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On compact operators on the Riesz B^m-difference sequence space

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Abstract

In this paper, we give the characterization of some classes of compact operators given by matrices on the normed sequence space $r_p^q(B^m)$, which is a special case of the paranormed Riesz B^m-difference sequence space $r^q(p, B^m)$. For this purpose, we apply the Hausdorff measure of noncompactness and use some results.

Keywords: B^m-difference sequence spaces; Hausdorff measure of noncompactness; compact operators

1. Introduction

Let ω be the space of all real valued sequences. Any vector subspace of ω is called a sequence space. We write ℓ_∞, c, c_0 and ϕ the sets of all bounded, convergent, null and finite sequences, respectively. Also, by ℓ_1 and ℓ_p $(1 < p < \infty)$ we denote the sequence spaces of all absolutely and $p-$absolutely convergent series, respectively. Throughout this paper, if $x \in \omega$, then we write $x = (x_k)$ instead of $x = (x_k)_{k=0}^\infty$. Further, we use the conventions that $e = (1,1,\dots)$ and $e^{(k)}$ is the sequence whose only non-zero term is 1 in the kth place for each $k \in \mathbb{N}$, where $\mathbb{N} = \{0,1,2,\dots\}$. Morever, by $\mathcal{F}_r (r \in \mathbb{N})$, we denote the subcollection of $\mathcal{F}$ consisting of all nonempty and finite subsets of N with elements that are greater than r, that is

$$\mathcal{F}_r = \{N \in \mathcal{F}: n > r \text{ for all } n \in N\};\ (r \in \mathbb{N}).$$

It is quite natural to find conditions for a matrix map between BK-spaces to define a compact operator since a matrix transformation between BK-spaces is continuous.

This can be achieved by applying the Hausdorff measure of noncompactness. In the past, several authors characterized classes of compact operators given by infinite matrices on some sequence spaces by using this method [1-12].

Recently, Malkowsky and Rakočević [13], Djolović and Malkowsky [14] and Mursaleen and Noman [15] have established some identities or estimates for the operator norms and Hausdorff measures of noncompactness of linear operators given by infinite matrices that map an arbitrary BK-space or the matrix domains of triangles in arbitrary BK-spaces. Also, Mursaleen [16] has determined the Hausdorff measure of noncompactness on the sequence space $n(\phi)$ of W. L. C. Sargent and applied the technique of measure of noncompactness to the theory of infinite systems of differential equations.

In this paper, by taking a special case of the paranormed Riesz B^m-difference sequence space $r^q(p, B^m)$, we obtain a BK-space and investigate the classes of some compact operators given by matrices on this space by applying the Hausdorff measure of noncompactness and using some results in [15] and [13].

2. Preliminaries and notations

The β-dual of a subset X of ω is defined by

$$X^\beta = \left\{a \in \omega: \sum_k a_k x_k \text{ converges for all } x \in X\right\}.$$

If A is an infinite matrix with real entries a_{nk} $(n, k \in \mathbb{N})$, then we write A_n for the sequence in the nth row of A, that is, $A_n = (a_{nk})_{k=0}^\infty$. Also, if $x = (x_k) \in \omega$, then we define the A-transform of x as the sequence $Ax = (A_n(x))_{n=0}^\infty$, where

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$$A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k \, ; \quad (n \in \mathbb{N})$$

provided the series on the right side converges for each $n \in \mathbb{N}$.

Let X and Y be subsets of ω. We say that A defines a matrix mapping from X into Y, and we denote it by writing $A: X \to Y$, if for every sequence $x = (x_n) \in X$ the sequence $Ax = (A_n(x))$ is in Y. Furthermore, the set

$$X_A = \{x \in \omega : Ax \in X\} \tag{1}$$

is called the matrix domain of A in X and (X, Y) denotes the class of matrices that maps X into Y, that is $A \in (X, Y)$ if and only if $X_A \subset Y$, or equivalently $A_n \in X^{\beta}$ for all $n \in \mathbb{N}$ and $Ax \in Y$ for all $x \in X$.

