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APPROXIMATE SOLUTION TO BOUNDARY VALUE PROBLEMS BY THE MODIFIED VIM[*]

M. HEYDARI AND G. B. LOGHMANI[**]

Department of Mathematics, Yazd University, Yazd, I. R. of Iran
Emails: m.heydari85@gmail.com & loghmani@yazduni.ac.ir

Abstract – This paper presents an efficient modification of the variational iteration method for solving boundary value problems using the chebyshev polynomials. The proposed method can be applied to linear and nonlinear models. The scheme is tested for some examples and the obtained results demonstrate the reliability and efficiency of the proposed method.

Keywords – Variational iteration method, chebyshev polynomials, boundary value problems

1. INTRODUCTION

The variational iteration method was developed by He in [1-4]. Over the years, variational iteration method has been applied to Klein-Gordon equation [5], Helmholtz equation [6], differential algebraic equations [7], epidemic and prey predators models [8], nonlinear boundary value problems [9], and many others problems [10-12]. The main idea of variational iteration method is to construct a correction functional by a general Lagrange multiplier. The purpose of the present paper is to introduce a modification of the variational iteration method using orthogonal Chebyshev polynomials.

Consider the following differential equation

$$Lu(t) + Nu(t) = g(t), \tag{1}$$

Where L is a linear operator, N is a nonlinear operator, and $g(t)$ is an inhomogeneous term. Then, we can construct a correction functional as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(Lu_n(x) + N\ \tilde{u}_n(x) - g(x))dx\ , \tag{2}$$

where λ is a general Lagrange multiplier [3, 4] which can be determined optimally via the variational theory. The second term on the right is called the correction and $\tilde{u}_n$ is considered as a restricted variation, i.e. $\delta\tilde{u}_n = 0$. The successive approximations u_{n+1}, $n \geq 0$, of the solution u will be readily obtained upon using the determined Lagrangian multiplier and the selective function u_0. Consequently, the solution is given by $u = \lim_{n\to+\infty} u_n$.

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[**]Corresponding author

2. MODIFIED VARIATIONAL ITERATION METHOD

In cases in which *g(t)* is a complicated function and its integration is cumbersome, Taylor series expansion may be used up to an arbitrary natural number M, and the variational iteration method is then applied. Now, if *g(t)* is expressed in Taylor series as follows:

$$g(t) \approx \sum_{k=0}^{M} g_k(t)\,, \tag{3}$$

by substituting Eq.(3) in Eq.(2) we have

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(Lu_n(x) + N\ \tilde{u}_n(x) - \sum_{k=0}^{M} g_k(x))dx \tag{4}$$

Thus, for an arbitrary natural number *m*, from Eq.(4) we can obtain the approximate solution,

$$u_T(t) = u_m(t)\,. \tag{5}$$

In this paper, *g(t)* is expressed in Chebyshev series instead, that is:

$$g(t) \approx \sum_{k=0}^{M} a_k T_k(t)\,, \tag{6}$$

where $T_k(t)$ denotes the Chebyshev polynomials of the first kind,

$$T_0(t) = 1,\ T_1(t) = t\,, T_2(t) = 2t^2 - 1\,,$$

and in general,

$$T_{k+1}(t) = 2tT_k(t) - T_{k-1}(t)\,, \qquad k \geq 1$$

By substituting Eq.(6) in Eq.(2) we have

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(Lu_n(x) + N\ \tilde{u}_n(x) - \sum_{k=0}^{M} a_k T_k(x))dx\,. \tag{7}$$

So, for an arbitrary natural number *m*, from Eq.(7) we can obtain the approximate solution,

$$u_C(t) = u_m(t)\,. \tag{8}$$

3. NUMERICAL RESULTS

In this section, we demonstrate the effectiveness of the proposed modification of the Variational iteration method with three illustrative examples. All of the computations have been done using the Maple*12* with *10* digits precision.

