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Analytical and numerical solutions of different parabolic heat equations presented in the form of multi-term fractional differential equations

M. Kazemi and G. H. Erjaee*

Mathematics Department, Shiraz University, Shiraz, Iran
E-mails: kazemimahsa63@gmail.com & erjaee@shirazu.ac.ir

Abstract

In this article, we study the analytical solutions of different parabolic heat equations with different boundary conditions in the form of multi-term fractional differential equations. Then we compare these analytical solutions with numerical finite difference methods. This comparison demonstrates the accuracy of the analytical and numerical methods presented here.

Keywords: Parabolic heat equations; fractional differential equations; finite difference methods

1. Introduction

Fractional differential equations have attracted many researchers ([1-4]) due to their important applications in science and engineering, including modeling of anomalous diffusive and sub-diffusive systems, description of fractional random walks, and unification of diffusion and wave propagation phenomena.

Time fractional diffusion-wave equations are obtained from classical diffusion or wave equations by replacing the first or second order time derivative with a fractional derivative of order $0 < \alpha < 1$ or $1 < \alpha < 2$. Such models have been used to analyze electromagnetic acoustics and mechanical responses [5].

Various methods have been used to solve the Fractional Boundary Value Problems (FBVPs). Daftardar-Gejji and Bhalekar [6] have used separation of variables method to solve multi-term diffusion–wave equations. Agrawal [7] has solved fractional diffusion on a bounded domain using the finite sine transform technique. Gaber and El-Sayed [8] have solved FBVPs involving Dirichlet boundary conditions by the Adomian Decomposition Method (ADM). Several iterative methods have also been used for solving linear and non-linear fractional equations. For example, recently Varsha Daftardar-Gejji and Sachin Bhalekar [9] have solved FBVPs with Dirichlet boundary conditions by New Iterative Method, and Odibat and Momani [10] have used ADM to solve time fractional wave equations.

In this article, we first discuss analytical solutions of fractional heat equations with various boundary conditions. For this, we have used the method of separation of variables, similar to [6]. Then we use the Finite Difference Method (FDM) to find numerical solutions to various examples, and compare our analytical solutions to their corresponding numerical solutions.

Our paper is organized as follows: In section 2, basic definitions and notations have been presented, and in section 3, we have discussed analytical solutions of multi-term fractional heat equations by separating variables. In that section, we have derived the solutions to some fractional heat equations with various boundary conditions. In section 4, we have compared our analytical solutions for particular examples with those obtained by the numerical FDM method. We have summarized in section 5.

2. Basic definitions and notations

We begin with some preliminaries and notation from fractional calculus [11-14].

Definition 2.1. A real function $f(x),\ x > 0$, is said to be in space $C_\alpha, \alpha \in \Re$, if there exist a real number P $(> \alpha)$, such that $f(x) = x^P f_1(x)$ where $f_1(x) \in C[0, \infty)$.

Definition 2.2. A real function $f(x),\ x > 0$, is said to be in space $C_\alpha^m, m \in \mathbb{N} \cup \{0\}$, if $f^{(m)} \in C_\alpha$.

*Corresponding author
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Definition 2.3. Let $f \in C_\alpha$ and $\alpha \geq -1$, then the Riemann-Liouville integral of order μ, (μ> 0) is given by

$$I_t^\mu f(x,t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} f(x,t)\, d\tau, \, t > 0.$$

Definition 2.4. let $f \in C_{-1}^m, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, then the expression ${}_{R}D_t^\mu f(x,t) = \frac{\partial^m}{\partial t^m}\left(I_t^{m-\mu} f(x,t)\right)$, $m-1 < \mu < m,\ m \in \mathbb{N},\ t > 0$ is called (left sided) Riemann-Liouville fractional derivative *f*.

Definition 2.5. The (left sided) Caputo fractional derivative of f, $f \in C_{-1}^m$, $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, is defined as

$$D_t^\mu f(x,t) = \frac{\partial^m}{\partial t^m} f(x,t), \quad \mu = m,$$
$$= I_t^{m-\mu} \frac{\partial^m f(x,t)}{\partial t^m}, \quad m-1 < \mu < m, \quad m \in \mathbb{N}.$$

Definition 2.6. Two-parameter Mittag-Leffler function [12] is defined as:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\beta+\alpha k)}, \, (z, \beta \in \mathbb{C};\ \Re(\alpha) > 0).$$

Note that, if $\beta = 1$, then $E_{\alpha,1}(z) = E_\alpha(z)$, and for $\alpha = \beta = 1$ $E_{1,1}(z) = \mathrm{Exp}(z)$.

