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INEQUALITIES FOR MEROMORPHICALLY P-VALENT FUNCTIONS*

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Abstract – The aim of this paper is to prove some inequalities for p-valent meromorphic functions in the punctured unit disk Δ* and find important corollaries.

Keywords – p-valent functions, meromorphic starlike function, convex, close-to-convex

1. INTRODUCTION

Let Σ_p denote the class of functions *f(z)* of the form

$$f(z) = z^{-p} + \sum_{k=p}^{\infty} a_k z^k \tag{1}$$

which are analytic meromorphic multivalent in the punctured unit disk

$$\Delta^* = \{z : 0 < |z| < 1\}.$$

We say that f(z) is p-valently starlike of order $\gamma (0 \le \gamma < p)$ if and only if for $z \in \Delta^*$

$$-\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \gamma, \tag{2}$$

Also, *f(z)* is p-valently convex of order γ $(0 \le \gamma < p)$ if and only if

$$-\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \gamma, \ (z \in \Delta^*). \tag{3}$$

Definition 1.1: A function $f(z) \in \Sigma_p$ is said to be in the subclass $X_p^*(j)$ if it satisfies the inequality

$$\left| \frac{(p-1)!}{(-1)^j (p+j-1)!} \frac{f^{(j)}(z)}{z^{-p-j}} - 1 \right| < 1 \tag{4}$$

where

$$f^{(j)}(z) = (-1)^j \frac{(p+j-1)!}{(p-1)!\, z^{p+j}} + \sum_{k=p}^{\infty} \frac{k!}{(k-j)!} a_k z^{k-j} \tag{5}$$

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is the j-th differential of *f (z)* and a function *f (z)* $\in \Sigma_p$ is said to be in the subclass $Y_p^*(j)$ if it satisfies the inequality

$$\left| -\frac{z[f^{(j)}(z)]'}{f^{(j)}(z)} - (p+j) \right| < p+j. \tag{6}$$

To establish our main results we need the following lemma due to Jack [1].

Lemma 1.2. Let w(z) be analytic in $\Delta = \{z : |z| < 1\}$ with w(0) = 0. If | w (z) | attains its maximum value on the circle | z | = r < 1 at a point z_0, then

$$z_0 w'(z_0) = c w(z_o)$$

where *c* is a real number and $c \geq 1$.

Some different inequalities on p-valent holomorphic and p-valent meromorphic functions by using operators were studied in [2-5].

2. MAIN RESULTS

In the first theorem we give a sufficient condition for *f* $\in \Sigma_p$ to be in the chass $X_p^*(j)$.

Theorem 2.1. If *f (z)* $\in \Sigma_p$ satisfies the inequality

$$\mathrm{Re}\left\{ \frac{z[f^{(j)}(z)]'}{f^{(j)}(z)} + p + j \right\} > 1 - \frac{1}{2p} \tag{7}$$

then *f (z)* $\in X_p^*(j)$.

Proof: Let *f (z)* $\in \Sigma_{p,}$ we define the function *w(z)* by

$$\frac{(p-1)!}{(-1)^j (p+j-1)!} \frac{f^{(j)}(z)}{z^{-p-j}} = 1 - w(z), \quad (z \in \Delta^*). \tag{8}$$

It is easy to verify that w(0) = 0.
From (8) we obtain

$$f^{(j)}(z) = \frac{(1)^j (p+j-1)!}{(p-1)!} z^{-p-j} - \frac{(-1)^j (p+j-1)!}{(p-1)!} z^{-p-j} w(z)$$

or

$$[f^{(j)}(z)]' = (-1)^{j+1} (p+j)\, z^{-p-j-1} \frac{(p+j-1)!}{(p-1)!} + (-1)^j\, (p+j) z^{-p-j-1}$$

$$\frac{(p+j-1)!}{(p-1)!} w(z) + (-1)^{j+1} \frac{(p+j-1)!}{(p-1)!} z^{-p-j} w'(z).$$

After a simple calculation we obtain

$$\frac{zw'(z)}{1-w(z)} = \frac{z[f^{(j)}(z)]'}{f^{(j)}(z)} + (p+j). \tag{9}$$

Now, suppose that there exists a point $z_0 \in \Delta^*$ such that

$$\max_{|z|\le|z_0|} |w(z)| = |w(z_0)| = 1.$$

Then by letting $w(z_0) = e^{i\theta}$ $(w(z_0) \neq 1)$ and using the Jack's lemma in the equation (9), we have

$$-\operatorname{Re}\left\{\frac{z[f^{(j)}(z)]'}{f^{(j)}(z)} + p + j\right\} = \operatorname{Re}\left\{\frac{z_0 w'(z_0)}{1-w)z_0)}\right\} = \operatorname{Re}\left\{\frac{cw(z_0)}{1-w(z_0)}\right\}$$

$$= c\operatorname{Re}\left\{\frac{e^{i\theta}}{1-e^{i\theta}}\right\} = \frac{-C}{2} < \frac{-1}{2},$$

which contradicts the hypothesis (7). Hence, we conclude that for all z, $|w(z)| < 1$ and from (8) we have

$$\left|\frac{(p-1)!\,f^{(j)}(z)}{(-1)^j (p+j-1)!\,z^{-p-j}} - 1\right| = |w(z)| < 1$$

and this gives the result.

