IJST (2012) A3 (Special issue-Mathematics): 391-394
Iranian Journal of Science & Technology

http://www.shirazu.ac.ir/en

The uniqueness theorem for discontinuous boundary value problems with aftereffect using the nodal points

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Abstract

In this paper, uniqueness theorem is studied for boundary value problem with "aftereffect" on a finite interval with discontinuity conditions in an interior point. The oscillation of the eigenfunctions corresponding to large modulus eigenvalues is established and an asymptotic of the nodal points is obtained. By using these new spectral parameters, uniqueness theorem is proved.

***Keywords**:* Uniqueness Theorem; nodal Points; discontinuous conditions; eigenvalues; eigenfunctions

1. Introduction

Inverse nodal problems exist in recovering operators from given nodes (zeros) of their eigenfunctions. Mclaughlin seems to have been the first to consider this sort of inverse problem for the one-dimensional Schrodinger equations on an interval with Dirichlet boundary conditions [1]. Later on, some remarkable results were obtained. For example, X. F. Yang got the uniqueness for general boundary conditions using the same method as McLaughlin [2]; C.K. Law and Ching-Fu Yang [3] have reconstructed the potential function and its derivatives from nodal data. We consider boundary value problem with "aftereffect" on a finite interval with discontinuity conditions in an interior point:

$$\begin{aligned} ly(x) &:= -y''(x) + q(x)y(x) \\ &+\int_0^x M(x-t)y(t)dt = \lambda y(x), \quad 0 < x < T, \end{aligned} \tag{1}$$

$$\begin{aligned} U(y) &:= y'(0) - hy(0) = 0, \\ V(y) &:= y'(T) + Hy(T) = 0, \end{aligned} \tag{2}$$

$$\begin{aligned} y(\tfrac{T}{2}+0) &= a_1 y(\tfrac{T}{2}-0), \\ y(\tfrac{T}{2}+0) &= a_1^{-1} y'(\tfrac{T}{2}-0) + a_2 y(\tfrac{T}{2}-0). \end{aligned} \tag{3}$$

Here λ is the spectral parameter. Let $\lambda = \rho^2$, $\rho = \sigma + i\tau$, $q(x)$, h, H, a_1, a_2 be real, $q(x) \in L(0,T)$ and $a_1 > 0$.

Without loss of generality we assume that $\int_0^T q(x)dx = 0$. We denote the boundary value problem (1)-(3) by $L(q, M, h, H)$. Boundary value problems with discontinuities inside the interval often appear in mathematics, mechanics, physics, geophysics and other branches of Natural sciences. For example, discontinuous inverse problems appear in electronics for constructing the parameters of heterogeneous electronic lines with desirable technical characteristics [4], [5]. As a rule, such problems are connected with discontinuous material properties. In [6], the authors considered the inverse nodal problem for the differential equation $-y'' + q(x)y = \lambda y$, $0 < x < T$ with discontinuity conditions inside the interval. In the present paper, we investigate uniqueness theorem from given nodes of their eigenfunctions for the boundary value problem L. In section 2, the eigenvalues and eigenfunctions corresponding to large modulus eigenvalues are obtained and in section 3, an asymptotic of the nodal points is calculated and the uniqueness theorem is proven.

2. Asymptotic of the eigenvalues and eigenfunctions

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Received: 23 January 2012 / Accepted: 7 March 2012

Let $\varphi(x,\lambda), C(x,\lambda)$, $S(x,\lambda)$ be the solutions of equation (1) under initial conditions $C(0,\lambda)=S'(0,\lambda)=\varphi(0,\lambda)=1$, $C'(0,\lambda)=S(0,\lambda)=0$, $\varphi'(0,\lambda)=h$ and under the jump conditions (3). Then $U(\varphi)=0$. Denote