Definition 2.7. The multivariate Mittag-Leffler function [11 & 12] is defined as:

$$E_{(a_1,a_2,\ldots,a_n),b}(z_1, z_2, \ldots, z_n) = \sum_{k=0}^{\infty} \sum_{\substack{l_1+l_2+\cdots+l_n=k \\ l_1,l_2,\ldots \geq 0}} (k; l_1, l_2, \ldots, l_n) \frac{\prod_{i=1}^n z_i^{l_i}}{\Gamma(b + \sum_{i=1}^n a_i l_i)},$$

where $z_1, \ldots, z_n \in \mathbb{C}$, $a_1, a_2, \ldots, a_n, b \in \mathbb{C}$ and multinomial coefficients $(k; l_1, l_2, \ldots, l_n) = \frac{k!}{l_1! l_2! \ldots l_n!}$.

Theorem 2.1. Let $\mu > 0, m-1 < \mu \leq m, m \in \mathbb{N} \cup \{0\}$, $\lambda \in \mathbb{R}$. Then the Initial Value Problem (IVP)

$$(D_x^\mu y)(x) - \lambda y(x) = g(x), \quad \text{with} \quad y^{(k)}(0) = c_k \in \mathbb{R}, \quad k = 0, \ldots, m-1 \tag{1}$$

has a unique solution in the space C_{-1}^m, of the form

$$y = y_g + y_h, \tag{2}$$

where $g \in C_{-1}$ if $\mu \in \mathbb{N}$ and $g \in C_{-1}^m$ if $\mu \notin \mathbb{N}$.
Here y_g is a solution related to the non-homogeneous part with zero initial conditions and can be evaluated by

$$y_g(x) = \int_0^t t^{\mu-1} E_{\mu,\mu}(\lambda t^\mu) g(x-t) dt \tag{3}$$

and

$$y_h(x) = \sum_{k=0}^{m-1} c_k\, x^k E_{\mu,k+1}(\lambda x^\mu) \tag{4}$$

is a solution for the homogeneous part with the given initial conditions.

Theorem 2.2. Let $\mu > \mu_1 > \cdots > \mu_n \geq 0$, $m_{i-1} < \mu_i \leq m_i, m_i \in \mathbb{N} \cup \{0\}, \lambda_i \in \mathbb{R}$ and $i = 1, \ldots, n$.
Then the IVP

$$(D_x^\mu y)(x) - \sum_{i=1}^n \lambda_i \left(D_x^{\mu_i} y\right)(x) = g(x), y^{(k)}(0) = c_k \in \mathbb{R}, k = 0, \ldots, m-1, m-1 < \mu \leq m, \tag{5}$$

where the function g is as in Theorem (2.1) above, has a unique solution in C_{-1}^m, of the form

$$y(x) = y_g(x) + \sum_{k=0}^{m-1} c_k\, u_k(x), \quad x \geq 0. \tag{6}$$

Here

$$y_g(x) = \int_0^x t^{\mu-1} E_{(\mu-\mu_1,\ldots,\mu-\mu_n),\mu} (\lambda_1 t^{\mu-\mu_1}, \ldots, \lambda_n t^{\mu-\mu_n}) g(x-t) dt \tag{7}$$

is a solution related to the non-homogeneous part with zero initial conditions. Furthermore, the set of functions

$$u_k(x) = \frac{x^k}{k!} + \sum_{i=l_k+1}^n \lambda_i\, x^{k+\mu-\mu_i} E_{(\mu-\mu_1,\ldots,\mu-\mu_n),k+1+\mu-\mu_i} (\lambda_1 t^{\mu-\mu_1}, \ldots, \lambda_n t^{\mu-\mu_n}) \tag{8}$$

fulfill the initial conditions $u_k^{(l)}(0) = \delta_{kl},\ k, l = 0, \ldots, m-1$. The natural numbers l_k, $k = 0, \ldots, m-1$ can be determined from the conditions $m_{l_k} \geq k+1, m_{l_{k+1}} \leq k$.

