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Existence and local attractivity of solutions of a nonlinear quadratic functional integral equation

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Abstract

In this paper, using the tools involving measures of noncompactness and Darbo fixed point theorem for condensing operator, we study the existence of solutions for a large class of generalized nonlinear quadratic functional integral equations. Also, we show that solutions of these integral equations are locally attractive. Furthermore, we present an example to show the efficiency and usefulness of our results.

Keywords: Quadratic integral equations; measure of noncompactness; modulus of continuity; uniformly locally attractive

1. Introduction

In this paper, we discuss the problem of the existence of solutions for a generalized nonlinear quadratic functional equation of the form

$$x(t) = q(t) + f(t, x(\alpha(t)))\psi(\int_0^{\beta(t)} g(t, s, x(\gamma(s)))ds) \quad (1)$$

where f, g, ψ, α and β are appropriate given functions. Dhage and Bellale [1] investigated this problem, when $\psi(x) = x$ and $f, g, \psi, \alpha, \beta$ satisfy the following conditions.

(A_1) The functions $\alpha,\ \beta,\ \gamma : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and $\alpha(t) \to \infty$ as $t \to \infty$.

(A_2) The function $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a bounded function $\ell : \mathbb{R}_+ \to \mathbb{R}$ with bound L such that

$$|f(t,x) - f(t,y)| \le \ell(t)|x - y| \quad (2)$$

for any $t \in \mathbb{R}_+$ and for all $x, y \in \mathbb{R}$.

(A_3) The function $F : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $F(t) = |f(t,0)|$ is bounded on $\mathbb{R}_+$ with $F_0 = \sup_{t \in \mathbb{R}_+} |F(t)|$.

(B_2) The function $g : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist continuous functions $a, b : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|g(t,s,x)| \le a(t)b(s) \quad (3)$$

for $t, s \in \mathbb{R}_+$. Moreover, assume that

$$\lim_{t \to \infty} a(t) \int_0^t b(s)ds = 0 \quad (4)$$

and $K_2 L < 1$ where

$$K_2 = \sup_{t \in \mathbb{R}_+} a(t) \int_0^t b(s)ds. \quad (5)$$

They gave their main result under the above conditions as follows.

Theorem 1.1. Assume that the hypotheses (A_1) through (A_3) and (B_1) through (B_2) hold. Then the functional integral equation

$$x(t) = q(t) + f(t, x(\alpha(t)))\int_0^{\beta(t)} g(t, s, x(\gamma(s)))ds \quad (6)$$

has at least one solution in the space $BC(\mathbb{R}_+)$. Moreover, solutions of this equation are uniformly locally attractive.

The aim of this paper is to study the existence of solutions for Eq. (1) under conditions that are

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weaker than Conditions (B_1) and (B_2). Tools used in this paper are the technique of measure of noncompactness and Darbo fixed point theorem for condensing operators. In 1930, Kuratowski [2] introduced the concept of measure of noncompactness. Later, Banaś and Goebel [3] generalized this concept axiomatically which is more convenient in application and will be accepted in this paper. They also presented applications of their results (see [4-8]. Subsequently, applications of the measure of noncompactness and many other techniques to nonlinear integral equations were considered by many investigators and some basic results have been obtained (see [9-19] and references cited therein). Finally, we give an example to validate our main results in this work.

2. Preliminaries

In this section, we recall some notations, definitions and theorems to obtain all the results of this work.

In what follows, let E be a real Banach space and X be a subset of E. We denote by $\bar{X}$ the closure of X and by $co(X)$ the closed convex hull of X in E. Also, let $\bar{B}_r$ be the closed ball in E centered at zero and with radius r and we write $B(x_0, r)$ to denote the closed ball centered at x_0 with radius r. Moreover, we symbolize by $\mathfrak{M}_E$ and $\mathfrak{N}_E$ the family of all nonempty bounded subsets and its subfamily consisting of all relatively compact subsets of E, respectively.

Definition 2.1. ([3]) A mapping $\mu : \mathfrak{M}_E \to 0, \infty)$ is said to be a measure of noncompactness in E if it satisfies the following conditions.

(H_1) The family $Ker\,\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and Ker $\mu \subseteq \mathfrak{N}_E$.

(H_2) $X \subseteq Y \Rightarrow \mu(X) \le \mu(Y)$,

(H_3) $\mu(\overline{X}) = \mu(X)$,

(H_4) $\mu(CoX) = \mu(X)$,

(H_5) $\mu(\lambda X + (1-\lambda)Y) \le \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in 0,1)$,

(H_6) If (X_n) is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subseteq X_n$, $(n \ge 1)$ and if $\lim_{n\to\infty}\mu(X_n) = 0$, then the intersection set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

The family $Ker\,\mu$ described in (H_1) said to be the kernel of the measure of noncompactness μ. Observe that the intersection set X_∞ from (H_6) is a member of the family $Ker\,\mu$. In fact, since $\mu(X_\infty) \le \mu(X_n)$ for any n, we infer that $\mu(X_\infty) = 0$. This yields that $X_\infty \in Ker\,\mu$.

