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JOHNSON AMENABILITY FOR TOPOLOGICAL SEMIGROUPS*

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Abstract –A notion of amenability for topological semigroups is introduced. A topological semigroup S is called Johnson amenable if for every Banach S-bimodule E, every bounded crossed homomorphism from S to E^* is principal. In this paper it is shown that a discrete semigroup S is Johnson amenable if and only if $\ell^1(S)$ is an amenable Banach algebra. Also, we show that if a topological semigroup S is Johnson amenable, then it is amenable, but the converse is not true.

Keywords – Amenability, crossed homomorphism, topological semigroup

1. INTRODUCTION

The Johnson's Theorem [1] asserts that a locally compact Hausdorff group G is amenable if and only if the Banach algebra $L^1(G)$ is amenable. This is not true for discrete semigroups.

Duncan and Nomioka [2] showed that if $\ell^1(S)$ is amenable, then S is amenable, and for a wide class of inverse semigroups S, they showed that $\ell^1(S)$ fails to be amenable if E_S (the set of idempotent elements of S) is infinite.

Amini [3] has recently introduced the nation of module amenability for Banach algebras and showed that, under some action for an inverse semigroup S, $\ell^1(S)$ is module amenable if and only if S is amenable.

In this paper we introduce the concept of Johnson amenability for topological semigroups. In particular we show that a discrete semigroup S is Johnson amenable if and only if $\ell^1(S)$ is an amenable Banach algebra.

2. PRELIMINARIES

Let S be a topological semigroup, that is, a semigroup which is a topological space and the semigroup multiplication is separately continuous. A Banach space E is called a Banach S-bimodule, if there exists a two sided linear transitive action of S on E such that,

i. $s\cdot(x\cdot t)=(s\cdot x)\cdot t$ for all $s,t\in S$, $x\in E$,

ii. if $s_i\to s$ in S and $x\in E$, then $s_i\cdot x\to s\cdot x$ and $x\cdot s_i\to x\cdot s$ in the norm topology, and

iii. the action is bounded, that is, there is a $M>0$ such that for every $x\in E$ and $s\in S$, we have

$$\|s\cdot x\|\leq M\|x\|,\quad \|x\cdot s\|\leq M\|x\|.$$

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We say that the right (respectively, left) action of S on E is trivial, if $x \cdot s = x$ (respectively, $s \cdot x = x$) for $s \in S$ and $x \in E$; Also, the right (respectively, left) action is called zero, if $x \cdot s = 0$ (respectively, $s \cdot x = 0$).

If E is a Banach S-bimodule, then the topological dual E^* of E is also an S-bimodule, where the action is defined by

$$\langle s \cdot f, x \rangle = \langle f, x \cdot s \rangle, \quad \langle f \cdot s, x \rangle = \langle f, s \cdot x \rangle \quad (s \in S, f \in E^*, x \in E).$$

Note that if $s_i \to s$ in S, $f_i \to f$ in E^* in the weak*-topology and $\sup_i \|f_i\| < \infty$, then $s_i \cdot f_i \to s \cdot f$ and $f_i \cdot s_i \to f \cdot s$ in the weak*-topology and the dual action is also bounded.

Let $\mathbf{C}(S)$ be the Banach algebra of complex valued continuous bounded functions on S. Then $\mathbf{C}(S)$ is an S-bimodule via the following actions

$$a \cdot s(t) = a(st), \quad s \cdot a(t) = a(ts) \quad (s, t \in S, a \in \mathbf{C}(S)).$$

We call these actions the right and the left function module actions respectively.

A function $f \in \mathbf{C}(S)$ is right uniformly continuous if $\lim_i \|f \cdot s_i - f \cdot s\|_\infty = 0$ whenever $s_i \to s$. The Banach algebra of all right uniformly continuous functions on S is denoted by $\mathbf{RUC}(S)$. Similarly the Banach algebra of all left uniformly continuous functions on S is denoted by $\mathbf{LUC}(S)$.

Note that $\mathbf{RUC}(S)$ (respectively, $\mathbf{LUC}(S)$) is a Banach S-bimodule with the right (respectively, left) function module action and trivial left (respectively, right) action.

Let E be a linear subspace of $\mathbf{C}(S)$ which contains the constant function 1_S. A mean on E is a functional m in E^*, such that $m(1_S) = \|m\| = 1$. Suppose that E is also closed under right function module action. Then the mean m is called left invariant if $s \cdot m = m$ for all $s \in S$. Right invariant means are defined similarly.