Theorem 2.2. If $f(z) \in \Sigma_p$ satisfies the inequality

$$\operatorname{Re}\left\{\frac{z[f^{(j)}(z)]'}{f^{(j)}(z)} - \left(1 + \frac{z[f^{(j)}(z)]''}{[f^{(j)}(z)]'}\right)\right\} > \frac{2p+1}{2(p+1)}, \tag{10}$$

then $f(z) \in Y_p^*(j)$.

Proof: Let $f(z) \in \Sigma_p$. We consider the function $w(z)$ as follows:

$$-\frac{z[f^{(j)}(z)]'}{f^{(j)}(z)} = (p+j)(1-w(z)). \tag{11}$$

It is easy to see that w(0)= 0. Furthermore, by differentiating both sides of (11) we get

$$-\left[1 + \frac{z[f^{(j)}(z)]''}{[f^{(j)}(z)]'}\right] = (p+j)(1-w(z)) + \frac{zw'(z)}{1-w(z)}.$$

Now suppose there exists a point $z_0 \in \Delta^*$ such that $\max_{|z|\le|z_0|} |w(z)| = |w(z_0)| = 1$. Then by letting $w(z_0)=e^{i\theta}$ and using Jack's lemma we have

$$\operatorname{Re}\left\{\frac{z[f^{(j)}(z)]'}{f^{(j)}(z)} - \left(1 + \frac{z[f^{(j)}(z)]''}{[f^{(j)}(z)]'}\right)\right\} = \operatorname{Re}\left\{\frac{z_0 w'(z_0)}{1-w(z_0)}\right\}$$

$$= c\operatorname{Re}\left\{\frac{e^{i\theta}}{1-e^{i\theta}}\right\} = -\frac{c}{2} < -\frac{1}{2}$$

which contradicts the condition (10). So we conclude that $|w(z)| < 1$ for all $z \in \Delta^*$. Hence, from (11) we obtain

$$\left|-\frac{z[f^{(j)}(z)]'}{f^{(j)}(z)} - (p+j)\right| < p+j.$$

This completes the proof.

By taking $j = 0$ in Theorems 2.1 and 2.2, we obtain the following corollaries.

Corollary 1. If $f(z) \in \Sigma_p$ satisfies the inequality

$$-\operatorname{Re}\left\{\frac{zf'}{f}+p\right\}>1-\frac{1}{2p} \quad ,$$

then

$$\left|\frac{f(z)}{z^{-p}}-1\right|<1.$$

Corollary 2. If $f(z) \in \Sigma_p$ satisfies the inequality

$$-\operatorname{Re}\left\{\frac{zf'}{f}-\left(1+\frac{zf''}{f'}\right)\right\}>\frac{2p+1}{2(p+1)},$$

then $\left|-\frac{zf'}{f}-p\right|<p$ or equivalently $f(z)$ is meromorphically p-valent starlike with respect to the origin.

By taking $j= 1$ in theorems 2.1 and 2.2, we obtain the following corollaries.

Corollary 3. If $f(z) \in \Sigma_p$ satisfies the inequality

$$-\operatorname{Re}\left\{\frac{zf''}{f'}+p+1)\right\}>1-\frac{1}{2p}.$$

Then $\left|-\frac{f'(z)}{z^{-p-1}}-p\right|<p$ or equivalently $f(z)$ is meromorphically p-valent close-to-convex with respect to the origin.

Corollary 4. If $f(z) \in \Sigma_p$ satisfies the inequality

$$-\operatorname{Re}\left\{\frac{zf''}{f'}-\left(1+\frac{zf'''}{f''}\right)\right\}>\frac{2p+1}{2(p+1)},$$

then

$$\left|-\frac{zf''}{f'}-(p+1)\right|<p+1$$

or equivalently $f(z)$ is meromorphically multivalent convex.

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REFERENCES

1. Jack, I. S. (1971). Functions starlike and convex of order α. *J. London Math. Soc.*, *3*, 469-474.

2. Irmak, H. & Cetin, F. (1999). Some theorems involving inequalities on p-valent functions. *Turk. J. Math.*, *23*, 453-

459.

3. Irmak, H., Cho, N. E., Cetin, F. & Raina, R. K. (2001). Certain inequalities involving meromorphically functions. *Hacet. Bull. Nat. Sci. Eng. Ser*., *B 30*, 39-43.

4. Irmak, H. & Raina, R. K. (2006). On certain classes of functions associated with multivalently analytic and multivalently meromorphic functions. *Soochow J. Math*. *32*(3), 413-419.

5. Irmak, H. & Raina, R. K. (2007). New classes of non-normalized meromorphically multivalent functions. *Sarajevo J. Math*. *3*(16), no. 2, 157-162.