In section 2 we will apply the the following theorem:

Theorem 2.1. (Darbo [4]) Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space E and let $G : \Omega \to \Omega$ be a continuous mapping. Assume that there exists a constant $k \in 0,1)$ such that

$$\mu(G(X)) \le k\mu(X) \tag{7}$$

for any $X \subset \Omega$. Then G has a fixed point.

Throughout this paper, $BC(\mathbb{R}_+)$ is the set of all real functions defined, bounded and continuous on $\mathbb{R}_+$. Let X be a nonempty, bounded subset of $BC(\mathbb{R}_+)$. For any $x \in X, T > 0$ and $\varepsilon \ge 0$, let

$$\|x\| = \sup\{|x(t)| : t \ge 0\}$$

and

$$\omega^L(x,\varepsilon) = \sup\{|x(t) - x(s)| : t, s \in [0, L], |t - s| \le \varepsilon\}, \tag{8}$$

$$\omega^L(X,\varepsilon) = \sup\{\omega^L(x,\varepsilon) : x \in X\},$$

$$\omega_0^L(X) = \lim_{\varepsilon\to 0}\omega^L(X,\varepsilon),$$

$$\omega_0(X) = \lim_{L\to\infty}\omega_0^L(X)$$

Moreover, for $t \in \mathbb{R}_+$

$$X(t) = \{x(t) : x \in X\},$$

$$diamX(t) = \sup\{|x(t) - y(t)| : x, y \in X\},$$

and

$$\mu(X) = \omega_0(X) + \limsup_{t\to\infty} diamX(t). \tag{12}$$

Banas′ and Goebel proved that $\mu(X)$ is a measure of noncompactness in the sense of the above accepted definition (for details see [3]).

Now let $\Omega \subseteq BC(\mathbb{R}_+)$ and F be a map from Ω into itself and consider the equation

$$x(t) = F(x(t)). \tag{13}$$

Definition 1.2. ([7]) Solutions of equation (13) are locally attractive if there exists a ball $B(x_0, r)$ in the space $BC(\mathbb{R}_+)$ such that for arbitrary solutions $x = x(t)$ and $y = y(t)$ of equation (13) belonging to $B(x_0, r) \cap \Omega$ we have

$$\lim_{t\to\infty}(x(t) - y(t)) = 0. \tag{14}$$

When the limit in (14) is uniform with respect to $B(x_0, r) \cap \Omega$, solutions of equation (13) are said to be uniformly locally attractive.

3. Main results and Examples

In this section, we are going to study the existence and uniform local attractivity of solutions of the integral equation (1).

Theorem 3.1. Let the hypotheses $(A_1)-(A_3)$ hold. If we replace the assumptions (B_1) and (B_2) of Theorem 1.1 by the following assumptions,

(B_1') The function $q:\mathbb{R}_+ \to \mathbb{R}$ is continuous and bounded.

(B_2') Suppose that

$$\lim_{t\to\infty}\int_0^{\beta(t)} |g(t,s,x(\gamma(s))) - g(t,s,y(\gamma(s)))|\,ds = 0 \tag{15}$$

uniformly with respect to $x, y \in BC(\mathbb{R}_+)$ and $K_2' L < 1$ where

$$K_2' = \sup_{\substack{t\in\mathbb{R}_+ \\ x,y\in BC(\mathbb{R}_+)}} \left|\psi\left(\int_0^{\beta(t)} g(t,s,x(\gamma(s)))ds\right)\right|. \tag{16}$$

Furthermore, suppose that $\psi:\mathbb{R}\to\mathbb{R}$ is a continuous function and there exist some positive constants λ, p such that

$$|\psi(x) - \psi(y)| \le \lambda |x-y|^p \tag{17}$$

for any $x, y \in \mathbb{R}$.