Definition 2.1. A semigroup S is called left (respectively, right) amenable if there exists a left (respectively, right) invariant mean on $\mathbf{RUC}(S)$ (respectively, $\mathbf{LUC}(S)$); S is called amenable if it is left and right amenable.

Recall that if S is left amenable with respect to some topology, then it is left amenable with respect to all topologies which are coarser than that. Thus a commutative topological semigroup is amenable, since it is amenable with discrete topology [4, page 16].

We now give the definition of amenability for Banach algebras. Recall that, for a Banach algebra A, a Banach space E is a Banach A-bimodule if E is an A-bimodule and there is a constant M such that $\|a \cdot x\| \leq M \|a\| \|x\|$ and $\|x \cdot a\| \leq M \|x\| \|a\|$ for each a in A and x in E.

If E is a Banach A-bimodule, then the dual space E^* is a Banach A-bimodule with the actions defined by $\langle a \cdot f, x \rangle = \langle f, x \cdot a \rangle$ and $\langle f \cdot a, x \rangle = \langle f, a \cdot x \rangle$, for a in A, x in E and f in E^*. A derivation of A into an A-bimodule E is a linear map $D : A \to E$ such that $D(ab) = a \cdot D(b) + D(a) \cdot b$, for all a, b in A. For x in E, the map $a \mapsto a.x - x.a$ is a derivation. Such derivations are called inner.

Definition 2.2. A Banach algebra A is amenable if for any Banach A-bimodule E, every continuous derivation $D : A \to E^*$ is inner.

3. JOHNSON AMENABILITY

Let S be a topological semigroup and let E be a Banach S-bimodule. A bounded crossed homomorphism is a weak*-continuous map $D : S \to E^*$, such that $D(st) = s \cdot D(t) + D(s) \cdot t$ for every

$s,t \in S$ and $\sup_{s\in S}\|D(s)\| < \infty$. If f is in E^*, then $d_f : S \to E^*$ defined by $ad_f(s) = s \cdot f - f \cdot s$ is a bounded crossed homomorphism. Such bounded crossed homomorphisms are called principal.

Definition 3.1. Let S be a topological semigroup. Then S is called Johnson amenable if for every Banach S-bimodule E, every bounded crossed homomorphism from S to E^* is principal.

Proposition 3.2. For a topological semigroup S the following are equivalent.

i. S is left amenable.

ii. For every Banach S-bimodule E with trivial left action, any bounded crossed homomorphism $D : S \to E^*$ is principal.

Proof: First, suppose that S is left amenable. Let E be a Banach S-bimodule and let $D : S \to E^*$ be a bounded crossed homomorphism. For every $x \in E$ we define $\omega_x : S \to \mathbb{C}$ by $\omega_x(s) = \langle D(s), x\rangle$. Then $\|\omega_x\|_\infty = \sup_{s\in S} | \omega_x(s) | \leq \sup_{s\in S}\|D(s)\|\|x\| \leq M\|x\|$, where $M > 0$ is a bound for D. Let $s_\lambda \to s$ in S. Then $D(s_\lambda) \to D(s)$ in the weak*-topology. Thus $\omega_x(s_\lambda) \to \omega_x(s)$, that is, ω_x is continuous. Let $t_\lambda \to t$ in S. Then

$$\begin{aligned}\|\omega_x \cdot t_\lambda - \omega_x \cdot t\|_\infty &= \sup_{s\in S} | \omega_x(t_\lambda s) - \omega_x(ts) | \\ &= \sup_{s\in S} | \langle D(t_\lambda s), x\rangle - \langle D(ts), x\rangle | \\ &\leq | \langle D(t_\lambda) - D(t), x\rangle | + \sup_{s\in S} | \langle D(s), x \cdot t_\lambda - x \cdot t\rangle | .\end{aligned}$$

