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On Matsumoto metrics of special Ricci tensor

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Abstract

In this paper, the Matsumoto metric with special Ricci tensor has been investigated. It is proved that, if α is of positive (negative) sectional curvature and F is of α-parallel Ricci curvature with constant killing 1-form β, then (M,F) is a Riemannian Einstein space. In fact, we generalize the Riemannian result established by Akbar-Zadeh.

***Keywords**:* Matsumoto metric; Ricci parallel tensor; Einstein Finsler space

1. Introduction

One of the most important problems in Finsler geometry is to understand the geometric meanings of various quantities and their impacts on the global geometric structures. The flag curvature K, which is obtained by the Riemannian curvature, tells us how curved the Finsler manifold is at a specific point. Moreover, there are several important non-Riemannian quantities in Finsler geometry: the Cartan torsion C, the Berwald curvature B, the Landsberg curvature L, and the well-known S curvature, etc. They all vanish for Riemannian metrics, hence they are said to be non-Riemannian. These quantities interact with the flag curvature in a fragile way.

(α,β)-Metrics were introduced in 1972 by M.Matsumoto [1]. The study of Finsler spaces with (α,β)-metrics is quite old, but it is a very important aspect of Finsler geometry and its applications (see [2-5]). An (α,β)-metric is a scalar function on TM defined by $F=\Phi(\frac{\beta}{\alpha})\alpha$, $s=\beta/\alpha$ where $\Phi=\Phi(s)$ is a C^{∞} on (b_0,b_0) with certain regularity, $\alpha:=\sqrt{a_{ij}(x)y^iy^i}$ is a Riemannian metric and $\beta:=b_i(x)y^i$ is a 1-form in the manifold M. Therefore, (M,α) is called the associated Riemannian manifold. A Finsler space is a manifold M equipped with a family of smoothly varying Minkowsky norms; one on each tangent space, Riemannian metrics are examples of Finsler norms that are induced from an inner-product.

Some especially interesting examples of (α,β)-metrics are the Randers metric, Matsumoto metric and Berwald metric, $F=\frac{(\alpha+\beta)^2}{\alpha}$. Randers metric and its Ricci tensor are related via their history in physics. The well-known Ricci tensor was introduced in 1904 by G. Ricci. Nine years later Ricci's work was used to formulate Einstein's theory of gravitation. Einstein metrics are defined in the next section but, loosely, we will say a Finsler metric F is Einstein if the average of its flag curvatures at a flag pole y is a function of position x alone, rather than the a priori position x and flag pole y. C. Robles investigated Randers Einstein metrics in her thesis in 2003. She obtained the necessary and sufficient conditions for Randers metric to be Einstein and by using Einstein *Zermelo navigation* description, she proved the pair (h,W) of a Riemannian metric and an appropriate vector field W has been founded in [6].

Put $\mathrm{H}_{\mathrm{ij}}=\mathrm{H}^{\mathrm{k}}_{\mathrm{ikj}}$; denote the canonical section of the vector bundle π^*TM and the vertical derivation with respect to y^i by v and $\dot{\partial}_i$, respectively. For an (α,β)-metric $F=\Phi(\beta/\alpha)\alpha$, by using the geodesic coefficient of α, we can introduce a new geometric quantity. Let us denote the Levi-Civita connection of α by $\tilde{\nabla}$. We define the Ricci tensor $\overline{H}$ and $\tilde{H}$ on π^*TM as follows:

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$$\overline{H}_{ij} = \frac{1}{2}\dot{\partial}_i \dot{\partial}_j H(v,v)$$

$$\tilde{H}(X,Y) := \tilde{\nabla}_{\hat{V}} \overline{H}(X,Y), \qquad \hat{X}, \hat{Y} \in \chi(TM_0),$$

$$X = \pi_*(\hat{X}), \qquad Y = \pi_*(\hat{Y}),$$

where, $\hat{v}$ is the geodesic spray associated with α. The curvature $\tilde{H}$ is closely related to the Ricci curvature and its related to (α,β)-metrics, especially to the associated Riemannian manifold (M,α). In this paper we investigate an (α,β)-metric of α-parallel Ricci curvature, and we prove the following theorem:

Theorem 1.1. Let $F = \dfrac{\alpha^2}{\alpha-\beta}$ be a Matsumoto metric on a connected manifold M of dimension n. Suppose that α is of positive (negative) sectional curvature and $\tilde{H} = 0$, $(H(v,v) \neq 0)$ and β is a constant killing 1-form. Then, (M,F) is a Riemannian Einstein space.