Then equation (1) has at least one solution in the space $BC(\mathbb{R}_+)$. Also, these solutions are uniformly locally attractive.

proof: Define the function Q by

$$Qx(t) = q(t) + f(t, x(\alpha(t)))\psi\left(\int_0^{\beta(t)} g(t,s,x(\gamma(s)))ds\right) \tag{18}$$

where $x \in E = BC(\mathbb{R}_+)$. By definition (18) of Q, for any $x \in E$ and $t \in \mathbb{R}_+$, we have

$$\begin{aligned}
|Qx(t)| &\le |q(t)| + |f(t,x(\alpha(t)))|\left|\psi\left(\int_0^{\beta(t)} g(t,s,x(\gamma(s)))ds\right)\right| \\
&\le K_1 + \left(|f(t,x(\alpha(t))) - f(t,0)| \right. \\
&\quad \left. + |f(t,0)|\right)\left|\psi\left(\int_0^{\beta(t)} g(t,s,x(\gamma(s)))ds\right)\right| \\
&\le K_1 + (\ell(t)|x(\alpha(t))| + F_0)\left|\psi\left(\int_0^{\beta(t)} g(t,s,x(\gamma(s)))ds\right)\right| \\
&\le K_1 + (L\|x\| + F_0)K_2' = r
\end{aligned} \tag{19}$$

where $r = \dfrac{K_1 + F_0 K_2'}{1 - LK_2'}$ and $K_1 = \sup_{t\in\mathbb{R}_+}|q(t)|$. Hence, Q maps E into E. Moreover, from the inequality (19) we conclude that $Q(\bar{B}_r) \subset \bar{B}_r$.

Now, we shall show that the map $Q:\bar{B}_r \to \bar{B}_r$ is continuous. To prove this, assume that $\varepsilon > 0$ and pick $x, y \in \bar{B}_r$ with $\|x - y\| \le \varepsilon$. Then, using (2), $(16), (17)$ and the triangle inequality, we get

$$\begin{aligned}
|Qx(t) - Qy(t)| &\le |f(t,x(\alpha(t))) - f(t,y(\alpha(t)))|\left|\psi\left(\int_0^{\beta(t)} g(t,s,x(\gamma(s)))ds\right)\right| \\
&\quad + |f(t,y(\alpha(t)))|\left|\psi\left(\int_0^{\beta(t)} g(t,s,x(\gamma(s)))ds\right) - \psi\left(\int_0^{\beta(t)} g(t,s,y(\gamma(s)))ds\right)\right| \\
&\le \ell(t)|x(\alpha(t)) - y(\alpha(t))|K_2' \\
&\quad + [|f(t,y(\alpha(t))) - f(t,0)| + |f(t,0)|]\lambda\left(\left|\int_0^{\beta(t)} g(t,s,x(\gamma(s)))ds \right.\right. \\
&\quad \left.\left. - \int_0^{\beta(t)} g(t,s,y(\gamma(s)))ds\right|^p\right)
\end{aligned}$$

$$\le L\varepsilon K_2^{'}+[L\left|y\left(\alpha(t)\right)\right|+F_0]\lambda(\int_0^{\beta(t)}\left|\begin{matrix}g\left(t,s,x\left(\gamma(s)\right)\right)ds\\-g\left(t,s,y\left(\gamma(s)\right)\right)\end{matrix}\right|ds)^p$$

$$\le L\varepsilon K_2^{'}+[L\left\|y\right\|+F_0]\lambda(\int_0^{\beta(t)}\left|\begin{matrix}g\left(t,s,x\left(\gamma(s)\right)\right)ds\\-g\left(t,s,y\left(\gamma(s)\right)\right)\end{matrix}\right|ds)^p. \quad (20)$$

On the other hand, using (15), there exists $T>0$ such that

$$\int_0^{\beta(t)}\left|g(t,s,x(\gamma(s)))ds-g(t,s,y(\gamma(s)))\right|ds\le(\frac{\varepsilon}{\lambda})^{\frac{1}{p}} \quad (21)$$

for any $t>T$. Now, we have two the following cases.

(i) If $t>T$, then from (20) and (21), we get

$$\left|Qx(t)-Qy(t)\right|\le(K_2^{'}+Lr+F_0)\varepsilon.$$

(ii) If $0\le t\le T$, then, using uniform continuity of g on $[0,T]\times 0,\beta_T]\times -r,r]$, we obtain

$$\int_0^{\beta_T}\left|g(t,s,x(\gamma(s)))ds-g(t,s,y(\gamma(s)))\right|ds\to 0$$

as $\varepsilon\to 0$, where

$$\beta_T=\sup\{\beta(t):t\in[0,T]\}.$$

So Q is continuous. In the sequel, we show that Q satisfies the property (7) of Theorem 2.1. For this, suppose that X is a nonempty subset of $\bar{B}_r$ and fix $T>0$ and $\varepsilon>0$ arbitrarily. In addition, assume that $x\in X$ and $t_1,t_2\in[0,T]$ with $\left|t_1-t_2\right|\le\varepsilon$. Moreover, without loss of generality, we can assume that $\beta(t_1)<\beta(t_1)$. Then from $(16),(17),(18)$ and the triangle inequality, we get

