + \frac{\tau'}{\tau} \right),$$

$$\lambda_2 = -\frac{\kappa''}{\kappa} + \frac{\kappa' \tau'}{\kappa \tau} + 2\left(\frac{\kappa'}{\kappa} \right)^2 - \varepsilon_3 \kappa^2 - \varepsilon_1 \tau^2,$$

$$\lambda_3 = -\varepsilon_3 \kappa \tau \left(\frac{\kappa}{\tau} \right)'.$$

Proof: By (1) we get

$$V_2 = \frac{1}{\varepsilon_2 \kappa} \nabla_\gamma V_1. \tag{10}$$

Since $\nabla_\gamma V_3 = -\varepsilon_2 \tau V_2$, we have, by (10), that

$$\nabla_\gamma V_3 = -\frac{\tau}{\kappa} \nabla_\gamma V_1. \tag{11}$$

Since we have that $\nabla_\gamma V_2 = -\varepsilon_1 \kappa V_1 - \varepsilon_3 \tau V_3$, we get

$$\varepsilon_3 V_3 = -\frac{1}{\tau} \nabla_\gamma V_2 - \varepsilon_1 \frac{\kappa}{\tau} V_1. \tag{12}$$

Now combining (10) with (12) we may write

$$\varepsilon_3 V_3 = \frac{\varepsilon_2 \kappa'}{\tau \kappa^2} \nabla_\gamma V_1 - \frac{\varepsilon_2}{\tau \kappa} \nabla_\gamma^2 V_1 - \varepsilon_1 \frac{\kappa}{\tau} V_1. \tag{13}$$

Taking the covariant derivative of (13) and considering (11) we obtain the equation (9).

Corollary 4.2. Let γ be a unit speed non-null Frenet curve in a Lorentz 3-space.
(1) The diffential equation characterizing a timelike curve γ is

$$\nabla_\gamma^3 V_1 - \left(2\frac{\kappa'}{\kappa} + \frac{\tau'}{\tau}\right)\nabla_\gamma^2 V_1$$

$$+ \left[-\frac{\kappa''}{\kappa} + \frac{\kappa'\tau'}{\kappa\tau} + 2\left(\frac{\kappa'}{\kappa}\right)^2 - \kappa^2 + \tau^2\right]\nabla_\gamma V_1 \tag{14}$$

$$- \kappa\tau\left(\frac{\kappa}{\tau}\right)' V_1 = 0.$$

(2) The differential equation characterizing a spacelike curve γ whose principal normal is timelike is

$$\nabla_\gamma^3 V_1 - \left(2\frac{\kappa'}{\kappa} + \frac{\tau'}{\tau}\right)\nabla_\gamma^2 V_1$$

$$+ \left[-\frac{\kappa''}{\kappa} + \frac{\kappa'\tau'}{\kappa\tau} + 2\left(\frac{\kappa'}{\kappa}\right)^2 - \kappa^2 - \tau^2\right]\nabla_\gamma V_1 \tag{15}$$

$$- \kappa\tau\left(\frac{\kappa}{\tau}\right)' V_1 = 0.$$

(3) The differential equation characterizing a spacelike curve γ whose binormal is timelike is

$$\nabla_\gamma^3 V_1 - \left(2\frac{\kappa'}{\kappa} + \frac{\tau'}{\tau}\right)\nabla_\gamma^2 V_1$$

$$+ \left[-\frac{\kappa''}{\kappa} + \frac{\kappa'\tau'}{\kappa\tau} + 2\left(\frac{\kappa'}{\kappa}\right)^2 + \kappa^2 - \tau^2\right]\nabla_\gamma V_1 \tag{16}$$

$$- \kappa\tau\left(\frac{\kappa}{\tau}\right)' V_1 = 0.$$

Proof: (1). (9) gives us (14) since we have $\varepsilon_1 = -1$, $\varepsilon_2 = \varepsilon_3 = 1$, because the curve γ is timelike.

(2). (9) gives us (15) since we have $\varepsilon_2 = -1$, $\varepsilon_1 = \varepsilon_3 = 1$, because the curve γ is spacelike and has a principal normal that is timelike.

(3). (9) gives us (16) since we have $\varepsilon_3 = -1$, $\varepsilon_1 = \varepsilon_2 = 1$, because the curve γ is spacelike and has a binormal that is timelike.