Since the net $D(t_\lambda)$ is weak* convergent to $D(t)$, then $| \langle D(t_\lambda) - D(t), x\rangle | \to 0$. Also, $\sup_{s\in S} | \langle D(s), x \cdot t_\lambda - x \cdot t\rangle | \leq M\|x \cdot t_\lambda - x \cdot t\|$. Therefore, $\sup_{s\in S} | \langle D(s), x \cdot t_\lambda - x \cdot t\rangle | \to 0$ since $x \cdot t_\lambda \to x \cdot t$ in norm and so $\|\omega_x \cdot t_\lambda - \omega_x \cdot t\|_\infty \to 0$, which implies that $\omega_x \in \mathbf{RUC}(S)$. Now, let m be a left invariant mean on $\mathbf{RUC}(S)$. Define a linear functional f on E by $\langle f, x\rangle = m(\omega_x)$ for every $x \in E$. Then we have,

$$\|f\| = \sup_{\|x\|\leq 1} | \langle f, x\rangle | = \sup_{\|x\|\leq 1} | m(\omega_x) | \leq \sup_{\|x\|\leq 1} \|\omega_x\|_\infty \leq M.$$

Thus $f \in E^*$. For all $x \in E$ and $s,t \in S$, we have

$$\begin{aligned}\omega_{x\cdot s}(t) &= \langle D(t), x \cdot s\rangle \\ &= \langle s \cdot D(t), x\rangle \\ &= \langle D(st), x\rangle - \langle D(s), x\rangle \\ &= \omega_x(st) - \langle D(s), x\rangle 1_S(t).\end{aligned}$$

Therefore, $\omega_{x\cdot s} = \omega_x \cdot s - \langle D(s), x\rangle 1_S$. This implies that

$$\begin{aligned}\langle f - s \cdot f, x\rangle &= \langle f, x\rangle - \langle f, x \cdot s\rangle \\ &= m(\omega_x - \omega_{x\cdot s}) \\ &= m(\omega_x - \omega_x \cdot s - \langle D(s), x\rangle 1_S) \\ &= \langle D(s), x\rangle,\end{aligned}$$

for every $x \in E$ and $s \in S$. Thus $D(s) = f - s \cdot f$ for all $s \in S$, and D is principal.

Conversely, consider the Banach S-bimodule $E = \mathbf{RUC}(S)$ with trivial left action. Let $F = E / \mathbb{C}1_S$. Then F is a Banach S-bimodule and F^* is canonically isometrically isomorphic with the submodule $L = \{f \in E^* : \langle f, 1_S\rangle = 0\}$ of E^*. In particular, L is the dual of a Banach S-bimodule. Let

$f \in E^* \setminus L$ be arbitrary (note that by the Hann-Banach theorem $E^* \setminus L \neq \varnothing$). Define $D : S \to L$ by $D(s) = s \cdot f - f$. Clearly D is a bounded crossed homomorphism. Thus D is principal and so for some $g \in L$ we have $D(s) = s \cdot g - g$. Thus for $h = g - f$, we have $h \neq 0$ and $s \cdot h = h$ for every $s \in S$. Since $\mathbf{RUC}(S)$ is a commutative C*-algebra, there exists a compact Hausdorff space Δ with a canonical left action of S, such that $\mathbf{C}(\Delta)$ and $\mathbf{RUC}(S)$ are isometrically *-isomorphic C*-algebras and isomorphic S-modules. Thus one can consider h as a S-invariant complex Borel regular measure on Δ. Now, $|h| / |h|(\Delta)$ is an invariant mean for S, where $|h|$ denotes total variation measure of h.

With the same argument of Proposition 3.2 one can prove that, S is right amenable, if and only if for every Banach S-bimodule E, with trivial right action, every bounded crossed homomorphism from S to E^* is principal. Thus we have,

Theorem 3.3. Let S be a topological semigroup. If S is Johnson amenable, S is amenable.

Lemma 3.4. Let S be a topological semigroup with a unit element e and let E be a Banach S-bimodule with either zero right action or zero left action. Then any bounded crossed homomorphism from S to E^* is principal.

Proof: Suppose that the right action is zero. Then the left action of S on E^* is zero. If $D : S \to E^*$ is a bounded crossed homomorphism, then for every $s \in S$ we have, $D(s) = D(es) = e \cdot D(s) + D(e) \cdot s = D(e) \cdot s = D(e) \cdot s - s \cdot D(e) = ad_{-D(e)}(s)$. Proof for the other case is similar.

Let S be a topological semigroup with a unit element e. A Banach S-bimodule E is called left-unital (respectively, right-unital) if $e \cdot x = x$ (respectively, $x \cdot e = x$) for all $x \in E$. Also, E is called unital if it is both left and right-unital.