$$\Delta(\lambda)=-V(\varphi). \tag{4}$$

Let $C_0(x,\lambda)$ and $S_0(x,\lambda)$ be smooth solutions of (1) on the initial $[0,T]$ under the initial conditions $C_0(0,\lambda)=S_0'(0,\lambda)=1$, $C_0'(0,\lambda)=S_0(0,\lambda)=0$. Then, using the jump conditions (3) we get [7]:

$$C(x,\lambda)=C_0(x,\lambda),\ \ S(x,\lambda)=S_0(x,\lambda),\ \ x<\frac{T}{2} \tag{5}$$

$$\begin{aligned}C(x,\lambda)&=A_1C_0(x,\lambda)+B_1S_0(x,\lambda),\\ S(x,\lambda)&=A_2C_0(x,\lambda)+B_2S_0(x,\lambda),\ \ x>\frac{T}{2}\end{aligned} \tag{6}$$

where

$$\begin{aligned}A_1&=a_1C_0(\tfrac{T}{2},\lambda)S_0'(\tfrac{T}{2},\lambda)\\ &-a_1^{-1}C_0'(\tfrac{T}{2},\lambda)S_0(\tfrac{T}{2},\lambda)\\ &-a_2C_0(\tfrac{T}{2},\lambda)S_0(\tfrac{T}{2},\lambda),\\ B_1&=(a_1^{-1}-a_1)C_0(\tfrac{T}{2},\lambda)S_0'(\tfrac{T}{2},\lambda)+a_2C_0^2(\tfrac{T}{2},\lambda),\\ A_2&=(a_1-a_1^{-1})S_0(\tfrac{T}{2},\lambda)S_0'(\tfrac{T}{2},\lambda)-a_2S_0^2(\tfrac{T}{2},\lambda),\\ B_2&=a_1^{-1}C_0(\tfrac{T}{2},\lambda)S_0'(\tfrac{T}{2},\lambda)\\ &-a_1C_0'(\tfrac{T}{2},\lambda)S_0(\tfrac{T}{2},\lambda)\\ &+a_2S_0(\tfrac{T}{2},\lambda)C_0(\tfrac{T}{2}\lambda).\end{aligned} \tag{7}$$

The function $C_0(x,\lambda)$ satisfies the following integral equation:

$$\begin{aligned}C_0(x,\lambda)&=\cos\rho x+\int_0^x\frac{\sin\rho(x-t)}{\rho}(q(t)C_0(t,\lambda)\\ &+\int_0^t M(t-s)C_0(s,\lambda)ds)dt\end{aligned} \tag{8}$$

and for $|\rho|\to\infty$

$$C_0(x,\lambda)=\cos\rho x+O(\frac{1}{\rho}e^{|\tau|x}). \tag{9}$$

Then (8) implies

$$\begin{aligned}C_0(x,\lambda)&=\cos\rho x+\frac{\sin\rho x}{2\rho}\int_0^x q(t)dt\\ &+\frac{1}{2\rho}\int_0^x q(t)\sin\rho(x-2t)dt\\ &+\int_0^x\frac{\sin\rho(x-t)}{\rho}\int_0^t M(t-s)\cos\rho s ds dt+O(\frac{1}{\rho^2}e^{|\tau|x})\end{aligned} \tag{10}$$

$$\begin{aligned}C_0'(x,\lambda)&=-\rho\sin\rho x+\frac{\cos\rho x}{2}\int_0^x q(t)dt\\ &+\frac{1}{2}\int_0^x q(t)\cos\rho(x-2t)dt\\ &+\int_0^x\cos\rho(x-t)\int_0^t M(t-s)\cos\rho s ds dt+O(\frac{1}{\rho}e^{|\tau|x}).\end{aligned} \tag{11}$$

Analogously,

$$\begin{aligned}S_0(x,\lambda)&=\frac{\sin\rho x}{\rho}+\int_0^x\frac{\sin\rho(x-t)}{\rho}(q(t)S_0(t,\lambda)\\ &+\int_0^t M(t-s)S_0(s,\lambda)ds)dt.\end{aligned} \tag{12}$$

and for $|\rho|\to\infty$

$$S_0(x,\lambda)=\frac{\sin\rho x}{\rho}+O(\frac{1}{\rho^2}e^{|\tau|x}). \tag{13}$$