Theorem 4.3. Let γ be a unit speed non-null Frenet curve in Lorentz 3-space. Then γ is a general helix if and only if

$$\nabla_\gamma^3 V_1 + \lambda\nabla_\gamma^2 V_1 + \mu\nabla_\gamma V_1 = 0, \tag{17}$$

where $\lambda = -3\dfrac{\kappa'}{\kappa}$ and $\mu = -\dfrac{\kappa''}{\kappa} + 3\left(\dfrac{\kappa'}{\kappa}\right)^2 - \varepsilon_3\kappa^2 - \varepsilon_1\tau^2$.

Proof: Suppose that γ is a general helix, i.e., $\dfrac{\kappa}{\tau}$ is constant, in other words $\kappa'\tau = \kappa\tau'$. If we replace the value $\dfrac{\kappa'}{\kappa} = \dfrac{\tau'}{\tau}$ in (9), then we get (17).

Conversely, assume that (17) holds. We show that the curve γ is a general helix. To obtain (9) from (17), $\dfrac{\kappa}{\tau}$ must be constant. Thus γ is a general helix.

Corollary 4.4. The equation (17) is a general form of the equation (7).

Proof: If γ is a circular helix, then κ and τ are constants. By Theorem 4.3 we get (7) as a special case.

Corollary 4.5. Let γ be a unit speed non-null Frenet curve in Lorentz 3-space.

(1) The differential equation characterizing a timelike general helix is

$$\nabla_\gamma^3 V_1 - 3\frac{\kappa'}{\kappa}\nabla_\gamma^2 V_1$$
$$+\left[-\frac{\kappa''}{\kappa}+3\left(\frac{\kappa'}{\kappa}\right)^2-\kappa^2+\tau^2\right]\nabla_\gamma V_1 = 0 \tag{18}$$

(2) The differential equation characterizing a spacelike general helix whose principal normal is timelike is

$$\nabla_\gamma^3 V_1 - 3\frac{\kappa'}{\kappa}\nabla_\gamma^2 V_1$$
$$+\left[-\frac{\kappa''}{\kappa}+3\left(\frac{\kappa'}{\kappa}\right)^2-\kappa^2-\tau^2\right]\nabla_\gamma V_1 = 0 \tag{19}$$

(3) The differential equation characterizing a spacelike curve γ whose binormal is timelike is

$$\nabla_\gamma^3 V_1 - 3\frac{\kappa'}{\kappa}\nabla_\gamma^2 V_1$$
$$+\left[-\frac{\kappa''}{\kappa}+3\left(\frac{\kappa'}{\kappa}\right)^2+\kappa^2-\tau^2\right]\nabla_\gamma V_1 \tag{20}$$
$$-\kappa\tau\left(\frac{\kappa}{\tau}\right)' V_1 = 0.$$

Proof: (1). (17) gives us (18) since we have $\varepsilon_1=-1$, $\varepsilon_2=\varepsilon_3=1$, because the curve γ is timelike.

(2). (17) gives us (19) since we have $\varepsilon_2=-1$, $\varepsilon_1=\varepsilon_3=1$, because the curve γ is spacelike and has a principal normal that is timelike.

(3). (17) gives us (20) since we have $\varepsilon_3=-1$, $\varepsilon_1=\varepsilon_2=1$, because the curve γ is spacelike and has a binormal that is timelike.

Note that K. Ilarslan provided these results in another way [11].

Theorem 4.6. Let γ be a unit speed non-null Frenet curve in Lorentz 3-space. Then γ is a general helix if and only if

$$\Delta H + \lambda\nabla_\gamma H + \mu H = 0, \tag{21}$$

where

$$\lambda = 3\varepsilon_1\frac{\kappa'}{\kappa}, \quad \mu = \varepsilon_1\frac{\kappa''}{\kappa}+\varepsilon_1\varepsilon_3\kappa^2+\tau^2-3\varepsilon_1\left(\frac{\kappa'}{\kappa}\right)^2.$$

Proof: According to (17), (4) and (6) we have (21). Sufficiency is clear.