Test problem 1. Consider the linear boundary value problem

$$u''(t) + t\ u'(t) + t^2 u(t) = g(t), \qquad 0 \leq t \leq 1 \tag{9}$$

$$u(0) = 0\,, u(1) = \ln(2), \tag{10}$$

With $g(t) = t^2\ \ln(1 + t^2) + \frac{2(1 + t^4)}{(1 + t^2)^2}$ and the exact solution $u(t) = \ln(1 + t^2)$. According to the variational iteration method, we can construct the correction functional of Eq.(9) as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(u''_n(x) + x\,\tilde{u}'_n(x) + x^2\tilde{u}_n(x) - g(x))dx\,, \tag{11}$$

Where λ is general Lagrange multiplier and $\tilde{u}'_n(x)$,$\tilde{u}_n(x)$ denote restricted variation, i.e. $\delta\tilde{u}'_n(x) = \delta\tilde{u}_n(x) = 0.$ Making the above correction functional as the initial guess, we can begin with the following stationary conditions:

$$1 - \lambda'(x)|_{x=t} = 0,\ \lambda(x)|_{x=t} = 0,\ \lambda''(x)|_{x=t} = 0, \tag{12}$$

The Lagrange multiplier, therefore, can be identified as

$$\lambda = x - t. \tag{13}$$

Therefore, the following iteration formula is obtained:

$$u_{n+1}(t) = u_n(t) + \int_0^t (x-t)(u''_n(x) + x\ u'_n(x) + x^2u_n(x) - g(x))dx\ , \tag{14}$$

According to Eq.(10) we start with initial approximation $u_0(t) = At$, where A is constant to be determined. From Eq.(14), we can obtain the following result:

$$u_1(t) = At + \int_0^t (x-t)(Ax + Ax^3 - g(x))dx. \tag{15}$$

Considering $g(t)$, it is clear how difficult it is to calculate $u_1(t)$. So, by setting $M = 10$ in Eq.(3), the Taylor series of $g(t)$ becomes,

$$g(t) \approx 2 - 4t^2 + 9t^4 - \frac{25}{2}t^6 + \frac{49}{3}t^8 - \frac{81}{4}t^{10}. \tag{16}$$

Thus, by Eq.(4) and $m = 8$, we have:

$$u_1(t) = At + t^2 - \frac{A}{6}t^3 - \frac{1}{3}t^4 - \frac{1}{20}At^5 + \frac{3}{10}t^6 + \cdots,$$

$$u_2(t) = At + t^2 - \frac{A}{6}t^3 - \frac{1}{2}t^4 - \frac{1}{40}At^5 + \frac{14}{45}t^6 + \cdots,$$

$$\vdots$$

$$u_8(t) = At + t^2 - \frac{A}{6}t^3 - \frac{1}{2}t^4 - \frac{1}{40}At^5 + \frac{1}{3}t^6 + \cdots.$$

By imposing the boundary condition at $t = 1$, we obtain $A = 0.08161535111$, and
$u_T(t) = u_8(t) = 0.08161535111t + t^2 - 0.0136025585t^3 - 0.5t^4 - 0.002040383778t^5$
$+0.3333333333t^6 + \cdots.$

Now, the Chebyshev polynomials are used to expand $g(t)$. By setting $M = 10$ in Eq.(6), the Chebyshev expansion of $g(t)$ becomes,

$$g(t) \approx \sum_{k=0}^{10} a_k T_k(2t-1), \qquad 0 \le t \le 1$$

Where

$$a_0 = \frac{1}{\pi}\int_{-1}^{1} \frac{g(0.5t+0.5)T_0(t)}{\sqrt{1-t^2}}dt,$$

and

$$a_k = \frac{2}{\pi}\int_{-1}^{1} \frac{g(0.5t+0.5)T_k(t)}{\sqrt{1-t^2}}dt, \qquad k = 1,2,\ldots,10.$$

So, we have

$$g(t) \approx 1.999998592 + 0.0003541638800t - 4.014741800t^2 + \ldots - 1.223069532t^{10}. \tag{17}$$

Thus, by Eq.(7) and $m = 8$, we obtain,

$u_1(t) = At + 0.9999992960t^2 + (0.00005902731335 - 0.16666666667A)t^3 - 0.3345618165t^4$
$+(0.01197517288 - 0.0500000000000A)t^5 + 0.2336299475t^6 + \cdots,$