Theorem 2.3. Let $f \in C_{-1}^m$, $m \in \mathbb{N}$ and $m-1 < \mu \leq m$. Then the Riemann-Liouville and Caputo fractional derivatives are connected by the relation

$${}_{R}D_t^\mu f(x,t) = D_t^\mu f(x,t) + \sum_{k=0}^{m-1} \frac{\partial^k f}{\partial t^k}(x,0) \frac{t^{k-\mu}}{\Gamma(1+k-\mu)}, \quad t > 0.$$

3. Fractional BVPs

Case 1 Homogenous case with homogenous boundary conditions

In this section, first we consider the following multi-term homogeneous fractional heat equations on a bounded domain

$$P(D)u(x,t) = k\frac{\partial^2 u(x,t)}{\partial x^2}, 0 < x < \pi, \ t > 0, \tag{9}$$

with homogenous boundary conditions and initial value

$$u_x(0,t) = u_x(\pi,t) = 0, \quad t > 0, \tag{10}$$

$u(x,0) = f(x), \quad 0 \le x \le \pi, \qquad (11)$

where $P(D) = D_t^{\mu} - \sum_{i=1}^{r-1} \lambda_i D_t^{\mu_i}$, $0 < \mu_{r-1} < \mu_{r-2} < \cdots < \mu_1 < \mu \le 1$, with k and λ_i being constants. Now, similar to the method described in [6] and using separation variables method, we let $u(x,t) = X(x)T(t)$. Then (9) with the boundary conditions (10) implies

$$X''(x) + \omega\, X(x) = 0, \text{ with } X'(0) = X'(\pi) = 0, \qquad (12)$$

and

$$(P(D) + \omega k)T(t) = 0, \qquad (13)$$

where ω is a separation constant. It is well-known that, the Sturm-Liouville problem (12) has eigenvalues $\omega_n = n^2$ with the corresponding eigenfunctions $X_n(x) = \cos(nx)$ $(n = 1,2,\ldots)$. In this case, (13) becomes

$$(P(D) + n^2 k)T(t) = 0 \; (n = 1,2,\ldots). \qquad (14)$$

Now, using Theorem 2.2, the solution of (14) for any fixed positive integer n is

$$T_n(t) = 1 - n^2 k t^{\mu} E_{(\mu-\mu_1,\ldots,\mu-\mu_{r-1},\mu),1+\mu}(\lambda_1 t^{\mu-\mu_1}, \ldots, -n^2 k t^{\mu}).$$

Finally, the solution of the original problem (9) will be

$$u(x,t) \sum_{n=1}^{\infty} C_n \cos(nx) \, (1 - n^2 k t^{\mu} E_{(\mu-\mu_1,\ldots,\mu-\mu_{r-1},\mu),1+\mu}(\lambda_1 t^{\mu-\mu_1}, \ldots, -n^2 k t^{\mu})), \qquad (15)$$

where the C_n are obtained from (11) as follows:

$$C_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx)\, dx \qquad (n = 1,2,\ldots).$$

Case 2 Homogenous case with non-homogenous boundary conditions

For our second case, consider the following multi-term fractional heat equation

$$P(D)u(x,t) = k \frac{\partial^2 u(x,t)}{\partial x^2}, 0 < x < \pi, \;\; t > 0, \qquad (16)$$

with the following non-homogenous conditions

$$u(0,t) = T_1\,, \; u(\pi,t) = T_2, \quad t > 0 \qquad (17)$$

$$u(x,0) = f(x), 0 \le x \le \pi, \qquad (18)$$

where

$$P(D) = D_t^{\mu} - \sum_{i=1}^{r-1} \lambda_i D_t^{\mu_i}\,, \qquad 0 < \mu_{r-1} < \mu_{r-2} < \cdots < \mu_1 < \mu \le 1,$$

k and λ_i are constant, and assume T_1, T_2 are not both zero. Substituting $u(x,t) = U(x,t) + \psi(x)$ in (16)-(18), we obtain

$$P(D)\, U(x,t) = k \frac{\partial^2 U(x,t)}{\partial x^2} + k\psi''(x)\,, \quad 0 < x < \pi, \;\; t > 0\,, \qquad (19)$$

$$U(0,t) + \psi(0) = T_1,$$
$$U(\pi,t) + \psi(\pi) = T_2,$$
$$U(x,0) + \psi(x) = f(x). \qquad (20)$$