Lemma 3.5. Let S be a topological semigroup with a unit element e and let E be a Banach S-bimodule. Let $F = \{e \cdot x : x \in E\}$ (respectively, $G = \{x \cdot e : x \in E\}$). Then F (respectively, G) is a left-unital (respectively, right-unital) closed submodule of E. Also, if every bounded crossed homomorphism from S to F^* (respectively, G^*) is principal, then every bounded crossed homomorphism from S to E^* is principal.

Proof: We prove the Lemma for F, the other case is similar. Note that F is a left unital submodule of E. Let $x \in E$ be an accumulation point of F. Then there exists a net $(e \cdot x_i)$ in F such that $e \cdot x_i \to x$ in norm. Thus $e \cdot x_i = e \cdot (e \cdot x_i) \to e \cdot x$ in norm. This implies $x = e \cdot x \in F$, thus F is closed. Now, suppose every bounded crossed homomorphism from S to F^* is principal. Let $D : S \to E^*$ be a bounded crossed homomorphism and let $\pi : E^* \to F^*$ be the restriction map. Then π is a bimodule homomorphism and thus $\pi \circ D$ is a bounded crossed homomorphism from S to F^*. Thus there exists $f \in F^*$ such that $\pi \circ D = ad_f$. Let $\overline{f} \in E^*$ be such that $\pi(\overline{f}) = f$ and let $\tilde{D} = D - ad_{\overline{f}}$. Then for every $s \in S$, the functional $\tilde{D}(s)$ vanishes on F. By identification $\{f \in E^* : f|_F = 0\} \cong (E / F)^*$, we can suppose that $\tilde{D}$ is a bounded crossed homomorphism from S to $(E / F)^*$. On the other hand, the left action of S on E / F is zero. Thus by Lemma 3.4, $\tilde{D} = ad_g$ for some $g \in E^*$. Therefore we have $D = ad_{f+g}$.

Proposition 3.6. Let S be a topological semigroup with a unit element e. Then the following are equivalent.

i. S is Johnson amenable.

ii. For every unital Banach S-bimodule E, any bounded crossed homomorphism from S to E^* is principal.

Proof: (i)$\Rightarrow$(ii) is trivial.

(ii)$\Rightarrow$(i) Let E be a Banach S-bimodule and $F_1 = \{x \cdot e : x \in E\}$. By Lemma 3.5, F_1 is a closed right-unital submodule of E. Also, let $F_2 = \{e \cdot x \cdot e : x \in E\}$. By Lemma 3.5, F_2 is a closed (two sided) unital submodule of F_1 and hence any bounded crossed homomorphism from S to F_2^* is principal, thus by Lemma 3.5, any bounded crossed homomorphism from S to F_1^* is principal. Again by Lemma 3.5, any bounded crossed homomorphism from S to E^* is principal. This completes the proof.

Recall that for a topological group G, the map $(g,h) \mapsto gh^{-1}$ from $G \times G$ to G is jointly continuous.

Theorem 3.7. Let G be a topological group. Then the following are equivalent.

i. G is amenable.

ii. G is Johnson amenable.

Proof: (ii) $\Rightarrow$ (i) Follows from Theorem 3.3.

Suppose that G is amenable. By Proposition 3.6, it is enough to prove that if E is a unital Banach G-bimodule and $D : G \to E^*$ is a bounded crossed homomorphism, then D is principal. Let $E_\#$ be a Banach G-bimodule with underlying space E and trivial left action, and the right action defined by $x * g := g^{-1} \cdot x \cdot g$ for $x \in E$ and $g \in G$. Then the dual action of G on $E_\#^*$ becomes, $g * f = g \cdot f \cdot g^{-1}$ and $f * g = f$ for $f \in E^*$. Define $D_\# : G \to E_\#^*$ by $D_\#(g) = D(g) \cdot g^{-1}$ ($g \in G$). Then for any $g, h \in G$, we have

$$\begin{aligned} D_\#(gh) &= D(gh) \cdot (gh)^{-1} \\ &= D(g) \cdot (hh^{-1}g^{-1}) + g \cdot D(h) \cdot (h^{-1}g^{-1}) \\ &= D_\#(g) + g * D_\#(h) \\ &= D_\#(g) * h + g * D_\#(h). \end{aligned}$$

Thus $D_\#$ is a bounded crossed homomorphism and by Proposition 3.2, $D_\#$ is principal. Thus for some $f \in E^*$ and every $g \in G$, we have $D(g) \cdot g^{-1} = D_\#(g) = g * f - f = g \cdot f \cdot g^{-1} - f$, that implies $D(g) = g \cdot f - f \cdot g$. Thus D is principal.