A sequence space X is called FK-space if it is a complete linear metric space with continuous coordinates $p_n: X \to \mathbb{R}$ $(n \in \mathbb{N})$, where $\mathbb{R}$ denotes the real field and $p_n(x) = x_n$ for all $x = (x_k) \in X$ and every $n \in \mathbb{N}$. A BK-space is a normed FK-space, that is, a BK-space is a Banach space with continuous coordinates. An FK space $X \supset \phi$ is said to have AK if every sequence $x = (x_k) \in X$ has a unique representation $x = \sum_{k=0}^{\infty} x_k e^{(k)}$, that is $x = \lim_{n\to\infty} x^{[n]}$. Here, $x^{[n]}$ is called the n-section of x $(n \in \mathbb{N})$.

The sequence spaces c_0, c and ℓ_∞ are BK-spaces with the usual sup-norm given by $\|x\|_{\ell_\infty} = \sup_{k\in\mathbb{N}} |x_k|$ and the space ℓ_p is a BK- space with the usual ℓ_p-norm defined by $\|x\|_{\ell_p} = (\sum_{k=0}^{\infty} |x_k|^p)^{1/p}$, where $1 \le p < \infty$. Also, all of the c_0 and ℓ_p $(1 \le p < \infty)$ have AK [17, Example 1.13 and 1.20].

Let X be a normed space. Then, we write S_X for the unit sphere in X, that is $S_X = \{x \in X : \|x\| = 1\}$. If X and Y be Banach spaces then $B(X, Y)$ is the set of all continuous linear operators $L: X \to Y$; $B(X, Y)$ is a Banach space with the operator norm defined by $\|L\| = \sup\{\|L(x)\| : \|x\| \le 1\}$ for all $L \in B(X, Y)$.

If $(X, \|.\|)$ is a normed sequence space, then we write $\|a\|_X^* = \sup_{x\in S_X} |\sum_{k=0}^{\infty} a_k x_k|$ for $a \in \omega$, provided the expression on the right hand side exists and is finite, which is the case whenever X is a BK space and $a \in X^{\beta}$ [18, Theorem 7.2.9, p.107]. Throughout, let $1 \le p < \infty$ and q denote the conjugate of p, that is, $q = p/(p-1)$ for $1 < p < \infty$ or $q = \infty$ for $p = 1$.

The following well-known results are fundamental for our investigation.

Lemma 2.1. [18, Theorem 4.2.8]. Let X and Y be BK-spaces. Then we have $(X, Y) \subset B(X, Y)$, that is, every $A \in (X, Y)$ defines a linear operator $L_A \in B(X, Y)$, where $L_A(x) = Ax$ for all $x \in X$.

Lemma 2.2. [17, Theorem 1.29(b)]. Let $1 \le p < \infty$. Then, we have $\ell_p^{\beta} = \ell_q$ and $\|a\|_{\ell_p}^* = \|a\|_{\ell_q}$ for all $a = (a_k) \in \ell_q$.

3. The Riesz B^m-difference sequence space $r_p^q(B^m)$ $(1 \le p < \infty)$

Let us give the definition of some triangle limitation matrices which are used in the text. Let (q_k) be a sequence of positive numbers and

$$Q_n = \sum_{k=0}^{n} q_k \, ; \quad (n \in \mathbb{N}).$$

Then the matrix $R^q = (r_{nk}^q)$ of the Riesz mean is given by

$$r_{nk}^q = \begin{cases} \dfrac{q_k}{Q_n} & (0 \le k \le n) \\ 0 & (k > n). \end{cases}$$

The difference and generalized difference matrices $\Delta = (\Delta_{nk})$ and $B = (b_{nk})$ are defined by

$$\Delta_{nk} = \begin{cases} (-1)^{n-k} & (n-1 \le k \le n) \\ 0 & (0 \le k < n-1) \text{ or } (k > n) \end{cases}$$

and

$$b_{nk} = \begin{cases} r & (k = n) \\ s & (k = n-1) \\ 0 & (0 \le k < n-1) \text{ or } (k > n) \end{cases}$$

for all $k, n \in \mathbb{N}$ and $r, s \in \mathbb{R} - \{0\}$ (for the matrix B see [19], [20]). If we take $r = 1$ and $s = -1$ in the matrix B, then we have $B = \Delta$. Thus, for any sequence space X, the space X_B is more general and more comprehensive than the corresponding consequences of the space X_Δ.