$$\left|Qx\left(t_1\right)-Qx\left(t_2\right)\right|\le\left|q\left(t_1\right)-q\left(t_2\right)\right|$$

$$+\left|\begin{matrix}f\left(t_1,x\left(\alpha(t_1)\right)\right)\\-f\left(t_2,x\left(\alpha(t_2)\right)\right)\end{matrix}\right|\left|\psi(\int_0^{\beta(t_1)}g\left(t_1,s,x\left(\gamma(s)\right)\right)ds)\right|$$

$$+\left|f\left(t_2,x(\alpha(t_2))\right)\right|\left|\psi(\int_0^{\beta(t_1)}g(t_1,s,x(\gamma(s)))ds-\psi(\int_0^{\beta(t_2)}g(t_2,s,x(\gamma(s)))ds)\right|$$

$$\le\left|q(t_1)-q(t_2)\right|+[\left|f\left(t_1,x(\alpha(t_1))\right)-f\left(t_2,x(\alpha(t_1))\right)\right|$$

$$+\left|f\left(t_2,x(\alpha(t_1))\right)-f\left(t_2,x(\alpha(t_2))\right)\right|]\left|\psi(\int_0^{\beta(t_1)}g(t_1,s,x(\gamma(s)))ds)\right|$$

$$+[\left|f\left(t_2,x(\alpha(t_2))\right)-f\left(t_2,0\right)\right|+\left|f\left(t_2,0\right)\right|]\lambda(\left|\int_0^{\beta(t_1)}g(t_1,s,x(\gamma(s)))ds\right.$$

$$\left.-\int_0^{\beta(t_2)}g(t_2,s,x(\gamma(s)))ds\right|^p)$$

$$\le\left|q(t_1)-q(t_2)\right|+[\omega_r^T(f,\varepsilon)+\ell(t_2)\left|x(\alpha(t_1))-x(\alpha(t_2))\right|]K_2^{'}$$

$$+[\ell(t_2)\left|x(\alpha(t_2))\right|+F_0]\lambda(\left|\int_0^{\beta(t_1)}g(t_1,s,x(\gamma(s)))ds\right.$$

$$\left.-\int_0^{\beta(t_2)}g(t_2,s,x(\gamma(s)))ds\right|^p) \quad (22)$$

On the other hand,

$$\left|\int_0^{\beta(t_1)}g(t_1,s,x(\gamma(s)))ds-\int_0^{\beta(t_2)}g(t_2,s,x(\gamma(s)))ds\right|\le\int_0^{\beta(t_1)}\left|g(t_1,s,x(\gamma(s)))\right.$$

$$\left.-g(t_2,s,x(\gamma(s)))\right|ds+\int_{\beta(t_1)}^{\beta(t_2)}\left|g(t_2,s,x(\gamma(s)))\right|ds$$

$$\le\beta_T\omega_r^{\beta_T}(g,\varepsilon)+G_r^T\omega_r^T(\beta,\varepsilon), \quad (23)$$

where

$$\omega_r^{\beta_T}(g,\varepsilon)=\sup\left\{\begin{matrix}\left|g(t_1,s,x)-g(t_2,s,y)\right|:\\t_1,t_2\in[0,T],\left|t_1-t_2\right|\le\varepsilon,\\s\in[0,\beta_T],x,y\in[-r,r]\end{matrix}\right\}$$

and

$$\omega_r^T(\beta,\varepsilon)=\sup\{\left|\beta(t_1)-\beta(t_2)\right|:t_1,t_2\in[0,T],\left|t_1-t_2\right|\le\varepsilon\}.$$

Since x was arbitrary, from $(8),(9),(22)$ and (23), we obtain

$$\omega_0^T(QX,\varepsilon)\le\omega_0^T(q,\varepsilon)+[\omega_r^T(f,\varepsilon)+L\omega_r^T(X,\omega_r^T(\alpha,\varepsilon))]K_2^{'}$$

$$+(Lr+F_0)\lambda(\beta_T\omega_r^{\beta_T}(g,\varepsilon)+G_r^T\omega_r^T(\beta,\varepsilon))^p. \quad (24)$$

where

$$\omega^T(\alpha,\varepsilon)=\sup\{\left|\alpha(t_1)-\alpha(t_2)\right|:t_1,t_2\in[0,T],\left|t_1-t_2\right|\le\varepsilon\},$$

$$\omega^T(x,\omega^T(\alpha,\varepsilon))=\sup\{\left|x(t_1)-x(t_2)\right|:$$

$$t_1,t_2\in[0,T],\left|t_1-t_2\right|\le\omega^T(\alpha,\varepsilon)\},$$

$$\omega_r^T(f,\varepsilon)=\sup\left\{\begin{matrix}\left|f(t_1,x)-f(t_2,x)\right|:\\t_1,t_2\in[0,T],\left|t_1-t_2\right|\le\varepsilon,x\in-r,r]\end{matrix}\right\},$$