Suppose $\psi''(x) = 0$, $\psi(0) = T_1$ and $\psi(\pi) = T_2$. Then $\psi(x) = \left(\frac{T_2 - T_1}{\pi}\right) x + T_1$, and the problem (19)-(20) reduces to the following case

$$P(D)\, U(x,t) = k \frac{\partial^2 U(x,t)}{\partial x^2}, 0 < x < \pi, \;\; t > 0\,, \qquad (21)$$

with

$$U(0,t) = U(\pi,t) = 0, \qquad (22)$$

$$U(x,0) = f(x) - \left(\frac{T_2 - T_1}{\pi}\right) x - T_1. \qquad (23)$$

Now, if we let $U(x,t) = X(x)T(t)$, then (21) with the conditions (22) yields

$$X''(x) + \omega\, X(x) = 0, X(0) = X(\pi) = 0 \qquad (24)$$

and

$$(P(D) + \omega k)T(t) = 0, \qquad (25)$$

where ω is a separation constant, but the Sturm-Liouville problem (24) has the eigenvalues $\omega_n = n^2$ with the corresponding eigenfunctions $X_n(x) = \sin(nx)$ $(n = 1,2,\ldots)$. So the problem (25) becomes

$$(P(D) + n^2 k)T(t) = 0 \qquad (n = 1,2,\ldots). \qquad (26)$$

Now, for any fixed positive integer n, the solution (26) is (cf. Theorem 2.2.)

$$T_n(t) = 1 - n^2 k t^{\mu} E_{(\mu-\mu_1,\ldots,\mu-\mu_{r-1},\mu),1+\mu}(\lambda_1 t^{\mu-\mu_1}, \ldots, -n^2 k t^{\mu}).$$

Therefore, the general solution of (19) can now be presented in the form of

$$U(x,t) = \sum_{n=1}^{\infty} C_n \sin(nx) \, (1 - n^2 k t^{\mu} E_{(\mu-\mu_1,\ldots,\mu-\mu_{r-1},\mu),1+\mu}(\lambda_1 t^{\mu-\mu_1}, \ldots, -n^2 k t^{\mu})), \qquad (27)$$

where the C_n are obtained from condition (23) as follows:

$$C_n = \frac{2}{\pi} \int_0^{\pi} \left(f(x) - \left(\frac{T_2 - T_1}{\pi}\right) x - T_1\right) \sin(nx)\, dx \qquad (n = 1,2,\ldots).$$

Finally, the solution of the original problem (16) is

$$u(x,t) = U(x,t) + \left(\frac{T_2 - T_1}{\pi}\right)x + T_1, \quad (28)$$

where $U(x,t)$ is given in (27).

Case 3. Non-homogenous case with homogenous boundary conditions

Now, for the third form of multi-term fractional heat equations on a bounded domain, we consider the following non-homogeneous case

$$P(D)u(x,t) = k\frac{\partial^2 u(x,t)}{\partial x^2} + \mathrm{g}(t),$$
$$0 < x < \pi,\ t > 0, \quad (29)$$

with the homogenous boundary conditions

$$u(0,t) = 0,\ u(\pi,t) = 0, \quad t \geq 0 \quad (30)$$

and initial condition

$$u(x,0) = f(x),\ 0 \leq x \leq \pi, \quad (31)$$

where

$$P(D) = D_t^{\mu} - \sum_{i=1}^{r-1} \lambda_i D_t^{\mu_i},$$
$$0 < \mu_{r-1} < \mu_{r-2} < \cdots < \mu_1 < \mu \leq 1,$$

with k and λ_i are constant and $\mathrm{g}(t)$ is considered as a continuous function of *t*. It is easy to see that the separation of variables method will not work directly for this equation. Hence, we use the method of variation of parameters [15]. To do this, first we suppose $\mathrm{g}(t) = 0$ and substitute $u(x,t) = X(x)T(t)$ into (29)-(30) to get

$$X''(x) + \omega\, X(x) = 0, X(0) = X(\pi) = 0 \quad (32)$$

and

$$(P(D) + \omega k)T(t) = 0.$$

Here again, ω is a separation constant and the eigenvalues of Sturm-Liouville problem (32) are $\omega_n = n^2$ with the corresponding eigenfunctions $X_n(x) = \sin(nx)$ $(n = 1,2,\dots)$. Obviously, the problem for T can be solved as in the previous cases. Now, we attempt a solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} C_n(t)\sin(nx), \quad (33)$$

but we must determine each $C_n(t)$. Clearly

$$1 = \sum_{n=1}^{\infty} \frac{2[1-(-1)^n]}{n\pi} \sin(nx). \quad (34)$$

By substituting (33) into (29) and using (34), we get

$$\sum_{n=1}^{\infty}[P(D)C_n(t) + n^2 k C_n(t)]\sin(nx) = \sum_{n=1}^{\infty} \frac{2[1-(-1)^n]}{n\pi}\mathrm{g}(t)\sin(nx). \quad (35)$$