In fact, we generalize the Riemannian result established by Akbar-Zadeh in [7].

2. Preliminaries

Let M be an n-dimensional C^∞ manifold. Denote by $T_\chi M$ the tangent space at $x \in M$, and by $TM = \cup_{x\in M} T_\chi M$ the tangent bundle of M. Each element of TM has the form (x,y), where $x \in M$ and $y \in T_\chi M$. Let $TM_0 = TM \setminus \{0\}$. The natural projection $\pi : TM \to M$ is given by $\pi(x,y) = x.$ The pull-back tangent bundle π^*TM is a vector bundle over TM_0 whose fiber $\pi^*_v TM$ at $v \in TM_0$ is just $T_\chi M$ where $\pi(v) = x$. Then

$$\pi^*TM = \{(x,y,v) \setminus y \in T_x M_0, v \in T_x M\}.$$

A Finsler metric on a manifold M is a function $F : TM \to [0,\infty)$ which has the following properties:

(i) F is C^∞ on TM_0;

(ii) $F(x,\lambda y) = \lambda F(x,y) \qquad \lambda > 0;$

(iii) For any tangent vector $y \in T_x M$, the vertical Hessian of $\dfrac{F^2}{2}$ given by

$$g_{ij}(x,y) = [\frac{1}{2}F^2]_{y^i y^j}$$

is positive definite.

Every Finsler metric F induding a spray $G = y^i \dfrac{\partial}{\partial x^i} - 2G^i(x,y)\dfrac{\partial}{\partial y^i}$ is defined by

$$G^i(x,y) := \frac{1}{4} g^{il}(x,y)\{2\frac{\partial g_{jl}}{\partial x^k}(x,y) - \frac{\partial g_{jk}}{\partial x^l}(x,y)\} y^j y^k,$$

where the matrix (g^{ij}) means the inverse of matrix (g_{ij}), and the coefficients G^i_j, G^i_{jk} and hv-curvature G^i_{jkl} of the Berwald connection can be derived from the spray G^i as follows:

$$G^i_j = \frac{\partial G^i}{\partial y^j}, \qquad G^i_{jk} = \frac{\partial G^i_j}{\partial y^k}, \qquad G^i_{jkl} = \frac{\partial G^i_{jk}}{\partial y^l}.$$

When $F = \sqrt{a_{ij}(x) y^i y^j}$ is a Riemannian metric, $K^i_k = R^i_{jkl}(x) y^j y^l$ where $R^i_{jkl}(x)$ denote the coefficients of the usual Riemannian curvature tensor. Thus, the Ricci scalar function of F is given by

$$\rho := \frac{1}{F^2} K^i_{i,} \qquad H(v,v) = K^i_i.$$

Therefore, the Ricci scalar function is positive homogeneous of degree 0 in y. This means $\rho(x,y)$ depends on the direction of the flag pole y, but not its length.

$$H_{ij} := \frac{1}{2}\dot{\partial}_i \dot{\partial}_j H(v,v).$$

A Finsler manifold (M,F) is called an *Einstein space* if there exists a differentiable function c defined on M such that $H(v,v) = cF^2$. The Ricci identity for a tensor W_{jm} of π^*TM is given by the following formula:

$$D_k D_l W_{jm} - D_l D_k W_{jm} = -W_{rm} H^r_{jkl}$$
$$-W_{jr} H^r_{mkl} - \frac{\partial W_{jm}}{\partial y^r} H^r_{okl} ,$$