Corollary 4.7. Let γ be a unit speed non-null Frenet curve in Lorentz 3-space. Then γ is a general helix if and only if the equation (4.13) satisfies one of the following conditions:

(1) γ is a timelike general helix such that $\lambda = -3\dfrac{\kappa'}{\kappa}$ and $\mu = -\dfrac{\kappa''}{\kappa} - \kappa^2 + \tau^2 + 3\left(\dfrac{\kappa'}{\kappa}\right)^2$;

(2) γ is a spacelike general helix whose principal normal is timelike such that $\lambda = 3\dfrac{\kappa'}{\kappa}$ and $\mu = \dfrac{\kappa''}{\kappa} + \kappa^2 + \tau^2 - 3\left(\dfrac{\kappa'}{\kappa}\right)^2$;

(3) γ is a spacelike general helix whose binormal is timelike such that $\lambda = 3\dfrac{\kappa'}{\kappa}$ and $\mu = \dfrac{\kappa''}{\kappa} - \kappa^2 + \tau^2 - 3\left(\dfrac{\kappa'}{\kappa}\right)^2$.

The proof is obvious.

Corollary 4.8. In the special case that γ is a circular helix in a Lorentz 3-space, the equation (4.1) gives us the equation

$$\nabla_\gamma^3 V_1 + \varepsilon_2(\varepsilon_1\kappa^2 + \varepsilon_3\tau^2)\nabla_{\gamma'} V_1 = 0. \qquad (22)$$

Proof: Since γ is a circular helix, then κ and τ are constants. Then, by (9) and (3), we get (22) since $\kappa' = \tau' = 0$.

Corollary 4.9. Let γ be a unit speed curve in Lorentz 3-space. γ is a circular helix if and only if the equation (22) becomes one of the following equations:

(1) the differential equation characterizing a timelike circular helix is

$$\nabla_{\gamma'}^3 V_1 + (-\kappa^2 + \tau^2)\nabla_{\gamma'} V_1 = 0; \qquad (23)$$

(2) the differential equation characterizing a spacelike circular helix whose principal normal is timelike is

$$\nabla_{\gamma'}^3 V_1 - (\kappa^2 + \tau^2)\nabla_{\gamma'} V_1 = 0; \qquad (24)$$

(3) the differential equation characterizing a spacelike circular helix whose binormal is timelike is

$$\nabla_{\gamma'}^3 V_1 + (\kappa^2 - \tau^2)\nabla_{\gamma'} V_1 = 0. \qquad (25)$$

The proof is clear.

Note that Ikawa, Inoguchi and Ilarslan obtained the equations (23), (24) and (25) from another way.

As a result, we can say that 1-type (which is a circular helix) and biharmonic non-null curves (which are timelike helix and spacelike helix) can be studied by only one general differential equation (9) which is for a Frenet curve.

5. A GENERAL CHARACTERIZATION FOR A NULL FRENET CURVE AND SOME RESULTS

The following theorem gives a general characterization for a Frenet null curve in Lorenz 3-space.

Theorem 5.1. Let γ be a Frenet null curve in Lorentz 3-space. Then γ satisfies the following differential equation

$$\nabla_\gamma^3 V_1 + \lambda_1 \nabla_\gamma^2 V_1 + \lambda_2 \nabla_\gamma V_1 + \lambda_3 V_1 = 0, \tag{26}$$

where

$$\lambda_1 = -3\frac{\kappa'}{\kappa},$$

$$\lambda_2 = \kappa\left[-\left(\frac{\kappa'}{\kappa^2}\right)' + \frac{(\kappa')^2}{\kappa^3} - 2\tau\right],$$

$$\lambda_3 = \kappa'\tau - \kappa\tau'.$$

Proof: By a Cartan frame $\{V_1, V_2, V_3\}$ of γ we mean a family of vector field $V_1 = V_1(s)$, $V_2 = V_2(s)$, $V_3 = V_3(s)$, along the curve γ satisfying the following conditions:

$$\gamma'(s) = V_1, \ g(V_1, V_1) = g(V_1, V_1) = 0,$$

$$g(V_1, V_2) = -1, \ g(V_1, V_3) = g(V_2, V_3) = 0, \ g(V_3, V_3) = 1,$$

$$\begin{bmatrix} \nabla_\gamma V_1 \\ \nabla_\gamma V_2 \\ \nabla_\gamma V_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \kappa \\ 0 & 0 & \tau \\ \tau & \kappa & 0 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}, \tag{27}$$

where κ and τ are the curvature and torsion of γ, respectively [9]. Here V_1 and V_2 are null vectors and V_3 is a unit spacelike vector. (27) gives us (26).