$u_2(t) = At + 0.9999992959t^2 + (0.00005902731335 - 0.16666666667A)t^3 - 0.5012283655t^4$ $\vdots$
$+(0.01196631878 - 0.0250000000000A)t^5 + 0.244904880t^6 + \cdots,$

$u_8(t) = At + 0.9999992964t^2 + (0.00005902731335 - 0.16666666667A)t^3 - 0.5012283660t^4$
$+(0.01196631878 - 0.0250000000000A)t^5 + 0.2671270867t^6 + \cdots.$

By imposing the boundary condition at $t = 1$, we obtain A=$1.227117207 \times 10^{-10}$ and
$u_C(t) = u_8(t) = -1.840675810\times 10^{-9}t + 0.9999992960t^2 + 0.00005902762013t^3$
$-0.5012283660t^4 + 0.01196631883t^5 + 0.2671270867t^6 + \cdots.$

Table 1 shows the maximum norm errors of approximate solutions $u_T(t)$ and $u_C(t)$.

Table 1. Maximum norm errors for Test problem 1

m m	$\| u(t)- u_T(t) \|_{+\infty}$ in variational iteraton method using Taylor series	$\| u(t)- u_C(t) \|_{+\infty}$ in modified variational iteration method using Chebyshev polynomials
2	6.7(e-2)	1.5(e-2)
5	5.4 (e-2)	1.6(e-5)
8	5.4(e-2)	5.4(e-9)

Test problem 2. Consider the nonlinear boundary value problem

$$u''(t) + (u(t))^2 = g(t), \qquad 0 \le t \le 1 \tag{18}$$

$$u(0) = 0\ , \quad u(1) = \sin(1), \tag{19}$$

With $g(t) = -4\sin(t^2)t^2 + 2\cos(t^2) + \sin^2(t^2)$ and the exact solution $u(t) = \sin(t^2)$. According to the variational iteration method, we can construct the correction functional of

Eq.(18) as follows: $u_{n+1}(t) = u_n(t) + \int_0^t \lambda (u_n''(x) + (\tilde{u}_n(x))^2 - g(x))dx$, (20)

Where λ is general Lagrange multiplier and $(\tilde{u}_n(x))^2$ denote restricted variation, i.e. $\delta(\tilde{u}_n(x))^2 = 0$. so, we obtain following iteration formula:

$$u_{n+1}(t) = u_n(t) + \int_0^t (x - t)(u_n''(x) + (u_n(x))^2 - g(x))dx \ , \tag{21}$$

According to Eq.(19) we start with initial approximation $u_0(t) = Bt$. By setting $M = 8$ and $m = 5$ in Eq.(3), Eq.(6) and Eq.(4), Eq.(7), respectively, and imposing the boundary condition at $t = 1$ we obtain:
$B = 0.00004212166016$,
$u_T(t) = u_5(t) = 0.0000421216601t + t^2 - 1.478528546\times 10^{-10} t^4 - 0.00000421216601t^5$
$-0.1666666667t^6 + \cdots$.
and
$B = -6.1741267791\times 10^{-9}$,
$u_C(t) = u_5(t) = -6.0174126779\times 10^{-9} t + 0.9999996385t^2 + 0.00001758714667t^3$
$-0.0002061403733t^4 + 0.001076113193t^5 - 0.1696219915t^6 + \cdots$.
Table 2 shows the maximum norm errors of approximate solutions $u_T(t)$ and $u_C(t)$.

Table 2. Maximum norm errors for Test problem 2

m	$\lVert u(t)- u_T(t) \rVert_{+\infty}$ in variational iteraton method using Taylor series	$\lVert u(t)- u_C(t) \rVert_{+\infty}$ in modified variational iteration method using Chebyshev polynomials
1	1.6(e-2)	1.6(e-2)
3	2.5 (e-5)	5.4(e-6)
5	3.(e-5)	4.6(e-9)