where, D_k denotes the horizontal covariant derivative whit respect to $\{\frac{\delta}{\delta x^k}\}$ in the Berwald connection. Let (M,F) be an n-dimensional Finsler space. For every $x \in M$, assume $S_x M = \{y \in T_x M \setminus F(x,y)=1\}$. $S_x M$ is called the indicatrix of F at $x \in M$ and is a compact hyper surface of $T_x M$, for every $x \in M$. Let $v: S_x M \to T_x M$ be its canonical embedding; where $\|v\|=1$ and (t,U) be a coordinate system on $S_x M$. Then, $S_x M$ is represented locally by $v^i = v^i(t^\alpha), \quad \alpha = 1,2,...,(n-1).$ One can easliy show that:

$$\frac{\partial}{\partial v^i} = F \frac{\partial}{\partial y^i}$$

The $(n-1)$ vectors $\{(v^i_\alpha)\}$ from a basis for the tangent space of $S_x M$ in each point, where $v^i_\alpha = \frac{\partial v^i}{\partial t^\alpha}, \quad \alpha = 1,2,...,(n-1).$ For the sake of simplicity, put $\partial_\alpha = \frac{\partial}{\partial t^\alpha}$. It can be easily shown that

$$\partial_\alpha = F v^i{}_\alpha \frac{\partial}{\partial y^i}$$

$g = g_{ig}(x,y) dy^i dy^j$ is a Riemannian metric on $T_x M$. Inducing g in $S_x M$, one gets the Riemannian metric $\bar{g} = \bar{g}_{\alpha\beta} dt^\alpha dt^\beta$, where $\bar{g}_{\alpha\beta} = v^i_\alpha v^i_\beta g_{ij}$. The canonical unit vertical vector field $V(x,y) = y^i \frac{\partial}{\partial y^i}$ together the $(n-1)$ vectors ∂_α, from the local basis for $T_x M$, $B = \{u^1, u^2, ..., u^n\}$ where, $u^\alpha = (v^i_\alpha)$ and $u^n = V$. We conclude that $g(V, \partial_\alpha) = 0$ that is $y_i v^i_\alpha = 0$.

Let (M, F) be an n-dimensional Finsler space equipped with an (α,β)-metric F, where

$$a(x,y) = \sqrt{a_{ij}(x) y^i y^j}, \qquad \beta(x,y) = b_i(x) y^i ,$$

M. Matsumoto [2] showed that G^i of (α,β)-metric space are given by

$$2G^i = \gamma^i_{00} + 2B^i ,$$

where

$$B^i = (E/\alpha) y^i + (\alpha F_\beta / F_\alpha) s^i_0$$
$$- (\alpha F_{\alpha\alpha} / F_\alpha) C\{(y^i/\alpha) - (\alpha/\beta) b^i ,$$

$$E = (\beta F_\beta / F) C, \; C = \alpha\beta (r_{00} F_\alpha$$
$$- 2\alpha s_0 F_\beta) / 2(\beta^2 F_\alpha + \alpha \gamma^2 F_{\alpha\alpha}),$$

$$b^i = a^{ir} b_r , \qquad b^2 = b^r b_r , \qquad \gamma^2 = b^2 \alpha^2 - \beta^{2,}$$

$$r_{ij} := \frac{1}{2}(\tilde{\nabla}_j b_i + \tilde{\nabla}_i b_i), \qquad s_{ij} := \frac{1}{2}(\tilde{\nabla}_j b_i),$$

$$s^i_j := a^{ih} s_{hj} , \qquad s_j := b_i s^i_j .$$

The matrix (a^{ij}) means the inverse of matrix (a^{ij}). The function γ^i_{jk} stands for the Christoffel symbols in the space (M,α), and the suffix 0 means transacting length with respect to α, equivalently

$$r_{ij} = 0, \qquad and \qquad s_i = 0 .$$

In an n-dimensional coordinate neighborhood U, we consider a liner partial differential equation of second order,

$$L(\varphi) = g^{ik} \frac{\partial^2 \varphi}{\partial x^i \partial x^k} + h^i \frac{\partial \varphi}{\partial x^i}$$

where $g^{jk}(x)$ and $h^i(x)$ are continuous function of point x in U, and quadratic form $g^{jk} Z_j Z_k$ is supposed to be positive definite everywhere in U. Then we call L an elliptic differential operator.