Theorem 3.8. Let G be a locally compact Hausdorff group. Then the following are equivalent.

i. G is amenable.

ii. G is Johnson amenable.

iii. $L^1(G)$ is an amenable Banach algebra.

Proof: (i) and (iii) are equivalent by Johnson's Theorem [1]. (i) and (ii) are equivalent by Theorem 3.7.

Theorem 3.9. Let S be a discrete semigroup. Then S is Johnson amenable if and only if $\ell^1(S)$ is an amenable Banach algebra.

Proof: Suppose that $\ell^1(S)$ is amenable. Let E be a Banach S-bimodule and let $D : S \to E^*$ be a bounded crossed homomorphism. Then E by the actions

$$a \cdot x = \sum_{s \in S} a(s)(s \cdot x), \quad x \cdot a = \sum_{s \in S} a(s)(x \cdot s) \quad (a \in \ell^1(S), x \in E)$$

is a Banach $\ell^1(S)$-bimodule and D can be canonically extended to a bounded derivation $\bar{D}:\ell^1(S)\to E^*$, defined by $\bar{D}(\delta_s)=D(s)$. Thus there is $f\in E^*$ such that $\bar{D}(a)=a\cdot f-f\cdot a$ for all $a\in\ell^1(S)$. Thus $D=ad_f$ and S is Johnson amenable.

Conversely, suppose that S is Johnson amenable. Let E be a Banach $\ell^1(S)$-bimodule and let $D:\ell^1(S)\to E^*$ be a bounded derivation. Then E is a Banach S-bimodule by the actions

$$s\cdot x=\delta_s\cdot x,\quad x\cdot s=x\cdot\delta_s\quad(s\in S,x\in E).$$

Also, $\tilde{D}:S\to E^*$ defined by $\tilde{D}(s)=D(\delta_s)$ is a bounded crossed homomorphism. Thus there is $f\in E^*$ such that for all $s\in S$, $\tilde{D}(s)=s\cdot x-x\cdot s$. This implies that $D(a)=a\cdot f-f\cdot a$ for all $a\in\ell^1(S)$, and thus D is an inner derivation.

4. HEREDITARY PROPERTIES

Proposition 4.1. Let S and T be topological semigroups and let $\phi:T\to S$ be a continuous semigroup homomorphism with dense range. If T is Johnson amenable, then so is S.

Proof: Suppose that T is Johnson amenable. Let E be a Banach S-bimodule and $D:S\to E^*$ be a bounded crossed homomorphism. Then E is a Banach T-bimodule by the action,

$$t\cdot x=\phi(t)\cdot x,\quad x\cdot t=x\cdot\phi(t)\quad(t\in T,x\in E).$$

Also, $D\circ\phi:T\to E^*$ is a bounded crossed homomorphism. Thus there exists $f\in E^*$ such that for all $t\in T$, $D(\phi(t))=t\cdot f-f\cdot t$. Since $\phi(T)$ is dense in S and D is continuous, we have $D(s)=s\cdot f-f\cdot s$ for all $s\in S$.

Corollary 4.2. Let S be a topological semigroup and T be a dense topological subsemigroup of S. Then, if T is Johnson amenable, then so is S.

Proof: Apply Proposition 4.1 with the identity continuous homomorphism $id:T\to S$.

Let G be a locally compact Hausdorff non compact group, and $S=G\cup\{\infty\}$ be its one point compactification. Extend the semigroup operation of G to S by putting $g\infty=\infty g=\infty\infty=\infty$ ($g\in G$). Then S becomes a compact Hausdorff topological semigroup which is not a group and has G as a dense subsemigroup. Thus by Theorem 3.7 and Corollary 4.2, if G is an amenable group, then S is Johnson amenable.

Corollary 4.3. Let S be a semigroup and let τ and τ' be two topologies on S for which S is a topological semigroup, such that $\tau\subset\tau'$. If S is Johnson amenable with topology τ', then S is Johnson amenable with τ.