$$\omega_r^T(g,\varepsilon)=\sup\left\{\begin{matrix}\left|g(t_1,x,y)-g(t_2,x,y)\right|:\\t_1,t_2\in[0,T],\left|t_1-t_2\right|\le\varepsilon,x,y\in-r,r]\end{matrix}\right\},$$

$$\omega_0^T(q,\varepsilon)=\sup\{\left|q(t_1)-q(t_2)\right|:$$

$$t_1,t_2\in[0,T],\left|t_1-t_2\right|\le\varepsilon\}$$

and

$$G_r^T=\sup\{\left|g(t,s,x)\right|:t\in[0,T],s\in[0,\beta_T],x\in.[-r,r]\}.$$

Since q, f and g are uniformly continuous on the compact sets $[0,T]$, $[0,T]\times[-r,r]$ and $[0,T]\times[0,\beta_T]\times[-r,r]$ respectively, we have

$\omega_0^T(q,\varepsilon)\to 0$, $\omega_r^T(f,\varepsilon)\to 0$ and $\omega_r^T(g,\varepsilon)\to 0$ as $\varepsilon\to 0$. Similarly, $\omega^T(\alpha,\varepsilon)\to 0$ and $\omega_r^T(\beta,\varepsilon)\to 0$, as $\varepsilon\to 0$. Taking the limit from (24) as $\varepsilon\to 0$ and by (10) we get

$$\omega_0^T(QX)\le LK_2^{'}\omega_0^T(X). \tag{25}$$

Letting $T\to\infty$ in (25), then using (11), we obtain

$$\omega_0(QX)\le LK_2^{'}\omega_0(X). \tag{26}$$

Now let $x,y\in X$ and $t\in\mathbb{R}_+$. Then, using (2), (16), (17) and the triangle inequality

$$\begin{aligned}
&|Qx(t)-Qy(t)|\le|f(t,x(\alpha(t)))-f(t,y(\alpha(t)))|\left|\psi(\textstyle\int_0^{\beta(t)}g(t,s,x(\gamma(s)))ds)\right|\\
&+|f(t,y(\alpha(t)))|\left|\psi(\textstyle\int_0^{\beta(t)}g(t,s,x(\gamma(s)))ds)-\psi(\int_0^{\beta(t)}g(t,s,y(\gamma(s)))ds)\right|\\
&\le[\ell(t)|x(\alpha(t))-y(\alpha(t))|K_2^{'}]\\
&+[|f(t,y(\alpha(t)))-f(t,0)|+|f(t,0)|]\lambda\left|\int_0^{\beta(t)}g(t,s,x(\gamma(s)))ds\right.\\
&\left.-\int_0^{\beta(t)}g(t,s,y(\gamma(s)))ds\right|^p\\
&\le LK_2^{'}diamX(\alpha(t))+[\ell(t)|y(\alpha(t))|+F_0]\lambda(\left|\int_0^{\beta(t)}g(t,s,x(\gamma(s)))ds\right.\\
&\left.-\int_0^{\beta(t)}g(t,s,y(\gamma(s)))ds\right|^p)\\
&\le LK_2^{'}diamX(\alpha(t))+\\
&[L\|y\|+F_0]\lambda(\int_0^{\beta(t)}\left|\begin{matrix}g(t,s,x(\gamma(s)))ds\\-g(t,s,y(\gamma(s)))\end{matrix}\right|ds)^p\\
&\le LK_2^{'}diamX(\alpha(t))\\
&+[Lr+F_0]\lambda(\int_0^{\beta(t)}\left|\begin{matrix}g(t,s,x(\gamma(s)))ds\\-g(t,s,y(\gamma(s)))\end{matrix}\right|ds)^p.
\end{aligned}\tag{27}$$

Since, x,y and t were arbitrary in (27), we obtain

$$\begin{aligned}
&diamQX(t)\le LK_2^{'}diamX(\alpha(t))\\
&+[Lr+F_0]\lambda(\int_0^{\beta(t)}\left|\begin{matrix}g(t,s,x(\gamma(s)))ds\\-g(t,s,y(\gamma(s)))\end{matrix}\right|ds)^p.
\end{aligned}\tag{28}$$

Thus, taking the limit from (28) and using (15), we earn

$$\limsup_{t\to\infty}QX(t)\le LK_2^{'}\limsup_{t\to\infty}diamX(\alpha(t)). \tag{29}$$