Recently, the generalized B^m-Riesz difference sequence space $r^q(p, B^m)$ has been introduced by Başarır and Kayıkçı [21] as follows:

$$r^q(p, B^m) = \{x = (x_k) \in \omega : T^m(x) \in \ell(p)\}; \quad (1 \le p_n < H),$$

where $\ell(p)$ is the paranormed sequence space defined by Maddox [22] and the matrices $B^m = (b_{nk}^m)$ and $R^q B^m = T^m = (t_{nk}^m)$ are defined by

$$b_{nk}^m = \begin{cases} \binom{m}{n-k} r^{m-n+k} s^{n-k} & (\max\{0, n-m\} \le k \le n) \\ 0 & (0 \le k < \max\{0, n-m\}) \text{ or } (k > n) \end{cases}$$

and

$$t_{nk}^m = \begin{cases} \dfrac{1}{Q_n}\sum_{i=k}^{n}\binom{m}{i-k} r^{m-i+k}s^{i-k}q_i & (k<n) \\ \dfrac{r^m}{Q_n} & (k=n) \\ 0 & (k>n) \end{cases}$$

for all $k, n \in \mathbb{N}$. It is obvious that the matrix B^m reduced the difference matrix Δ^m in case $r = 1$ and $s = -1$, where $\Delta^m = \Delta(\Delta^{m-1})$.

If we take $p_n = p$ for all $n \in \mathbb{N}$, then we have

$$r_p^q(B^m) = \{x = (x_k) \in \omega \colon \textstyle\sum_{n=0}^{\infty}|T_n^m(x)|^p < \infty\};\ (1 \le p < \infty). \tag{2}$$

Also, if we take $m = 1$ in (2) then we have

$$r_p^q(B) = \left\{x = (x_k) \in \omega \colon \sum_{n=0}^{\infty}|T_n(x)|^p < \infty\right\};\ (1 \le p < \infty).$$

With the notation of (1), we can redefine the sequence spaces $r_p^q(B^m)$ and $r_p^q(B)$ as follows:

$$r_p^q(B^m) = (\ell_p)_{T^m} \text{ and } r_p^q(B) = (\ell_p)_T;\ (1 \le p < \infty).$$

It is easy to see that the spaces $r_p^q(B^m)$ and $r_p^q(B)$ are BK-spaces with the norm, respectively, as follows:

$$\|x\|_{r_p^q(B^m)} = \|T^m(x)\|_{\ell_p} = (\textstyle\sum_{n=0}^{\infty}|T_n^m(x)|^p)^{1/p};\ (1 \le p < \infty) \tag{3}$$

and

$$\|x\|_{r_p^q(B)} = \|T(x)\|_{\ell_p} = (\textstyle\sum_{n=0}^{\infty}|T_n(x)|^p)^{1/p};\ (1 \le p < \infty). \tag{4}$$

Throughout, for any sequence $x = (x_k)$, we define the associated sequence $y = (y_k)$, which will be frequently used, as the T^m-transform of x, i.e., $y = T^m(x)$ and so

$$y_k = \frac{1}{Q_k}\sum_{j=0}^{k-1}\left[\sum_{i=j}^{k}\binom{m}{i-j} r^{m-i+j}s^{i-j}q_i\right]x_j + \frac{r^m}{Q_k}q_k x_k;\ (k \in \mathbb{N}). \tag{5}$$

Obviously, if the sequences x and y are connected by the relation (5), then $x \in r_p^q(B^m)$ if and only if $y \in \ell_p$, further if $x \in r_p^q(B^m)$, then $\|x\|_{r_p^q(B^m)} = \|y\|_{\ell_p}$.