By indentifying the coefficients in the sine series on each side of (35), we get

$$P(D)C_n(t) + n^2 k C_n(t) = \frac{2[1-(-1)^n]}{n\pi}\mathrm{g}(t), \qquad n = 1,2,\dots. \quad (36)$$

Now referring to Theorem 2.2, this equation has the general solution

$$C_n(t) = y_{\mathrm{g}}(t) + c_n y_n(t), \quad (37)$$

where

$$y_{\mathrm{g}}(t) = \frac{2[1-1)}{n} n\pi \int_0^t x^{\mu-1} E_{(\mu-\mu_1,\dots,\mu-\mu_{r-1},\mu),\mu}$$
$$(\lambda_1 x^{\mu-\mu_1}, \dots, -n^2 k x^{\mu})\,\mathrm{g}(\mathrm{t}-x)\mathrm{d}x \quad (38)$$

and

$$y_n(t) = 1 - n^2 k t^{\mu} E_{(\mu-\mu_1,\dots,\mu-\mu_{r-1},\mu),1+\mu}(\lambda_1 t^{\mu-\mu_1}, \dots, -n^2 k t^{\mu}). \quad (39)$$

Obviously, $C_n(0) = c_n$ and $C_n(0)$ is obtained from condition (31) as follows:

$$C_n(0) = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$
$$(n = 1,2,\dots). \quad (40)$$

Case 4. Non-homogenous case with non-homogenous boundary conditions

As the fourth case, consider non-homogeneous multi-term fractional heat equation

$$P(D)u(x,t) = k\frac{\partial^2 u(x,t)}{\partial x^2} + \mathrm{g}(t),$$
$$0 < x < \pi,\ t > 0, \quad (41)$$

with non-homogeneous boundary and initial conditions

$$u(0,t) = 0,\ u_x(\pi,t) = B, \quad t > 0, \quad (42)$$

$$u(x,0) = f(x),\ 0 \leq x \leq \pi, \quad (43)$$

where

$$P(D) = D_t^{\mu} - \sum_{i=1}^{r-1} \lambda_i D_t^{\mu_i},$$
$$0 < \mu_{r-1} < \mu_{r-2} < \cdots < \mu_1 < \mu \leq 1.$$

To solve such system as the previous case we let $u(x,t) = U(x,t) + \psi(x)$. Then, substituting this into the equations (41)-(42) yields,

$$P(D)\,U(x,t) = k\left[\frac{\partial^2 U(x,t)}{\partial x^2} + \psi''(x)\right] + \mathrm{g}(t)\,, \quad 0 < x < \pi,\ \ t > 0\,, \tag{44}$$

$$U(0,t) + \psi(0) = 0,$$
$$U_x(\pi,t) + \psi'(\pi) = B,$$
$$U(x,0) + \psi(x) = f(x). \tag{45}$$

Suppose $\psi''(x) = 0,\ \psi(0) = 0$ and $\psi'(\pi) = B$. Then $\psi(x) = Bx$, and the boundary value problem (44)-(45) reduces to the homogeneous case

$$P(D)U(x,t) = k\frac{\partial^2 U(x,t)}{\partial x^2} + \mathrm{g}(t), \quad 0 < x < \pi\,,\ t > 0, \tag{46}$$

$$U_x(\pi,t) = U(0,t) = 0, \tag{47}$$

$$U(x,0) = f(x) - Bx. \tag{48}$$

Now, similar to the case of homogenous boundary condition, if we let $\mathrm{g}(t) = 0$ and substituting $U(x,t) = X(x)T(t)$ into (46)-(47) we get

$$X''(x) + \omega\, X(x) = 0,\ \ X(0) = X'(\pi) = 0 \tag{49}$$

$$(P(D) + \omega k)T(t) = 0,$$

with ω being a separation constant and the eigenvalues of the Sturm-Liouville problem (49) are $\omega_m = (\frac{2m-1}{2})^2$ with corresponding eigenfunctions $X_m(x) = \sin\left(\frac{2m-1}{2}x\right)$ $(m = 1,2,\dots)$. The problem for T is again solved as in the previous cases. We attempt to find a solution of the form

$$U(x,t) = \sum_{m=1}^{\infty} C_m\,(t) \sin\left(\frac{2m-1}{2}x\right), \tag{50}$$

and the problem is again to determine each $C_m(t)$. Clearly, $1 = \sum_{m=1}^{\infty} \frac{4}{(2m-1)\pi} \sin\,(\frac{2m-1}{2}x)$ and by substituting (50) into (46), we get

$$\sum_{m=1}^{\infty}\left[P(D)C_m(t) + k\left(\frac{2m-1}{2}\right)^2 C_m(t)\right]\sin\left(\frac{2m-1}{2}x\right) = \sum_{m=1}^{\infty} \frac{4}{(2m-1)\pi}\mathrm{g}(t)\sin\,(\frac{2m-1}{2}x) \tag{51}$$