Also, adding (26) and (29), we have

$$\begin{aligned}
&\omega_0(QX)+\limsup_{t\to\infty}diamQ(X)(t)\le LK_2^{'}(\omega_0(X)\\
&+\limsup_{t\to\infty}diamX(\alpha(t))).
\end{aligned}\tag{30}$$

Now (12) and (30) imply that

$$\mu(QX)\le LK_2^{'}\mu(QX), \tag{31}$$

So, by applying Theorem 1.2 we conclude that the operator Q has at least a fixed point and consequently the integral equation (1) has a solution in $BC(\mathbb{R}_+)$. Now, we shall show the uniform local attractivity of solutions of equation (1). To do this, we first consider the ball B_r with $r=\dfrac{K_1+F_0K_2^{'}}{1-LK_2^{'}}$. From (19) we have obtained that Q maps B_r into itself. Take

$$S=\{x\in BC(\mathbb{R}_+):\|x\|<r, x=Q(x)\}.$$

Define by induction $\Omega_0=Co(f(B_r))$ and $\Omega_n=Co(f(\Omega_{n-1}))$ for any $n\ge 1$. It is easy to see that

$$S\subset\Omega_n \tag{32}$$

for any $n\ge 0$. Furthermore, from (31), we have

$$\mu(\Omega_n)\le(LK_2^{'})^n\mu(\Omega_0) \tag{33}$$

for any $n\ge 1$. Therefore, from (33), $\mu(\Omega_n)\to 0$ as $n\to\infty$. Since $\{\Omega_n\}$ is a decreasing sequence and Ω_n is a bounded, closed, convex and nonempty subset in $BC(\mathbb{R}_+)$ for any $n\ge 0$, then (H_6) implies that $\Omega_\infty=\bigcap_{n=1}^{\infty}\Omega_n$ is nonempty and $\mu(\Omega_\infty)=0$. Thus (12) implies that

$$\limsup_{t\to\infty}diamX(\alpha(t))=0. \tag{34}$$

On the other hand, $S \subset \Omega_\infty$ by (32). Hence, by (12) and (34), the solutions of the equation (1) are uniformly locally attractive and the proof is complete.

Corollary 3.1. Theorem 1.1 can be deduced from Theorem 3.1.

Proof: Set $\psi(x) = x$. Thus ψ is Lipschitz with constant 1. On the other hand, using (3) and (5), we get

$$K_2' = \sup_{\substack{t\in\mathbb{R}_+ \\ x\in BC(\mathbb{R}_+)}} \left|\psi(\int_0^{\beta(t)} g(t,s,x(\gamma(s)))ds)\right|$$
$$= \sup_{\substack{t\in\mathbb{R}_+ \\ x\in BC(\mathbb{R}_+)}} \left|\int_0^{\beta(t)} g(t,s,x(\gamma(s)))ds\right|$$
$$\le \sup_{t\in\mathbb{R}_+} a(t)\int_0^{\beta(t)} b(s)ds = K_2.$$

Hence $K_2' L < 1$. Moreover, from (3) and (4) we have

$$\lim_{t\to\infty}\int_0^{\beta_T} |g(t,s,x(\gamma(s)))ds - g(t,s,y(\gamma(s)))|ds \le 2\lim_{t\to\infty} a(t)\int_0^{\beta(t)} b(s)ds = 0$$

uniformly with respect to $x, y \in BC(\mathbb{R}_+)$. Now, according to Theorem 3.1, the equation (6) has at least one solution in the space $BC(\mathbb{R}_+)$.

At the end of this section, we present an example to show how Theorem 3.1 can be successfully applied, and is especially more general than Theorem 1.1.

Examples: Let $m, n > 2$. Moreover, we assume that α is a positive constant such that $(\frac{\pi}{3})\frac{e^{-1}}{\alpha} < 1$. Consider the following generalized quadratic integral equation

$$x(t) = e^{-2t} + \ln(1+t^2e^{-\alpha t^2}|x(t)|)arctg(\int_0^{t^2} \frac{2\sqrt{3}m\ln(1+s^{\frac{m-2}{2}}|x^n(\sqrt{s})|) + m\sqrt{3}s^{m-1}(1+x^{2n}(\sqrt{s}))}{2(1+t^{2m})(1+x^{2n}(\sqrt{s}))}ds) \quad (35)$$

for $t \in \mathbb{R}_+$ and $x \in BC(\mathbb{R}_+)$. Comparing (35) with (1), we have

$$g(t,s,x) = \frac{2\sqrt{3}m\ln(1+s^{\frac{m-2}{2}}|x^n|) + m\sqrt{3}s^{m-1}(1+x^{2n})}{2(1+t^{2m})(1+x^{2n})},$$
$$f(t,x) = \ln(1+t^2e^{-\alpha t^2}), \psi(x) = arctg(x),$$
$$\alpha(t) = \sqrt{t}, \beta(t) = t^2, q(t) = e^{-2t}, \gamma(s) = \sqrt{s}.$$