Then (12) implies

$$\begin{aligned}S_0(x,\lambda)&=\frac{\sin\rho x}{\rho}-\frac{\cos\rho x}{2\rho^2}\int_0^x q(t)dt\\ &+\frac{1}{2\rho^2}\int_0^x q(t)\cos\rho(x-2t)dt\\ &+\frac{1}{\rho^2}\int_0^x\sin\rho(x-t)\int_0^t M(t-s)\sin\rho s ds dt+O(\frac{1}{\rho^3}e^{|\tau|x})\end{aligned} \tag{14}$$

$$\begin{aligned}S_0'(x,\lambda)&=\cos\rho x+\frac{\sin\rho x}{2\rho}\int_0^x q(t)dt\\ &-\frac{1}{2\rho}\int_0^x q(t)\sin\rho(x-2t)dt\\ &+\frac{1}{\rho}\int_0^x\cos\rho(x-t)\int_0^t M(t-s)\sin\rho s ds dt+O(\frac{1}{\rho^2}e^{|\tau|x}).\end{aligned} \tag{15}$$

By virtue of (7) and (10)-(15),

$$A_1=b_1+b_2\cos\rho T+(b_2\int_0^{\frac{T}{2}}q(t)dt-\frac{a_2}{2})\frac{\sin\rho T}{\rho}$$

$$+\frac{b_2}{\rho}\int_0^{\frac{T}{2}}\int_0^t\sin\rho(T+s-t)M(t-s)dsdt$$

$$+\frac{b_1}{\rho}\int_0^{\frac{T}{2}}\int_0^t \sin\rho(s-t)M(t-s)dsdt+O(\frac{1}{\rho^2}), \qquad 16)$$

$$B_1 = b_2(\rho\sin\rho T-\cos\rho T\int_0^{\frac{T}{2}}q(t)dt$$

$$-\int_0^{\frac{T}{2}}q(t)\cos\rho(T-2t)dt)+\frac{a_2}{2}(1+\cos\rho T)$$

$$+\int_0^{\frac{T}{2}}\int_0^t\cos\rho(T-t)\cos\rho sM(t-s)dsdt+O(\frac{1}{\rho}), \quad (17)$$

$$A_2 = b_2\frac{\sin\rho T}{\rho}+O(\frac{1}{\rho^2}), \qquad (18)$$

$$B_2 = b_1-b_2\cos\rho T+O(\frac{1}{\rho}), \qquad (19)$$

where $b_1=\dfrac{a_1+a_1^{-1}}{2}$, $b_2=\dfrac{a_1-a_1^{-1}}{2}$. Since $\varphi(x,\lambda)=C(x,\lambda)+hS(x,\lambda)$, for $|\lambda|\to\infty$ uniformly in x and using (10)-(15), (16)-(19) one has (see [8] or chapter 1 in [9]):

$$\phi(x,\lambda) = \cos\rho x+(h+\frac{1}{2}\int_0^x q(t)dt)\frac{\sin\rho x}{\rho}$$

$$+\int_0^x\frac{\sin\rho(x-t)}{\rho}\int_0^t M(t-s)\cos\rho sdsdt$$

$$+o(\frac{1}{\rho}e^{|\tau|x}) \quad x<\frac{T}{2}. \qquad (20)$$

$$\phi(x,\lambda) = b_1\cos\rho x+b_2\cos\rho(T-x)$$

$$+f_1(x)\frac{\sin\rho x}{\rho}+f_2(x)\frac{\sin\rho(T-x)}{\rho}$$

$$+b_1\int_0^x\frac{\sin\rho(x-t)}{\rho}\int_0^t M(t-s)\cos\rho sdsdt$$

$$+o(\frac{1}{|\rho|}e^{|\tau|x}) \quad x>\frac{T}{2}. \qquad (21)$$

$$\phi'(x,\lambda) = -\rho\sin\rho x+(h+\frac{1}{2}\int_0^x q(t)dt)\cos\rho x$$

$$+\int_0^x\cos\rho(x-t)\int_0^t M(t-s)\cos\rho sdsdt+o(e^{|\tau|x}) \quad x<\frac{T}{2}. \qquad (22)$$