In this paper, we characterize classes of compact operators given by infinite matrices from $r_p^q(B^m)$ to c_0, c, ℓ_∞ and ℓ_1. Also, we give the necessary and sufficient conditions for $A \in (r_1^q(B^m), \ell_p)$ to be compact, where $1 \le p < \infty$.

The following result is immediate by [13, Theorem 3.2].

Lemma 3.1. Let

$$\nabla(i,j,k) = (-1)^{j-k}\frac{s^{j-i}}{r^{m+j-i}}\binom{m+j-i-1}{j-i}\frac{1}{q_i};\ (i,j,k,m \in \mathbb{N}).$$

If $a = (a_k) \in \left(r_p^q(B^m)\right)^\beta$, then $\tilde{a} = (\tilde{a}_k) \in \ell_q$ and the equality

$$\textstyle\sum_{k=0}^{\infty} a_k x_k = \sum_{k=0}^{\infty} \tilde{a}_k y_k \tag{6}$$

holds for every $x = (x_k) \in r_p^q(B^m)$, where $y = T^m(x)$ is given by (5) and

$$\tilde{a}_k = Q_k\left(\frac{a_k}{r^m q_k} + \sum_{j=k+1}^{\infty}\left[\sum_{i=k}^{k+1}\nabla(i,j,k)\right]a_j\right);\ (k \in \mathbb{N}).$$

On the other hand, let $1 \le p < \infty$. Then, it can easily be shown that the inclusion $r_p^q(B^m) \supset \phi$ holds if and only if $1/Q \in \ell_p$, where $\frac{1}{Q} = \left(\frac{1}{Q_k}\right)$. So, we shall assume that $1/Q \in \ell_p$ whenever we study the space $r_p^q(B^m)$.

Lemma 3.2. Let $1 \le p < \infty$ and $\tilde{a} = (\tilde{a}_k)$ be defined as in Lemma 3.1. Then, we have

$$\|a\|_{r_p^q(B^m)}^* = \tilde{a}_k = \begin{cases} \left(\sum_{k=0}^{\infty}|\tilde{a}_k|^q\right)^{1/q} & (1 < p < \infty) \\ \sup_k |\tilde{a}_k| & (p = 1) \end{cases}$$

for all $a = (a_k) \in \left(r_p^q(B^m)\right)^\beta$.

Proof: Let $a = (a_k) \in \left(r_p^q(B^m)\right)^\beta$. Then we have from Lemma 3.1 that $\tilde{a} = (\tilde{a}_k) \in \ell_q$ and the equality (6) holds for all $x = (x_k) \in r_p^q(B^m)$ and $y = (y_k) \in \ell_p$, which are connected by the relation (5). Also, we can write by (3) that $x \in S_{r_p^q(B^m)}$ if and only if $y \in S_{\ell_p}$. Thus, we have from (6) that

$$\|a\|_{r_p^q(B^m)}^* = \sup_{x \in S_{r_p^q(B^m)}}\left|\sum_{k=0}^{\infty} a_k x_k\right| = \sup_{y \in S_{\ell_p}}\left|\sum_{k=0}^{\infty}\tilde{a}_k y_k\right|. \tag{7}$$

Further, since $\tilde{a} \in \ell_q$, we get by Lemma 2.2 and (7) that

$$\|a\|_{r_p^q(B^m)}^* = \|\tilde{a}\|_{\ell_p}^* = \|\tilde{a}\|_{\ell_q} < \infty$$

which concludes the proof.