Theorem 5.2. Let $\gamma : I \rightarrow M$ be a null Frenet curve in Lorentz 3-space. Denote the Laplace operator by Δ. Then γ satisfies $\Delta H = \lambda H$ if and only if γ is a circular null helix. In this case the eigenvalue λ of the operator Δ is

$$\lambda = -2\kappa\tau. \tag{28}$$

Proof: Suppose γ satisfies $\Delta H = \lambda H$. Let us denote by H the mean curvature vector field of γ and by the Laplace-Beltrami operator Δ of γ. Then the mean curvature vector field H is defined by

$$H = tr(\sigma) = \sigma(V_1, V_1) = \nabla_\gamma V_1 = \kappa V_3 \tag{29}$$

and the Laplacian operator along γ is given by

$$\Delta = -\nabla_\gamma^2 = -\nabla_\gamma \nabla_\gamma \tag{30}$$

[1, 6]. By (27), (29) and (30) we have

$$\Delta H = -2(\kappa'\tau + \kappa\tau')V_1 - \kappa'\kappa V_2 - (2\kappa^2\tau + \kappa'')V_3 \tag{31}$$

and

$$\lambda H = \lambda\kappa V_3. \tag{32}$$

From the equations (31), (32) and $\Delta H = \lambda H$, we get

$$\kappa'\kappa = 0, \ 2\kappa'\tau + \kappa\tau' = 0 \tag{33}$$

and

$$\lambda = -\left(2\kappa\tau + \frac{\kappa''}{\kappa}\right).$$ (34)

According to (33), γ is a circular null helix. Since γ is a circular null helix, the equation (34) reduces to $\lambda = -2\kappa\tau$.

Conversely, suppose that γ is a circular null helix and that $\lambda = -2\kappa\tau$. If we replace (34) and (33) in (31), we obtain $\Delta H = \lambda H$. In this case the curve is 1-type.

Corollary 5.3. Let γ be a Frenet null curve in Lorentz 3-space. Then γ is a biharmonic null curve if and only if γ is a pseudo null circle which is a degenerate null helix.

Proof: Since the curve γ is a Frenet curve in which $\kappa \neq 0$, $\Delta H = 0 \Rightarrow \lambda = -2\kappa\tau \Rightarrow \tau = 0$. This completes the proof.

Corollary 5.4. Let γ be a Frenet null curve in Lorentz 3-space. Then γ is a general null helix if and only if

$$\nabla_{\gamma'}^3 V_1 + \lambda\nabla_{\gamma'}^2 V_1 + \mu\nabla_{\gamma'}V_1 = 0,$$ (35)

where

$$\lambda = -3\frac{\kappa'}{\kappa} \text{ and } \mu = -\frac{\kappa''}{\kappa} + 3\left(\frac{\kappa'}{\kappa}\right)^2 - 2\kappa\tau.$$

Proof: Suppose that γ is a general null helix, i.e., $\frac{\kappa}{\tau}$ is constant but κ and τ are not constants, in other words, $\kappa'\tau = \kappa\tau'$. If we write the value $\frac{\kappa'}{\kappa} = \frac{\tau'}{\tau}$ in (26), then we get (35), $\frac{\kappa}{\tau}$ must be constant. Thus γ is a general null helix.

Note that Ilarslan obtained Corollary 5.4 in a different way [11].

Corollary 5.5. Let γ be a Frenet null curve in Lorentz 3-space. Then γ is a general null helix if and only if

$$\Delta H + \lambda\nabla_{\gamma'}H + \mu H = 0,$$ (36)

where

$$\lambda = 3\frac{\kappa'}{\kappa} \text{ and } \mu = \frac{\kappa''}{\kappa} - 3\left(\frac{\kappa'}{\kappa}\right)^2 + 2\kappa\tau.$$

Proof: From (35), (29) and (30), the proof is clear as the same method in Theorem 4.6.

Note that this corollary is a general case of Theorem 5.2.

Corollary 5.6. For a null-Frenet curve γ in Lorentz 3-space the differential equation

$$\nabla_{\gamma'}^3 V_1 - 2\kappa\tau\nabla_{\gamma'}V_1 = 0$$ (37)

characterizes that γ is a circular null helix.

Proof: By (26), we easily get (37) since κ and τ are constants.

Ilarslan obtained Corollary 5.6 in a different way [11].

1-type and biharmonic curves for a Frenet null curve can also be studied easily by Equation (26)

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