$$\phi'(x,\lambda) = \rho(-b_1\sin\rho x+b_2\sin\rho(T-x))$$

$$+f_1(x)\cos\rho x-f_2(x)\cos\rho(T-x)$$

$$+b_1\int_0^x\cos\rho(x-t)\int_0^t M(t-s)\cos\rho sdsdt+o(e^{|\tau|x}) \quad x>\frac{T}{2}, \qquad (23)$$

where

$$f_1(x) = b_1h+\frac{b_1}{2}\int_0^x q(t)dt+\frac{a_2}{2},$$

and

$$f_2(x)=b_2h-\frac{b_2}{2}\int_0^x q(t)dt+b_2\int_0^{\frac{T}{2}}q(t)dt-\frac{a_2}{2}.$$

It follows from (20)-(23) that for $|\lambda|\to\infty$

$$\Delta(\lambda) = b_1\rho\sin\rho T-\omega_1\cos\rho T-\omega_2+\kappa(\rho), \qquad (24)$$

where

$$\omega_1 = b_1(H+h+\frac{1}{2}\int_0^T q(t)dt)+\frac{a_2}{2},$$

$$\omega_2 = b_2(H-h+\frac{1}{2}\int_0^T q(t)dt-\int_0^a q(t)dt)+\frac{a_2}{2},$$

$$\kappa(\rho)=-b_1\int_0^T\cos\rho(T-t)\int_0^t M(t-s)\cos\rho sdsdt+o(e^{|\tau|T}).$$

Using (24) by the well-known method (see, for example, [7]) one has that for $n\to\infty$,

$$\rho_n=\sqrt{\lambda_n}=\frac{n\pi}{T}+\frac{1}{n\pi b_1}(\omega_1+(-1)^{n-2}\omega_2)+\frac{\kappa_n}{n},\{\kappa_n\}\in l_2 \qquad (25)$$

The eigenfunctions of the boundary value problem L have the form $y_n(x)=\varphi(x,\lambda_n)$. Substituting (25) into (20) and (21) we obtain the following asymptotic formulae for $n\to\infty$ uniformly in x (see [10]):

$$y_n(x)=\cos\frac{n\pi}{T}x+\frac{1}{2n\pi}(T(2h+\int_0^x q(t)dt)-\frac{2}{b_1}(\omega_1+(-1)^{n-2}\omega_2)x$$

$$+2T\int_0^x\cos\frac{n\pi}{T}t\int_0^t M(t-s)\cos\frac{n\pi}{T}sdsdt+\frac{\kappa_n}{n} \quad x<\frac{T}{2}. \qquad (26)$$

$$y_n(x) = \cos\frac{n\pi}{T}x(b_1+(-1)^n b_2)$$

$$+\frac{1}{n\pi}(Tf_1(x)+(-1)^{n-1}Tf_2(x)$$

$$-(\omega_1+(-1)^{n-2}\omega_2)(x+(-1)^{n-1}\frac{b_2}{b_1}(T-x))$$

$$+Tb_1\int_0^x\cos\frac{n\pi}{T}t\int_0^t M(t-s)\cos\frac{n\pi}{T}sdsdt)\sin\frac{n\pi}{T}x+\frac{\kappa_n}{n} \quad x>\frac{T}{2}. \qquad (27)$$