By indentifying the coefficients in the sine series on each side of (51), we get

$$P(D)C_m(t) + k(\tfrac{2m-1}{2})^2 C_m(t) = \frac{4}{(2m-1)\pi}\,\mathrm{g}(t). \tag{52}$$

So, the general solution of (52) becomes (cf. Theorem 2.2.)

$$C_m(t) = y_{\mathrm{g}}(t) + c_m y_m(t) \tag{53}$$

where

$$y_{\mathrm{g}}(t) = \frac{4}{(2m-1)\pi}\int_0^t x^{\mu-1}\, E_{(\mu-\mu_1,\dots,\mu-\mu_{r-1},\mu),\mu}\left(\lambda_1 x^{\mu-\mu_1},\dots,-(\tfrac{2m-1}{2})^2 k x^{\mu}\right)\mathrm{g}(\mathrm{t}-x)\mathrm{d}x \tag{54}$$

and

$$y_m(t) = 1 - (\frac{2m-1}{2})^2 k t^{\mu} E_{(\mu-\mu_1,\dots,\mu-\mu_{r-1},\mu),1+\mu}(\lambda_1 t^{\mu-\mu_1},\dots,-(\tfrac{2m-1}{2})^2 k t^{\mu}). \tag{55}$$

Obviously, $C_m(0) = c_m$ where $C_m(0)$ is obtained from condition (48) as follows:

$$C_m(0) = \frac{2}{\pi}\int_0^{\pi} (f(x) - Bx)\sin\left(\frac{2m-1}{2}x\right)dx \quad (m = 1,2,\dots). \tag{56}$$

Hence the solution of the original problem (41) is $u(x,t) = U(x,t) + Bx$ where $U(x,t)$ is given in (50).

Case 5. With a linear function of u as a non-homogenous part

As another case of non-homogeneous multi-term fractional heat equation, we consider the following equation with $-\beta u$ as the non-homogenous part.

$$P(D)u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} - \beta u, \quad 0 < x < \pi\,,\ t > 0, \tag{57}$$

with homogenous boundary conditions

$$u(0,t) = u(\pi,t) = 0,\quad t \geq 0, \tag{58}$$

and initial condition

$$u(x,0) = f(x),\ \ 0 \leq x \leq \pi, \tag{59}$$

where

$$P(D) = D_t^{\mu} - \sum_{i=1}^{r-1}\lambda_i D_t^{\mu_i}\,, \qquad 0 < \mu_{r-1} < \mu_{r-2} < \cdots < \mu_1 < \mu \leq 1.$$

Again, we assume here that $u(x,t) = X(x)T(t)$. Placing this into the (57)-(58), we obtain

$$X''(x) + (\omega - \beta)\,X(x) = 0, X(0) = X(\pi) = 0 \tag{60}$$

$$(P(D) + \omega)T(t) = 0, \tag{61}$$

where ω is a separation constant. In this case, the Sturm-Liouville problem (60) has the eigenvalues $\omega_n = n^2 + \beta$ and the corresponding eigenfunctions $X_n(x) = \sin(nx)$ $(n = 1,2,\dots)$. The problem for T becomes

$$(P(D) + (n^2 + \beta\,))T(t) = 0\ (n = 1,2,\dots) \tag{62}$$

Now, for any fixed positive integer n, the solution (62) is (cf. Theorem 2.2.)

$$T_n(t) = y_g(t) + c_n y_n(t) \tag{63}$$

where $y_g(t) = 0$, and

$$y_n(t) = 1 - (n^2+\beta)t^{\mu} E_{(\mu-\mu_1,\ldots,\mu-\mu_{r-1},\mu),1+\mu}(\lambda_1 t^{\mu-\mu_1},\ldots,-(n^2+\beta)t^{\mu}). \tag{64}$$

Clearly $T_n(0) = c_n$, where $T_n(0)$ is obtained from condition (59) as follows:

$$T_n(0) = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx \quad (n = 1,2,\ldots). \tag{65}$$

Hence the solution of the original problem is

$$u(x,t) = \sum_{n=1}^{\infty} \sin(nx)\, T_n(t), \tag{66}$$

where $T_n(t)$ is as given in (63).