Strong Maximum principle: In coordinate neighborhood U, if a function $\varphi(p)$ of class C^2

satisfies

$$L(\varphi) \geq 0$$

where $\varphi : M \to R^n$, and if there exists a fixed point p_o in U such that $\varphi(p) \leq \varphi(p_0)$, $\forall_P \in U$, then we have $\varphi(p) = \varphi(p_0)$, $\forall\ p \in U$. If φ have absolute maximum in U, then φ is constant on U.

3. Proof of Theorem 1.1

In this section, we consider the (α,β)-metrics where α is of positive (negative) sectional curvature. Let $\{\frac{\delta}{\delta x^i}\}$ and $\{\frac{\hat{\delta}}{\hat{\delta} x^i}\}$ be the natural locally horizontal basis of TTM_0 with respect to F and α, respectively. To prove the theorem 1.1, we need the following:

Proposition 3.1. Let F $= \Phi(\frac{\beta}{\alpha})\alpha$ be an (α,β)-metric on a connected manifold M. Suppose that α is of positive (negative) sectional curvature. Then, we have $H(v,v) = c\alpha$ $, c \in R,$ if and only if $\overline{H} = 0$.

Proof: Denote the Riemann curvature of α by $\tilde{R}^i_{jkl}$ *and* $\frac{\partial}{\partial y^i}$ denote $\dot{\partial}_i$ for the sake of simplicity. By using the Ricci identity for $\overline{H}_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j H(v,v)$, with respect to $\tilde{\nabla}$, one obtains:

$$\begin{aligned} &\tilde{\nabla}_k\tilde{\nabla}_l\overline{H}_{ij} - \tilde{\nabla}_l\tilde{\nabla}_k\overline{H}_{ij} = -\overline{H}_{rj}\tilde{R}^r_{ikl} \\ &-\overline{H}_{ir}\tilde{R}_{jkl} - \dot{\partial}_r\overline{H}_{ij}\tilde{R}^r_{okl} \end{aligned} \tag{1}$$

Multiply the above relation by v^i, we get:

$$\begin{aligned} &\tilde{\nabla}_k\tilde{\nabla}_l\overline{H}_{oj} - \tilde{\nabla}_l\tilde{\nabla}_k\overline{H}_{oj} = -\overline{H}_{rj}\tilde{R}^r_{ikl} \\ &-\overline{H}_{ir}\tilde{R}_{okl} - \overline{H}_{o}\tilde{R}^r_{jkl}. \end{aligned} \tag{2}$$

One can observe that:

$$0 = \tilde{H}_{ij} = \tilde{\nabla}_0\overline{H}_{ij} = \tilde{\nabla}_i\overline{H}_{oj} = \tilde{\nabla}_j\overline{H}_{io}, \tag{3}$$

Multiplying (2) by v^i :

$$0 = \tilde{\nabla}_k\tilde{\nabla}_0\overline{H}_{0j} - \tilde{\nabla}_0\tilde{\nabla}_k\overline{H}_{oj} = -\overline{H}_{rj}\tilde{R}^r_{0k0} - \overline{H}_{0r}\tilde{R}_{jk0} \tag{4}$$

By (4) we have:

$$0 = \tilde{R}^r{}_{0k0}\dot{\partial}_r\dot{\partial}_j H(v,v) + \tilde{R}^r{}_{jk0}\dot{\partial}_r H(v,v). \tag{5}$$

Multiplying (5) by a^{jk} :

$$\tilde{R}^{r\,k}_{0\,0}\dot{\partial}_r\dot{\partial}_k H(v,v) + a^{jk}\tilde{R}^r_{jko}\dot{\partial}_r H(v,v) = 0 \tag{6}$$

Define the operator γ as follows:

$$\Upsilon := \tilde{R}^{rk}_{00}\dot{\partial}_r\dot{\partial}_k + a^{jk}\tilde{R}^r_{jko}\dot{\partial}_r. \tag{7}$$