3. Computation the nodal points

The eigenfunction $y_n(x)$ has exactly n (simple) zeros inside the interval $(0,T)$ namely: $0<x_n^1<...<x_n^n<T$. The set $X_B=\{x_n^j\}_{n\geq 1,\, j=\overline{1,n}}$ is called the set of nodal points of the boundary value problem L. Denote $X_B^k:=\{x_{2m-k}^j\}_{m\geq 1,\, j=\overline{1,2m-k}}$, $k=0,1$. Clearly, $X_B^0\cup X_B^1=X_B$. Inverse nodal problems consist in recovering the $M(x)$ and coefficients h and H from the given set X_B of nodal points. Denote $\alpha_n^j=(j-\frac{1}{2})\frac{T}{n}$. Taking (26)-(27) into account, we obtain the following

asymptotic formulae for nodal points as $n \to \infty$ uniformly in j:

$$x_n^j = \alpha_n^j + \frac{T}{2n^2\pi^2}[T(2h + \int_0^{\alpha_n^j} q(t)dt) - \frac{2}{b_1}(\omega_1 + (-1)^{n-2}\omega_2)\alpha_n^j$$

$$+2T\int_0^{\alpha_n^j}\int_0^t M(t-s)dsdt] + \frac{\kappa_n}{n^2}, \ x_n^j \in (0, \frac{T}{2}) \qquad (28)$$

$$x_n^j = \alpha_n^j + \frac{T}{2n^2\pi^2}[T\int_0^{\alpha_n^j} q(t)dt - \frac{2}{b_1}(\omega_1 + (-1)^{n-2}\omega_2)\alpha_n^j$$

$$+2T\frac{b_1}{b_1 + (-1)^n b_2}\int_0^{\alpha_n^j}\int_0^t M(t-s)dsdt + C] + \frac{\kappa_n}{n^2}, \ x_n^j \in (\frac{T}{2}, T) \qquad (29)$$

where

$$C = \frac{1}{b_1 + (-1)^n b_2}(2Th(b_1 + (-1)^{n-1}b_2) + Ta_2(1 + (-1)^n)$$

$$+2(-1)^{n-1}Tb_2\int_0^{\frac{T}{2}} q(t)dt + 2(-1)^n \frac{b_2}{b_1}T(\omega_1 + (-1)^{n-2}\omega_2)). \qquad (30)$$

We note that the sets X_B^k, $k = 0,1$ are dense on $(0,T)$. Using these formulate we arrive at the following assertion.

Theorem 1. Fix $k = 0 \vee 1$ and $x \in [0,T]$. Let $\{x_n^{j_n}\} \in X_B^k$ be chosen such that

$$\lim_{n\to\infty} x_n^{j_n} = x.$$

Then there exists a finite limit

$$g_k(x) := \lim_{n\to\infty} \frac{2\pi^2 n}{T^2}(nx_n^{j_n} - (j_n - \frac{1}{2})T), \qquad (31)$$

and

$$g_k(x) = 2h + \int_0^x q(t)dt - \frac{2}{b_1 T}(\omega_1 + (-1)^{k-2}\omega_2)x$$

$$+2\int_0^x\int_0^t M(t-s)dsdt \qquad (32)$$

$$g_k(x) = \int_0^x q(t)dt - \frac{2}{b_1 T}(\omega_1 + (-1)^{k-2}\omega_2)x$$

$$+2\frac{b_1}{b_1 + (-1)^k b_2}\int_0^x\int_0^t M(t-s)dsdt + \frac{C}{T} \qquad (33)$$

where C are defined by (30).
Let us now prove uniqueness theorem.

Theorem 2. Fix $k = 0 \vee 1$. Let $X \subset X_B^k$ be a subset of nodal points which is dense on $(0,T)$. Let $X = \tilde{X}$ then $M(x) = \tilde{M}(x)$ a.e. on $(0,T)$, $h = \tilde{h}$, $H = \tilde{H}$.

Proof: If, $X = \tilde{X}$ then (31) yields $g_k(x) = \tilde{g}_k(x)$, $x \in [0,T]$. By virtue of (32)-(33) we get a.e. on $M(x) = \tilde{M}(x)$. From $h = \frac{g_k(0)}{2}$, we have $h = \tilde{h}$. Similarly, we can derive $H = \tilde{H}$.

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