4. Comparison between numerical and analytical solutions

In this section, we first solve some partial fractional differential homogenous and non-homogenous heat equations with different boundary conditions using the method described in the last section. Then we compare these solutions with those found by numerical FDM.

Example 1. Consider the partial fractional homogeneous differential equation

$$D_t^{0.8} u = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < \pi,\ t > 0, \tag{67}$$

along with the boundary and initial conditions

$$u_x(0,t) = u_x(\pi,t) = 0, \quad t > 0,$$
$$u(x,0) = \cos(x), \qquad 0 \le x \le \pi.$$

In view of (15) the solution of this equation is

$$u(x,t) = \sum_{n=1}^{\infty} C_n \cos(nx)\left(1 - n^2 t^{0.8} E_{0.8,1.8}(-n^2 t^{0.8})\right),$$
$$n = 1,2,\ldots,$$

where $C_n = \frac{2}{\pi}\int_0^{\pi} \cos(x)\cos(nx)\,dx$ $(n = 1,2,\ldots)$. Hence, $C_1 = 1$, $C_n = 0$ $(n = 2,3,\ldots)$, and

$$u(x,t) = \cos(x)\left(1 - t^{0.8} E_{0.8,1.8}(-t^{0.8})\right) = \cos(x)\,(1 - t^{0.8}\sum_{k=0}^{\infty}\frac{(-t^{0.8})}{\Gamma(1.8+0.8k)}). \tag{68}$$

Using MAPLE this solution is illustrated in Fig. 1(a).

Now, to compare the solution, we apply finite-difference method to (67). We obtain $u_{i,j+1} = ru_{i-1,j} + (1-2r)u_{i,j} + ru_{i+1,j}$, where $u_{i,j} = u(ih,jk)$, $x_i = ih$, $(i = 0,1,2,\ldots)$, $t_j = jk$, $(j = 0,1,2,\ldots)$ and $r = \frac{k}{h^2}$. Taking $h = \frac{\pi}{10}$ and $= \frac{\pi}{1000}$, the results are illustrated in Fig. 1(b).

As we can see in Fig. 1 both solutions are completely similar.

Example 2. Consider the following fractional non-homogeneous differential equation

$$D_t^{0.8} u = \frac{\partial^2 u}{\partial x^2} + t,\ 0 < x < \pi,\ t > 0 \tag{69}$$

$$u(0,t) = 0, \quad u_x(\pi,t) = 1, \qquad t \ge 0$$
$$u(x,0) = x, \quad 0 < x < \pi.$$

This is non-homogenous case with non-homogenous boundary conditions. Utilizing equations (54)-(56) yields

$$u(x,t) = \sum_{m=1}^{\infty} sin\left(\frac{2m-1}{2}x\right)\left[\frac{4}{2m-1}\int_0^t x^{-0.2} E_{0.8,0.8}\left(-\left(\frac{2m-1}{2}\right)^2 x^{0.8}\right)(t-x)dx\right] + x. \tag{70}$$

This solution illustrated in Fig. 2(a).

Now, using forward-difference and central-difference approximation for u_t and u_{xx}, respectively, (69) can be written as $u_{i,j+1} = ru_{i-1,j} + (1-2r)u_{i,j} + ru_{i+1,j} + jk^2$, where $u_{i,j} = u(ih,jk)$, $x_i = ih$, $(i = 0,1,2,\ldots)$, $t_j = jk$, $(j = 0,1,2,\ldots)$ and $r = \frac{k}{h^2}$. Taking $h = \frac{\pi}{10}$, $k = \frac{\pi}{1000}$ and $r = \frac{1}{10\pi}$ the solution result is plotted in Fig. 2(b). As we can see, again the results are so similar.