Lemma 3.3. Let X be a sequence space, $A = (a_{nk})$ an infinite matrix and $1 \le p < \infty$. If $A \in$

$(r_p^q(B^m), X)$, then $\tilde{A} \in (\ell_p, X)$ such that $Ax = \tilde{A}y$ for all $x \in (r_p^q(B^m), X)$ and $y \in \ell_p$, where the sequences x and y are connected by the relation (5) and $\tilde{A} = (\tilde{a}_{nk})$ is the associated matrix with $A = (a_{nk})$ defined by

$$\tilde{a}_{nk} = Q_k\left(\frac{a_{nk}}{r^m q_k} + \sum_{j=k+1}^{\infty}\left[\sum_{i=k}^{k+1} \nabla(i,j,k)\right] a_{nj}\right); \quad (n,k \in \mathbb{N}). \tag{8}$$

Proof: Let $x \in r_p^q(B^m)$ and $A \in (r_p^q(B^m), X)$. Then $A_n \in \left(r_p^q(B^m)\right)^\beta$ for all $n \in \mathbb{N}$. Thus, it follows by Lemma 3.1 that $\tilde{A}_n \in (\ell_p)^\beta = \ell_q$ for all $n \in \mathbb{N}$ and the equality $Ax = \tilde{A}y$ holds. Hence, $\tilde{A}y \in X$. Since every $y \in \ell_p$ is the associated sequence of $x \in r_p^q(B^m)$, we obtain that $\tilde{A} \in (\ell_p, X)$. This completes the proof.

4. The Hausdorff measure of noncompactness and compact operators on the space $r_p^q(B^m)$ $(1 \le p < \infty)$

The Hausdorff measure of noncompactness was defined by Goldenštein, Gohberg and Markus in 1957, and later studied by Goldenštein and Markus in 1968.

In this section, we give some classes of compact operators on the space $r_p^q(B^m)$ for $1 \le p < \infty$.

We recall that if X and Y are Banach spaces and L is a linear operator from X to Y, then L is said to be compact if its domain is all of X and for every bounded sequence (x_n) in X, the sequence $(L(x_n))$ has a convergent subsequence in Y. We denote the class of such operators by $\mathcal{C}(X,Y)$.

If (X,d) is a metric space, we write $\mathcal{M}_X$ for the class of all bounded subsets of X. By $B(x,r) = \{y \in X : d(x,y) < r\}$ we denote the open ball of radius $r > 0$ with the centre in x. Then the Hausdorff measure of noncompactness of the set $Q \in \mathcal{M}_X$, denoted by $\chi(Q)$, is given by

$$\chi(Q) = \inf\left\{\varepsilon > 0 : Q \subset \bigcup_{i=0}^{n} B(x_i, r_i),\ \ x_i \in X,\ \ r_i < \varepsilon\ (i = 0,1,\dots,n)\ \ n \in \mathbb{N}\right\}.$$

The function $\chi : \mathcal{M}_X \to [0,\infty)$ is called the Hausdorff measure of noncompactness.

The basic properties of the Hausdorff measure of noncompactness can be found in [17], for example if Q, Q_1 and Q_2 are bounded subsets of a metric space (X,d), then

$$\chi(Q) = 0 \text{ if and only if } Q \text{ is totally bounded}$$

$$Q_1 \subset Q_2 \text{ implies } \chi(Q_1) \le \chi(Q_2).$$

Further, if X is a normed space, then the function χ has some additional properties connected with the linear structure, e.g.

$$\chi(Q_1 + Q_2) \le \chi(Q_1) + \chi(Q_2),$$

$$\chi(\alpha Q) = |\alpha|\chi(Q) \text{ for all } \alpha \in \mathbb{C}.$$

The following lemma is related to the Hausdorff measure of noncompactness of a bounded linear operator.

Lemma 4.1. [17, Theorem 2.25, Corollary 2.26]. Let X and Y be Banach spaces and $L \in B(X,Y)$. Then we have

$$\|L\|_\chi = \chi\big(L(S_X)\big) \tag{9}$$

and

$$L \in \mathcal{C}(X,Y) \text{ if and only if } \|L\|_\chi = 0. \tag{10}$$

Lemma 4.2. [6, Lemma 5.5]. Let Q be a bounded subsets of the normed space X, where X is ℓ_p for $1 \le p < \infty$ or c_0. If $P_n : X \to X$ is the operator defined by $P_n(x) = x^{[n]} = (x_0, x_1, x_2, \dots, x_n, 0, 0, \dots)$ for all $x = (x_k) \in X$, then we have

$$\chi(Q) = \lim_{n\to\infty}\left(\sup_{x \in Q}\|(I - P_n)(x)\|\right).$$

Lemma 4.3. [15, Theorems 3.7 and 3.11]. Let $X \supset \phi$ be a BK-space. Then, we have