The Riemannian manifold (M,α) has a positive (negative) sectional curvature, it results in the second order partial differential operator Υ being elliptic. From expression $\rho = H(v,v)/\alpha^2$, we have:

$$\partial_\eta\rho = \alpha^2 v^i_\eta\dot{\partial}_i\rho, \tag{8}$$

and then

$$\begin{aligned} &\partial_\beta\partial_\eta\rho = \alpha\partial_\beta v^i_\eta\dot{\partial}_p\rho + \alpha^2 v^i_\eta v^j_\beta\dot{\partial}_i\dot{\partial}_{jp} \\ &+\alpha v^j_\beta(\dot{\partial}_j v^i_\eta)\dot{\partial}_i\rho. \end{aligned} \tag{9}$$

since

$$v^j_\beta\frac{\partial\alpha}{\partial y^j} = 0,$$

we get

$$\partial_B\partial_\eta\rho = \alpha\partial_\beta v^i_\eta\dot{\partial}_i\rho + \alpha^2 v^i_\eta v^j_\beta\dot{\partial}_i\dot{\partial}_j\rho. \tag{10}$$

Multiplying the two sides of (10) by $\tilde{R}^{\alpha\beta} := \tilde{R}^{\alpha\ \beta}_{n\ n}$, we obtain:

$$\tilde{R}^{\eta\beta}\partial_\beta\partial_\eta\rho = \tilde{R}^{ij}_{00}\dot{\partial}_i\dot{\partial}_j\rho + \alpha\tilde{R}^{\eta\beta}\partial_\beta v^i_\eta\partial_i\rho. \tag{11}$$

It follows that:

$$\begin{aligned} &\Upsilon(\rho) := \tilde{R}^{\eta\beta}\partial_\beta\partial_\eta\rho + B^\alpha\partial_a\rho = 0, \\ &(\alpha,\beta = 1,\dots,n-1) \end{aligned} \tag{12}$$

where $B^\eta := v^\eta_i\tilde{R}^i_{\beta\gamma}\tilde{a}^{\beta\gamma} - \alpha\tilde{R}^{\beta\gamma}\partial_\gamma v^\eta_\beta.$ The equation (12) can be viewed as elliptic PDE on each indicatrix $S_x M$ and using the maximum principle of Hopf, we find ρ as a function of x only. Therefore, there is a function $c(x)$ such that $H(v,v) = c(x)\alpha^2$. Since it must satisfy $\tilde{\nabla}_0 H(v,v) = 0$ it results in the converse being true, since $\tilde{\nabla}_0 H(v,v) = 0$.

Now we consider the case of Matsumoto spaces. Matsumoto metric is of the form $F = \frac{\alpha^2}{\alpha - \beta}$. In [8], the authors have obtained the following relation between $H(v,v)$ and $\tilde{R}(v,v)$ for Matsumoto metrics with constant killing 1-form β:

$$H(v,v) = \tilde{R}(v,v) + \frac{-2\alpha^3}{(\alpha - 2\beta)^3} s^i_0 s_{i0} + \frac{2\alpha^2}{\alpha - 2\beta} \tilde{\nabla}_i s^i_0 - \frac{\alpha^4}{(\alpha - 2\beta)^2} s^{ij} s_{ij}. \tag{13}$$

Let $H(v.v) = c\alpha^2$, where $c \in R$. We obtain:

$$0 = \tilde{R}(v,v) - \frac{2\alpha^3}{(\alpha - 2\beta)^3} s^i_0 s_{i0} + \frac{2\alpha^2}{\alpha - 2\beta} \tilde{\nabla}_i s^i_0 - \frac{\alpha^4}{(\alpha - 2\beta)^2} s^{ij} s_{ij} - c\alpha^2. \tag{14}$$

Multiplying (14) by $(\alpha - 2\beta)^3$ removes y from the denominators and we can derive the following identity:

$$Rat + \alpha Irrat = 0,$$

where *Rat* and *Irrat* are, respectively, degree 5 and degree 4 polynomials y given as follows:

$$\begin{aligned} Rat = & -(6\alpha^2\beta + 8\beta^3)\tilde{R}(v,v) + 2\alpha^2(\alpha^2 + 4\beta^2)\tilde{\nabla}_i s^i_0 \\ & + 2\beta\alpha^4 s^{ij} s_{ij} + c\alpha^2(6\alpha^2\beta + 8\beta^3), \end{aligned}$$

$$\begin{aligned} Irrat = & (\alpha^2 + 12\beta^2)\tilde{R}(v,v) - 2\alpha^2 s^i{}_0 s_{i0} \\ & - 8\alpha^2\beta\tilde{\nabla}_i s^i_0 - \alpha^4 s^{ij} s_{ij} - c(\alpha^4 - 12\alpha^2\beta^2). \end{aligned}$$