Test problem 3. Consider the following system of nonlinear boundary value problem

$$\begin{cases} u''(t) + t\ v'(t) + u(t) = g_1(t) \\ v''(t) + u(t)v(t) + v(t) = g_2(t) \end{cases} \qquad 0 \le t \le 1 \ , \tag{22}$$

with conditions,

$$u(0) = v(0) = 1,\ \ u(1) = v(1) = \sqrt{2}\ , \tag{23}$$

and the exact solution $u(t) = \sqrt{1+t^2}$ and $v(t) = \sqrt{1+t}$. According to the variational iteration method, the correction functionals are given by:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda_1 (u_n''(x) + x\ \tilde{v}_n'(x) + \tilde{u}_n(x) - g_1(x))dx\,, \tag{24}$$

$$v_{n+1}(t) = v_n(t) + \int_0^t \lambda_2 (v_n''(x) + \tilde{u}_n(x)\tilde{v}_n(x) + \tilde{v}_n(x) - g_2(x))dx\,, \tag{25}$$

where $\tilde{u}_n(x)$, $\tilde{v}_n(x)$ and $\tilde{v}'_n(x)$ denote restricted variations, i.e. $\delta\tilde{u}_n(x) = \delta\tilde{v}_n(x) = \delta\tilde{v}'_n(x) = 0$, can be easily identified as $\lambda_1 = x - t$, $\lambda_2 = x - t$ .Therefore, we have the following iteration formula:

$$u_{n+1}(t) = u_n(t) + \int_0^t (x-t)(u''_n(x) + x\ v'_n(x) + u_n(x) - g_1(x))dx, \tag{26}$$

$$v_{n+1}(t) = v_n(t) + \int_0^t (x-t)(v''_n(x) + u_n(x)v_n(x) + v_n(x) - g_2(x))dx, \tag{27}$$

According to Eq.(23) we start with initial approximation $u_0(t) = 1 + Ct$ and $v_0(t) = 1 + Dt$. By setting $M = 8$ and $m = 6$ in Eq.(3), Eq.(6) and Eq.(4), Eq.(7), respectively, and imposing the boundary conditions at $t = 1$ we obtain:

$C = -0.0124610863, D = 0.5033821290$,

$u_T(t) = u_6(t) = 1 - 0.01246108636t + 0.50000000000t^2 + 0.00151315956t^3$
$-0.12500000000t^4 - 0.00021807870t^5 + 0.06243030346t^6 + \cdots,$

$v_T(t) = v_6(t) = 1 + 0.5033821290t - 0.1250000000t^2 + 0.0634494714t^3 - 0.03853977598t^4$
$+0.0270107098t^5 + -0.02054169571t^6 + \cdots.$

and

$C = 6.760049962 \times 10^{-8}$, $D = 0.500000031$,

$u_C(t) = u_6(t) = 1 + 6.760049962\times 10^{-8}t + 0.5000019770t^2 - 0.00010458509t^3$
$-0.1236558835t^4 - 0.00775850784t^5 + 0.08604715582t^6 + \cdots,$

$v_C(t) = v_6(t) = 1 + 0.5000000311t - 0.1249999090t^2 + 0.0624944680t^3 - 0.03898251270t^4$
$+0.02679853388t^5 + -0.01837633575t^6 + \cdots.$

The obtained results are shown in Table 3.

Table 3. Maximum norm errors for Test problem 3

m	Variational iteration Method using Taylor series		Modified variational iteration method using Chebyshev polynomials	
	$\Vert u(t)- u_T(t) \Vert_{+\infty}$	$\Vert v(t)- v_T(t) \Vert_{+\infty}$	$\Vert u(t)- u_C(t) \Vert_{+\infty}$	$\Vert v(t)- v_C(t) \Vert_{+\infty}$
2	5.5(e-3)	5.7(e-3)	3.5(e-3)	3.2(e-3)
4	8.3(e-3)	2.8(e-3)	1.9(e-5)	1.2(e-5)
6	8.3(e-3)	2.7(e-3)	3.5(e-8)	1.4(e-8)

A comparison of the results mentioned in Tables 1-3 show the power of the proposed method in this paper, for these examples.

CONCLUSION

The main purpose of this paper was to employ Chebyshev polynomials to the variational iteration method for solving linear and nonlinear boundary value problems. The results were employed to solve both linear and nonlinear illustrative problems. The modification proposed in this paper has demonstrated that the nonlinear terms can be handled without much difficulty. The results confirm that a greater accuracy compared to the traditional Taylor expansion method is obtained.

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