(a) If $A \in (X, c_0)$, then

$$\|L_A\|_\chi = \limsup_{n\to\infty}\|A_n\|_X^*$$

and

$$L_A \text{ is compact if and only if } \lim_{n\to\infty}\|A_n\|_X^* = 0.$$

(b) If X has AK and $A \in (X, c)$, then

$$\frac{1}{2}.\limsup_{n\to\infty}\|A_n - \alpha\|_X^* \le \|L_A\|_\chi \le \limsup_{n\to\infty}\|A_n - \alpha\|_X^*$$

and

$$L_A \text{ is compact if and only if } \lim_{n\to\infty}\|A_n - \alpha\|_X^* = 0,$$

where $\alpha = (\alpha_k)$ with $\alpha_k = \lim_{n\to\infty} a_{nk}$ for all $k \in \mathbb{N}$.

(c) If $A \in (X, \ell_\infty)$, then

$$0 \le \|L_A\|_\chi \le \limsup_{n\to\infty}\|A_n\|_X^*$$

and

L_A is compact if $\lim_{n\to\infty}\|A_n\|_X^* = 0$.

(d) If $A \in (X, \ell_1)$, then

$$\lim_{r\to\infty}\left(\sup_{N\in\mathcal{F}_r}\left\|\sum_{n\in N} A_n\right\|_X^*\right) \le \|L_A\|_\chi$$
$$\le 4.\lim_{r\to\infty}\left(\sup_{N\in\mathcal{F}_r}\left\|\sum_{n\in N} A_n\right\|_X^*\right)$$

and

L_A is compact if and only if
$$\lim_{r\to\infty}\left(\sup_{N\in\mathcal{F}_r}\left\|\sum_{n\in N} A_n\right\|_X^*\right) = 0.$$

This lemma gives necessary and sufficient conditions for a matrix transformation from a BK-space X to c_0, c, ℓ_1 and ℓ_∞ to be compact (the only sufficient condition for ℓ_∞). Thus, we have:

Theorem 4.4. Let $1 < p < \infty$ and and $q = p/(p-1)$. Then we have

(a) If $A \in (r_p^q(B^m), c_0)$, then

$$\|L_A\|_\chi = \limsup_{n\to\infty}(\textstyle\sum_{k=0}^\infty |\tilde{a}_{nk}|^q)^{1/q} \tag{11}$$

and

L_A is compact if and only if $\lim_{n\to\infty}(\sum_{k=0}^\infty |\tilde{a}_{nk}|^q)^{1/q} = 0$. (12)

(b) If $A \in (r_p^q(B^m), c)$, then

$$\tfrac{1}{2}.\limsup_{n\to\infty}(\textstyle\sum_{k=0}^\infty |\tilde{a}_{nk} - \tilde{\alpha}_k|^q)^{1/q} \le \|L_A\|_\chi \le \limsup_{n\to\infty}(\sum_{k=0}^\infty |\tilde{a}_{nk} - \tilde{\alpha}_k|^q)^{1/q} \tag{13}$$

and

L_A is compact if and only if $\lim_{n\to\infty}(\sum_{k=0}^\infty |\tilde{a}_{nk} - \tilde{\alpha}_k|^q)^{\frac{1}{q}} = 0$, (14)

where $\tilde{\alpha} = (\tilde{\alpha}_k)$ with $\tilde{\alpha}_k = \lim_{n\to\infty}\tilde{a}_{nk}$ for all $k \in \mathbb{N}$.