Proof: Apply Proposition 4.1 with the identity continuous homomorphism $id:(S,\tau')\to(S,\tau)$.

Corollary 4.4. Let $\{S_i\}_{i\in I}$ be a class of topological semigroups. Consider the topological semigroup $\prod_{i\in I}S_i$, with product topology. If $\prod_{i\in I}S_i$ is Johnson amenable, then so is S_i for every $i\in I$.

Proof: For every $j\in I$, consider the canonical projection $\prod_{i\in I}S_i\to S_j$ and apply Proposition 4.1.

Proposition 4.5. Let S be a topological semigroup and let $(I,<)$ be a directed set. Suppose that for any $i \in I$, S_i is a topological subsemigroup of S such that

i. if $i < j$, then $S_i \subset S_j$,

ii. $S_0 = \cup_{i \in I} S_i$ is dense in S, and

iii. there exists a $K > 0$ such that for every $i \in I$, for every Banach S_i-bimodule E and for each bounded crossed homomorphism $D : S_i \to E^*$, there exists $f \in E^*$ with $D(s) = s \cdot f - f \cdot s$ ($s \in S_i$) and $\|f\| \le K \sup_{s \in S_i} \|D(s)\|$. Then S is Johnson amenable.

Proof: By Corollary 4.2, it is enough to prove that S_0 is Johnson amenable. Let E be a Banach S_0-bimodule and $D : S_0 \to E^*$ be a bounded crossed homomorphism. For every $i \in I$, let f_i be in E^* such that $D|_{S_i}(s) = s \cdot f_i - f_i \cdot s$ ($s \in S_i$) and $\|f\|_i \le K \sup_{s \in S_i} \|D(s)\| \le K \sup_{s \in S_0} \|D(s)\|$. Then $(f_i)_{i \in I}$ is a bounded net in E^*, and thus has a weak*-accumulation point f. By passing to a subnet we may assume $f_i w^* \to f$. Now, if $s \in S_0$, then for some i_0, we have $s \in S_i$ for all $i \ge i_0$, and thus for every $x \in E$,

$$\begin{aligned} \langle D(s), x \rangle &= \langle s \cdot f_i - f_i \cdot s, x \rangle \\ &= \langle f_i, x \cdot s - s \cdot x \rangle \\ &\to \langle f, x \cdot s - s \cdot x \rangle \\ &= \langle s \cdot f - f \cdot s, x \rangle. \end{aligned}$$

Thus for every $s \in S_0$, $D(s) = s \cdot f - f \cdot s$ and D is principal. This completes the proof.

Theorem 4.6. Let S and T be topological semigroups with unit elements. If S and T are Johnson amenable, then so is $S \times T$.

Proof: Let e and e' denote the units of S and T, respectively. Consider topological subsemigroups $\hat{S} = S \times \{e'\}$ and $\hat{T} = \{e\} \times T$ of $S \times T$. Clearly, $\hat{S}$ and $\hat{T}$ are Johnson amenable. Let E be a Banach $S \times T$-bimodule and $D : S \times T \to E^*$ be a bounded crossed homomorphism. Then E is canonically a Banach $\hat{S}$-bimodule and a Banach $\hat{T}$-bimodule.

Consider the bounded crossed homomorphism $D|_{\hat{S}} : \hat{S} \to E^*$. By Johnson amenability of $\hat{S}$, there is some $f_0 \in E^*$, such that for all $s \in S$,

$$D(s, e') = (s, e') \cdot f_0 - f_0 \cdot (s, e') \tag{1}$$

Now, consider bounded crossed homomorphism $\tilde{D} := D - ad_{f_0}$ from $S \times T$ to E^*. Then $\tilde{D}|_{\hat{S}} = 0$ and for all $(s,t) \in S \times T$ we have,

$$\begin{aligned} \tilde{D}(s,t) &= \tilde{D}((s,e')(e,t)) \\ &= \tilde{D}(s,e') \cdot (e,t) + (s,e') \cdot \tilde{D}(e,t) \\ &= (s,e') \cdot \tilde{D}(e,t). \end{aligned}$$