Example 3. Consider the following fractional non-homogeneous differential equation

$$D_t^{0.9} u = \frac{\partial^2 u}{\partial x^2} + t, \qquad 0 < x < \pi,\ t > 0, \tag{71}$$

along with the boundary and initial conditions

$$u(0,t) = u(\pi,t) = 0, \qquad t \ge 0$$
$$u(x,0) = \sin(x), \quad 0 < x < \pi.$$

Using (33), (37) the solution of this problem turns out to be

$$u(x,t) = \left[\frac{4}{\pi}\int_0^t x^{-0.1}\left(E_{0.9,0.9}(-x^{0.9})\right)(t-x)dx + \left(1 - t^{0.9}E_{0.9,1.9}(-t^{0.9})\right)\right]\sin(x) + \sum_{n=2}^{\infty}\left[\frac{2\left(1+(-1)^{n+1}\right)}{n\pi}\int_0^t x^{-0.1}\left(E_{0.9,0.9}(-n^2 x^{0.9})\right)(t-x)dx\right]\sin(nx). \tag{72}$$

Using MAPLE this solution illustrated in Fig. 3.

Example 4. Consider the following fractional differential equation

$$D_t^{0.9}u = \frac{\partial^2 u}{\partial x^2} - u, \quad 0 < x < \pi, \ \ t > 0, \tag{73}$$

along with the boundary conditions

$$u(x,0) = \sin(x), \qquad 0 < x < \pi$$
$$u(0,t) = u(\pi,t) = 0, \quad t \ge 0.$$

Using (63), (66) we get the analytical solution of this boundary value problem as

$$u(x,t) = \sin(x)\left(1 - 2t^{0.9}\sum_{k=0}^{\infty}\frac{(-2t^{0.9})^k}{\Gamma(1.9+0.9k)}\right) = \sin(x)\sum_{k=0}^{\infty}\frac{(-2t^{0.9})^k}{\Gamma(1+0.9k)} = \sin(x)\,E_{0.9}(-2t^{0.9}). \tag{74}$$

$u(x,t)$ is plotted in Fig. 4.

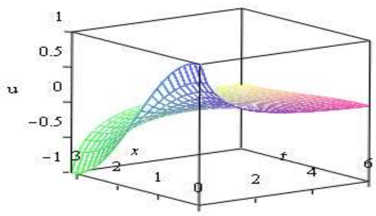

Fig. 1(a). Analytical solution of Example 1 using MAPLE

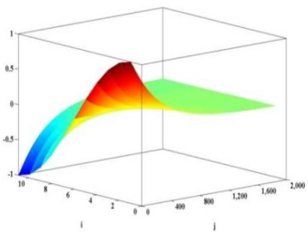

Fig. 1(b). Numerical solution of Example 1 using FDM

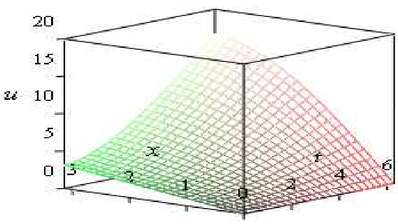

Fig. 2(a). Analytical solution of Example 2 using MAPLE

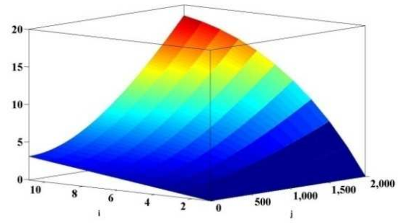

Fig. 2(b). Numerical solution of Example 2 using forward-difference and central-difference

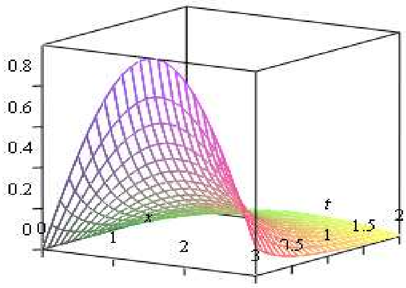

Fig. 3. Analytical solution of Example 3 using MAPLE

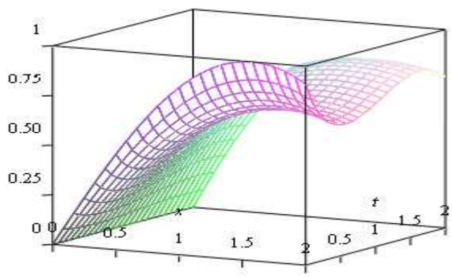

Fig. 4. Analytical solution of Example 4 using MAPLE

5. Conclusion

We have determined the analytical solutions of various homogeneous and non-homogeneous partial differential equations presented by fractional differential equations with various Dirichlet and Neumann boundary conditions. We have illustrated these solutions and compared them with the results found by finite difference method, demonstrating the accuracy of our solutions in these cases.

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