(c) If $A \in (r_p^q(B^m), \ell_\infty)$, then

$$0 \le \|L_A\|_\chi \le \limsup_{n\to\infty}(\textstyle\sum_{k=0}^\infty |\tilde{a}_{nk}|^q)^{1/q} \tag{15}$$

and

L_A is compact if $\lim_{n\to\infty}(\sum_{k=0}^\infty |\tilde{a}_{nk}|^q)^{1/q} = 0$. (16)

Proof: (a) Let $A \in (r_p^q(B^m), c_0)$. Since $A_n \in (r_p^q(B^m))^\beta$ for all $n \in \mathbb{N}$, we have from Lemma 3.2 that

$$\|A_n\|_{r_p^q(B^m)}^* = \|\tilde{A}_n\|_{\ell_q} = (\textstyle\sum_{k=0}^\infty |\tilde{a}_{nk}|^q)^{1/q} \tag{17}$$

for all $n \in \mathbb{N}$. Hence, we get (11) and (12) from (17) and Lemma 4.3(a).
Parts (b) and (c) can be proved similarly by using Lemma 4.3(b) and (c).

The conclusions of Theorem 4.4 still hold for $r_1^q(B^m)$ instead of $r_p^q(B^m)$ with $q = 1$, and on replacing the summations over k by the supremums over k.

Theorem 4.5. Let $1 \le p < \infty$. If $A \in (r_1^q(B^m), \ell_p)$, then

$$\lim_{r\to\infty}\left(\sup_k \sum_{n=r}^\infty |\tilde{a}_{nk}|^p\right)^{1/p}. \tag{18}$$

Proof: Let $S = S_{r_1^q(B^m)}$. Then, we have by Lemma 2.1 that $L_A(S) = AS \in \ell_p$. Thus, from (9) and Lemma 4.2 we can write that
a 4.2 we can write that

$$\|L_A\|_\chi = \chi(AS) = \lim_{r\to\infty}\left(\sup_{x\in S}\|(I-P_r)(Ax)\|_{\ell_p}\right), \tag{19}$$

where $P_r: \ell_p \to \ell_p$ $(r \in \mathbb{N})$ is the operator defined by $P_r(x) = (x_0, x_1, \ldots, x_r, 0,0,\ldots)$ for all $x = (x_k) \in \ell_p$.

Now, let $x = (x_k) \in r_1^q(B^m)$. Since $A \in (r_1^q(B^m), \ell_p)$, we obtain from Lemma 3.3 that $\tilde{A} \in (\ell_1, \ell_p)$ and $Ax = \tilde{A}y$, where $y = (y_k) \in \ell_1$ is the associated sequence defined by (5). Therefore, we have that

$$\|(I-P_r)(Ax)\|_{\ell_p} = \|(I-P_r)(\tilde{A}y)\|_{\ell_p}$$
$$= \left(\sum_{n=r+1}^\infty |\tilde{A}_n(y)|^p\right)^{\frac{1}{p}}$$
$$= \left(\sum_{n=r+1}^\infty \left|\sum_{k=0}^\infty \tilde{a}_{nk}y_k\right|^p\right)^{\frac{1}{p}}$$
$$\le \sum_{k=0}^\infty\left(\sum_{n=r+1}^\infty |\tilde{a}_{nk}y_k|^p\right)^{1/p}$$
$$\le \|y\|_{\ell_1}\left(\sup_k\left(\sum_{n=r+1}^\infty |\tilde{a}_{nk}|^p\right)^{1/p}\right)$$

$$= \|x\|_{r_1^q(B^m)} \left(\sup_k \left(\sum_{n=r+1}^{\infty} |\tilde{a}_{nk}|^p \right)^{1/p} \right)$$

for every $n \in \mathbb{N}$. This yields that

$$\sup_{x\in S} \|(I-P_r)(Ax)\|_{\ell_p} \le \sup_k \left(\sum_{n=r+1}^{\infty} |\tilde{a}_{nk}|^p \right)^{1/p}$$

for every $n \in \mathbb{N}$. Hence, from (19) we have that

$$\|L_A\|_\chi \le \lim_{r\to\infty} \left(\sup_k \left(\sum_{n=r+1}^{\infty} |\tilde{a}_{nk}|^p \right)^{1/p} \right). \qquad (20)$$