Similarly, $\tilde{D}(s,t) = \tilde{D}(e,t) \cdot (s,e')$. Thus if $F = \{f \in E^* : (s,e') \cdot f = f \cdot (s,e') \text{ for all } s \in S\}$, then the range of $\tilde{D}$ is in F. On the other hand if L is the closed linear span of $\{(s,e') \cdot x - x \cdot (s,e') : s \in S, x \in E\}$, then L is a Banach $\hat{T}$-submodule of E. Thus F is the dual of a Banach $\hat{T}$-bimodule, since F is identical with $(E / L)^*$. Therefore, $\tilde{D}|_{\hat{T}} : \hat{T} \to F$ is a bounded crossed homomorphism and thus for some $f_1 \in F \subset E^*$, we have $\tilde{D}|_{\hat{T}} = ad_{f_1}|_{\hat{T}}$, or equivalently, for all $t \in T$,

$$\tilde{D}(e,t) = (e,t) \cdot f_1 - f_1 \cdot (e,t), \tag{2}$$

and for all $s \in S$,

$$(s,e') \cdot f_1 = f_1 \cdot (s,e'). \tag{3}$$

Now from (1), (2) and (3) we have for all $(s,t) \in S \times T$,

$$\begin{aligned} D(s,t) &= D((s,e')(e,t)) \\ &= D(s,e') \cdot (e,t) + (s,e') \cdot D(e,t) \\ &= ((s,e') \cdot f_0 - f_0 \cdot (s,e')) \cdot (e,t) \\ &\quad + (s,e') \cdot ((e,t) \cdot f_1 - f_1 \cdot (e,t) + (e,t) \cdot f_0 - f_0 \cdot (e,t)) \\ &= (s,t) \cdot f_1 - f_1 \cdot (s,t) + (s,t) \cdot f_0 - f_0 \cdot (s,t) \\ &= ad_{f_0 + f_1}(s,t). \end{aligned}$$

This completes the proof.

5. SOME EXAMPLES AND APPLICATIONS

Let A be a Banach algebra. By a *structural semigroup* of A, we mean a subset S of A, such that
i. S is closed under multiplication,
ii. the linear span of S is norm dense in A, and
iii. $\sup_{s\in S}\|s\| < \infty$.
We consider S as a topological semigroup with topology induced by the norm of A.

Theorem 5.1. Let A be a Banach algebra and let S be a structural semigroup of A. If S is Johnson amenable, then A is amenable.

Proof: Let E be a Banach A-bimodule and let $D : A \to E^*$ be a bounded derivation. Then it is easily checked that E is a Banach S-bimodule with the same action as A, and the map $D|_S : S \to E^*$ is a crossed homomorphism. Since D is a bounded derivation and $\sup_{s\in S}\|s\| < \infty$, then $\sup_{s\in S}\|D|_S(s)\| < \infty$. Also, $D|_S$ is continuous in the weak*-topology of E^*, since it is continuous in the norm topology of E^*. Therefore, $D|_S$ is a bounded crossed homomorphism. Since S is Johnson amenable, then there is $f \in E^*$ such that $D(s) = s \cdot f - f \cdot s$ for all $s \in S$. Since A is the closed linear span of S, we have $D(a) = a \cdot f - f \cdot a$ for all $a \in A$, and the proof is complete.

Let A be a non-amenable commutative Banach algebra. Let S be a structural semigroup of A defined by,

$$S = \{a \in A : \|a\| < 1\}.$$

Then S is an amenable topological semigroup since S is commutative. But by Theorem 5.1, S is not Johnson amenable.

Let B be a unital commutative C*-algebra and G be the unitary group of B. Then G is a structural semigroup of B. On the other hand, G is abelian and thus an amenable group. Then by Theorem 3.7, G is Johnson amenable. Thus by Theorem 5.1, B is an amenable Banach algebra.

A Banach algebra A is called *dual*, if there exits a closed submodule A_* of A^* such that $A = (A_*)^*$, see [5, Section 4.4]. Let A be a dual Banach algebra. In what follows, we shall therefore suppose that A always comes with a fixed A_*. It is easily checked that the multiplication of A is separately weak*-continuous. Let E be a Banach A-bimodule. We call E *pre normal* A-bimodule if for each $x \in E$, the maps $a \mapsto a \cdot x$ and $a \mapsto x \cdot a$ from A to E^* are weak*-continuous.

The dual Banach algebra A is called *Connes-amenable,* if for every pre normal Banach A-bimodule E, every weak*-continuous derivation $D: A \to E^*$ is inner. For more details on Connes-amenability, we refer the reader to [5-8].