Now we verify the assumptions of Theorem 3.1. Obviously α, β and γ satisfy the assumption (A_1) of Theorem 1.1. Also, f is continuous on $\mathbb{R}_+ \times \mathbb{R}$ and $|f(t,0)|$ is bounded with

$$|f(t,x) - f(t,y)| = \left|\begin{matrix}\ln(1+t^2e^{-\alpha t^2}|x(t)|) \\ -\ln(1+t^2e^{-\alpha t^2}|x(t)|)\end{matrix}\right|$$
$$= \left|\ln\frac{(1+t^2e^{-\alpha t^2}|x|)}{(1+t^2e^{-\alpha t^2}|y|)}\right|$$
$$= \ln(\frac{1+t^2e^{-\alpha t^2}|x| - t^2e^{-\alpha t^2}|y| + t^2e^{-\alpha t^2}|y|}{1+t^2e^{-\alpha t^2}|y|})$$
$$\le \ln(1+\frac{t^2e^{-\alpha t^2}|x-y|}{1+t^2e^{-\alpha t^2}|y|})$$
$$\le \ell(t)|x-y|$$

for any $t \in \mathbb{R}_+$ and $x, y \in \mathbb{R}$ where $\ell(t) = t^2e^{-\alpha t^2}$. Moreover, we can easily verify that $L = \max_{t\in\mathbb{R}_+} \ell(t) = \frac{e^{-1}}{\alpha}$. These mean that the Assumptions (A_2) and (A_3) are satisfied. It is easy to see that Assumption (B_1') is satisfied for q. Since

$$g(t,s,x) = \frac{\sqrt{3}m\ln(1+s^{\frac{m-2}{2}}|x^n|)}{(1+t^{2m})(1+x^{2n})} + \frac{m\sqrt{3}s^{m-1}}{2(1+t^{2m})} \quad (36)$$

and

$$\int_0^{t^2}\left|\frac{\sqrt{3}m\ln(1+s^{\frac{m-2}{2}}|x^n|)}{(1+t^{2m})(1+x^{2n})}\right|ds \le \int_0^{t^2}\frac{\sqrt{3}ms^{\frac{m-2}{2}}|x^n|}{(1+t^{2m})(1+x^{2n})}ds$$
$$= \frac{2}{m}\frac{\sqrt{3}mt^m}{(1+t^{2m})}\frac{|x^n|}{(1+x^{2n})}$$
$$\le \frac{\sqrt{3}t^m}{(1+t^{2m})}. \quad (37)$$

Thus by (36) and (37) we obtain

$$\lim_{t\to\infty}\int_0^{t^2}|g(t,s,x)|ds = \lim_{t\to\infty}\int_0^{t^2}\frac{m\sqrt{3}s^{m-1}}{2(1+t^{2m})}ds$$
$$= \lim_{t\to\infty}\frac{\sqrt{3}t^{2m}}{2(1+t^{2m})}$$
$$= \frac{\sqrt{3}}{2}.$$

This shows that assumption (B_2) of Theorem 1.1 does not hold. Therefore, Theorem 1.1 is inapplicable. But, using (36), we have

$$\int_0^{t^2}\left|\frac{\sqrt{3}m\ln(1+s^{\frac{m-2}{2}}|x^n|)}{(1+t^{2m})(1+x^{2n})}+\frac{m\sqrt{3}s^{m-1}}{2(1+t^{2m})}\right|ds \le \frac{\sqrt{3}t^m}{(1+t^{2m})}+\int_0^{t^2}\frac{m\sqrt{3}s^{m-1}}{2(1+t^{2m})}ds$$
$$\le \frac{\sqrt{3}}{2}+\frac{m\sqrt{3}t^{2m}}{2m(1+t^{2m})}$$
$$\le \frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2} = \sqrt{3}.$$

It follows that

$$\int_0^{t^2}|g(t,s,x)|ds \le \sqrt{3} \quad (38)$$

for any $t\in\mathbb{R}_+$ and uniformly with respect to $x, y\in\mathbb{R}$. On the other hand, from (37) we get

$$\int_0^{t^2}|g(t,s,x)-g(t,s,y)|ds = \int_0^{t^2}\left|\frac{\sqrt{3}m\ln(1+s^{\frac{m-2}{2}}|x^n|)}{(1+t^{2m})(1+x^{2n})}-\frac{\sqrt{3}m\ln(1+s^{\frac{m-2}{2}}|y^n|)}{(1+t^{2m})(1+y^{2n})}\right|ds$$
$$\le \int_0^{t^2}\left(\frac{\sqrt{3}ms^{\frac{m-2}{2}}|x^n|}{(1+t^{2m})(1+x^{2n})}+\frac{\sqrt{3}ms^{\frac{m-2}{2}}|y^n|}{(1+t^{2m})(1+y^{2n})}\right)ds$$
$$\le 2\sqrt{3}\frac{t^m}{(1+t^{2m})}.$$