Conversely, let $e_B^{(k)} \in r_1^q(B^m)$ such that $T^m(e_B^{(k)}) = e^{(k)}$ $(k \in \mathbb{N})$, that is, $e^{(k)}$ is the associated sequence of $e_B^{(k)}$ for each $k \in \mathbb{N}$. Then, we have by Lemma 3.3 that $Ae_B^{(k)} = \tilde{A}e^{(k)} = (\tilde{a}_{nk})_{n=0}^{\infty}$ for every $k \in \mathbb{N}$. Now, let $E = \{e_B^{(k)}: k \in \mathbb{N}\}$. Then, $E \subset S$ and hence $AE \subset AS$ which implies that

$$\chi(AE) \le \chi(AS) = \|L_A\|_\chi. \qquad (21)$$

Morever, we can write from Lemma 4.2 and (21) that

$$\chi(AE) = \lim_{r\to\infty} \left(\sup_k \left(\sum_{n=r+1}^{\infty} \left|A_n(e_B^{(k)})\right|^p \right)^{1/p} \right) = \lim_{r\to\infty} \left(\sup_k \left(\sum_{n=r+1}^{\infty} |\tilde{a}_{nk}|^p \right)^{1/p} \right) \le \|L_A\|_\chi.$$

Thus, we get (18) from (20) and (21).

Corollary 4.6. Let $1 \le p < \infty$. If $A \in (r_1^q(B^m), \ell_p)$, then

L_A is compact if and only if

$$\lim_{r\to\infty} \left(\sup_k \sum_{n=r}^{\infty} |\tilde{a}_{nk}|^p \right)^{1/p} = 0.$$

Proof: This is an immediate consequence of (10) and Theorem 4.5.

Theorem 4.7. Let $1 < p < \infty$ and $q = p/(p-1)$. If $A \in (r_p^q(B^m), \ell_1)$, then

$$\lim_{r\to\infty} \left(\sup_{N\in\mathcal{F}_r} \left(\sum_{k=0}^{\infty} \left| \sum_{n\in N} \tilde{a}_{nk} \right|^q \right)^{1/q} \right) \le \|L_A\|_\chi \le 4. \lim_{r\to\infty} \left(\sup_{N\in\mathcal{F}_r} \left(\sum_{k=0}^{\infty} \left| \sum_{n\in N} \tilde{a}_{nk} \right|^q \right)^{1/q} \right) \qquad (22)$$

and

L_A is compact if and only if

$$\lim_{r\to\infty} \left(\sup_{N\in\mathcal{F}_r} \left(\sum_{k=0}^{\infty} \left| \sum_{n\in N} \tilde{a}_{nk} \right|^q \right)^{1/q} \right) = 0. \qquad (23)$$

Proof: Let $A \in (r_p^q(B^m), \ell_1)$. Since $A_n \in r_p^q(B^m)^\beta$ for all $n \in \mathbb{N}$, we derive from Lemma 3.2 that

$$\left\|\sum_{n\in N} A_n\right\|_{r_p^q(B^m)}^* = \left\|\sum_{n\in N} \tilde{A}_n\right\|_{\ell_q}^*. \qquad (24)$$

Thus, we get (22) and (23) from Lemma 4.3(d) and (24).

Remark: Let

$$\nabla(j,k) = (-1)^{j-k} \left(\frac{s^{j-k-1}}{r^{j-k} q_{k+1}} + \frac{s^{j-k}}{r^{j-k+1} q_k} \right); \ (j,k \in \mathbb{N}).$$

If we take

$$\tilde{a}_{nk} = Q_k \left(\frac{a_{nk}}{r q_k} + \sum_{j=k+1}^{\infty} \nabla(j,k) a_{nj} \right); \ (n,k \in \mathbb{N})$$

then, we can obtain the above same results for the sequence space $r_p^q(B)$ $(1 \le p < \infty)$.

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