For the dual Banach algebra A, we call a subset S of A, *dual structural semigroup* of A, if it satisfies (i) and (iii) of the definition of structural semigroup and also satisfies
ii'. the linear span of S is weak*-dense in A.

We always consider the dual structural semigroup S as a topological semigroup with induced weak*-topology of the dual Banach algebra A.
The proof of the following is similar to the proof of Theorem 5.1.

Theorem 5.2. Let A be a dual Banach algebra and S be a dual structural semigroup of A. If S is Johnson amenable, then A is Connes-amenable.

In [8] it was shown that for any locally compact group G, the measure algebra $\mathbf{M}(G)$ is Connes-amenable if and only if G is amenable. Now, we can prove the ``if" part of this result by our method.

Theorem 5.3. Let G be an amenable locally compact group. Then $\mathbf{M}(G)$ is Connes-amenable.

Proof: Let $\delta : G \to \mathbf{M}(G)$ be the usual pointmass measure map. Then δ is a continuous homomorphism in the weak*-topology and convolution product of $\mathbf{M}(G)$. Thus by Theorem 3.7 and Proposition 4.1, the topological semigroup $\delta(G)$ is Johnson amenable. Also, $\delta(G)$ is a dual structural semigroup for dual Banach algebra $\mathbf{M}(G)$. Thus by Theorem 5.2, $\mathbf{M}(G)$ is Connes-amenable.

The following is a Hann-Banach theorem. This is similar to Proposition 2 of [9] in the Banach algebra case, see also [10]. We call an element x of S-module E *symmetric* if for every s in S, $s \cdot x = x \cdot s$.

Theorem 5.4. Let S be a Johnson amenable topological semigroup. Let E be a Banach S-bimodule and F be a Banach submodule of E. Then any symmetric functional in F^* has an extension to a symmetric functional in E^*.

Proof: The quotient Banach space $Y = E / F$ is a Banach S-bimodule by the actions $s \cdot (x + F) = s \cdot x + F$ and $(x + F) \cdot s = x \cdot s + F$ for s in S and x in E. Let f be a symmetric element of F^* and $\hat{f}$ be any continuous extension of f on E. For every s in S, $s \cdot \hat{f} - \hat{f} \cdot s$ is in $F^\perp$, the Banach space of all functional in E^* that vanish on F. Let Q be the canonical isometry from $F^\perp$ onto $(E / F)^*$. Then the map $\delta(s) = Q(s \cdot \hat{f} - \hat{f} \cdot s)$ is a bounded crossed homomorphism, since Q is weak*-weak* continuous and S-bimodule homomorphism. Since S is Johnson amenable, there exists h in $F^\perp$ such that $\delta(s) = s \cdot Q(h) - Q(h) \cdot s$, for all s in S. It follows that $\overline{f} = \hat{f} - h$ is a symmetric extension of f on E.

REFERENCES

1. Johnson, B. E. (1972). Cohomology in Banach algebras. *Mem. Amer. Math. Soc., 127*, 96.
2. Duncan, J. & Namioka, I. (1978). Amenability of inverse semigroups and thier semigroup algebras, *Proceedings of the Royal Society of Edinburgh* 80A, 309-321.
3. Amini, M. (2004). Module amenability for semigroup algebras. *Semigroup Forum*, 69, 243-254.
4. Paterson, A. L. T. (2002). *Amenability*. Amer. Math. Soc. Providence, 1978.
5. Runde, V. (2002). *Lectures on amenability*. Lecture Notes in Mathematics: Vol. *1774*, Berlin, Springer-Verlag.

6. Runde, V. (2001). Amenability for dual Banach algebras. *Studia Math, 148*(1), 47-66.
7. Runde, V. (2004). Dual Banach algebras: Connes-amenability, normal, virtual diagonals, and injectivity of the predual bimodule. *Math.cand.*, 95(1), 124-144.
8. Runde, V. (2003). Connes-amenability and normal, virtual diagonals for measure algebras I. *J. London Math. Soc.*, 67(3), 643-656.
9. Bunce, J. W. (1972). Characterizations of amenable and strongly amenable C*-algebras. *Pacific J. Math, 43*, 563-572.
10. Lau, A. T. (1978). Characterization of Amenable Banach Algebras, *Proc. Amer. Math. Soc.*, *70*(2), 156-160.