Lemma 3.1. Let F be a Matsumoto metric with constant killing from β, and $H(v,v) = c\alpha^2$ for some constants $c \in R$. Then, (M,F) is a Riemannian Einstein space.

Proof: We know that α can never be a polynomial in y. Otherwise the quadratic $\alpha^2 = a_{ij}(x) y^i y^j$ would have been factored into two linear terms. Its zero set would then consist of a hyper-plan, contradicting the positive definiteness of a_{ij}. Now suppose the polynomial Rat were not zero. The above equation would imply that it is the product of polynomial $Irrat$ with a non-polynomial factor α. This is not possible. So Rat must vanish and, since α is positive at all $y \neq 0$, we see that $Irrat$ must be zero as well. Notice that $Rat = 0$ shows that α^2 divides $\beta^3 \tilde{R}(v,v)$. Since α^2 is an irreducible degree two polynomial in y, and β^3 factors into three linear terms, it must be the case that α^2 divides $\tilde{R}(v,v)$.

That is, (M, α) is an Einstein space. Therefore, $\tilde{R}(v,v) = k\alpha^2$, where the function k must be a constant by the Riemannian Schur's Lemma for the case $n > 2$. But, we can easily reform $Rat = 0$ as the following formula:

$$\begin{aligned} & (-8k\beta^3 + 8\beta^2\tilde{\nabla}_i s^i_0 + 8c\beta^3) \\ & = \alpha^2(6k\beta - 2\tilde{\nabla}_i s^i_0 - 2\beta s^{ij} s_{ij} - 6c\beta), \end{aligned}$$

which results in, α^2 divides β^2. From the irreducibility of α^2, it results that, $\beta = 0$ and F is a Riemannian Einstein metric.

Proof of theorem 1.1. By theorem 3.1 it results in $H(v,v) = c\alpha^2$, where c is a non-zero constant and by lemma 3.1 it results in $\tilde{R}(v,v) = k\alpha^2$, where k is differentiable function defined on M and $M = \alpha$. That is to say that, (M,F) is a Riemannian Einstein space.

References

[1] Matsumoto, M. (1992). Theory of Finsler spaces with (α, β)-metrics. *Rep. on Math. Phys.*, *31*, 43-83.

[2] Antonelli, P. L., Ingarden, R. S. & Matsumoto, M. (1993). T*he theory of spray and finsler spaces with Applications in Physics and Biology*, FTPH *58*, Kluwer Academic Publishers.

[3] Bao, D. & Robles, C. (2003). On Randers spaces of constant flag curvature, *Rep. on Math Phys.*, 51, 9-42.

[4] Bao, D. & Robles, C. (2004). *On Ricci and flag curvatures in Finsler geometry*. PP. 199-261 in A sampler of Riemann-Finsler Geometry, edited by D. Bao et al. Math. Sci. Res. Inst. Publ. *50*, Cambridge, Cambridge Univ. Press,

[5] Bao, D. & Shen, Z. (2002). *Finsler metrics of constant positive curvature on the Lie group S^3. J. London Math. Soc.*, *66*, 453-467.

[6] Bidabad, B. & Rafie-rad, M. (2009). Pure pursuit navigation on Riemannian manifolds. *Nonlinear Analysis: Real World Applications*, *10*, 1265-1269.
[7] Akbar-Zadeh, H. & Couty, R. (1978). Espaces à tenseurs de Ricci parallèle admettant des transformations projectives. *Rendiconti di Matematica*, *11* serie VI.
[8] Rezaei, B. Rezavi, A. & Sadeghzadeh, N. (2007). *On Einstein (α, β)-metrics. IJST*, *Trans. A*, *31*(A4).