uniformly with respect to $x, y\in\mathbb{R}$ for any $t\in\mathbb{R}_+$ and $x, y\in\mathbb{R}$. This implies that

$$\lim_{t\to\infty}\int_0^{t^2}|g(t,s,x)-g(t,s,y)|ds = 0. \quad (39)$$

Also, using (38), it is easy to check that $\max_{t\in[-\sqrt{3},\sqrt{3}]}|\psi(t)| = \frac{\pi}{3}$. Therefore

$$K_2'L = \sup_{t\in\mathbb{R}_+}\{arctg\left|\int_0^{\beta(t)}g(t,s,x(\gamma(s)))ds\right|\}\frac{e^{-1}}{\alpha}$$
$$\le \frac{\pi e^{-1}}{3\alpha} < 1. \quad (40)$$

Furthermore, ψ is Lipshitz with constant 1. Hence, using (39) and (40), Assumption (B_2') is satisfied. Then, we conclude that all of the Assumptions of Theorem 3.1 are satisfied. Hence the equation (35) has at least one solution and all the solutions are uniformly locally attractive.

References

[1] Kuratowski, K. (1930). Sur les espaces complets. *Fund. Math*, *15*, 301-309.

[2] Banaś, J. & Goebel, K. (1980). *Measures of noncompactness in Banach Space*. *vol. 60*, Lecture Notes in pure and Applied Mathematics, New York, Marcel Dekker.

[3] Banaś, J. (1980). On Measures of noncompactness in Banach spaces. *Commentations Mathematical Universitatis Carolinae*, *21*, 131-143.

[4] Banaś, J. (1981). Measures of noncompactness in the spaces of continuous tempered functions. *Demonstration Math*, *14*, 127-133.

[5] Banaś, J. & Dhage, B. C. (2008). Global asymptotic stability of solutions of a functional integral equation. *Nonlinear Analysis*, *69*, 1945-1952.

[6] Banaś, J. & Regan, D. O'. (2008). On existence and local attractivity of solutions of a quadratic Volterra integral equation of fractional order. *J. Math, Anal. Appl*., *34*, 573-82.

[7] Banaś, J. & Rzepka, R. (2003). An application of a measure of noncompactness in the study of asymptotic stability. *Appl. Math. Lett*., *16*, 1-6.

[8] Aghajani, A., Banaś, J. & Jalilian, Y. (2011). Existence of solutions for a class of nonlinear Volterra sigular integral equations. *Computer and Mathematics with Applications*, *62*, 1215-1227.

[9] Aghajani, A. & Jalilian, Y. (2010). Existence and global attractivity of solutions of a nonlinear functional integral equation. *Communications in Nonlinear*

Science and Numerical Simulation, *15*, 3306-3312.
[10] Aghajani, A. & Jalilian, Y. (2010). Existence of nondecreasing positive solutions for a system of sigular integral equations. *Mediterr. J. Math*, *8*(4), 563-576.
[11] Darwish, M. A. (2005). On quadratic integral equation of fractional orders. *J. Math. Anal. Appl*., *311*, 112-119.
[12] Darwish, M. A. (2008). On monotonic solutions of a singular quadratic integral equation with supremum. *Dynamic Systems and Applications*., *17*, 539-550.
[13] Dhage, B. C. (2009). Nonlinear quadratic first order functional integro-differential equations with periodic boundary conditions. *Dynamic Systems and Applications*, *18*, 303-322.
[14] Dhage, B. C. & Imdad, M. (2009). Asymptotic behaviour of nonlinear quadratic functional integral equations involving Carathéodory. *Nonlinear Analysis*, *71*, 1285-1291.
[15] Falset, J. G. (2009). Existence of fixed points and measure of weak noncompactness. *Nonlinear Analysis*, *71*, 2625-2633.
[16] Maleknejad, K., Nouri, K. & Mollapourasl, R. (2009). Existence of solutions for some nonlinear integral equations. *Commun Nonlinear Sci Simulat*, *14*, 2559-2564.
[17] Salem, H. A. H. (2011). On the quadratic integral equations and their applications. *Computers and Mathematics with Applications*, *62*, 2931-2934.
[18] Xue, X. (2009). Nonlocal nonlinear differential equations with a measure of noncompactness in isometrics of Lipshitz spaces. *Nonlinear Analysis*, *70*, 2593-2601.