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Interesting dynamic behavior in some discrete maps

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Abstract

Different discrete models of population dynamics of certain insects have been investigated under various feasible conditions within the framework of nonlinear dynamics. Evolutionary phenomena are discussed through bifurcation analysis leading to chaos. Some tools of nonlinear dynamics, such as Lyapunov characteristic exponents (LCE), Lyapunov numbers, correlation dimension, etc. are calculated for numerical studies and to characterize regular and chaotic behavior. These results are produced through various graphics. Chaotic evolutions of such insect population have been discussed as the parameters attain certain set of critical values. The results obtained are informative and very significant. The correlation dimension for evolution of insect population signifies certain fractal structure.

***Keywords**:* Bifurcation; Lyapunov exponent; periodic attractor; correlation dimension

1. Introduction

Robert May has initiated pioneer investigation of population evolution [1, 2] followed by numerous research works for different types of populations, e.g. [3-7]. Appearance of chaos in various models has been observed through bifurcation phenomena and published in a number of recent literature, introduces the interesting analysis of systems related to the real world. There are a number of articles published in this regard and some are noteworthy such as that of Osinga [8 & 9]. Statistical measures in dynamical system show significant developments for evolutionary analysis. In this direction the work of Alseda and Costa [10] can be taken as very significant. Also, the recent article by Henson et al [11] provides a detailed study of evolutionary behavior in insect populations.

Certain important factors to be noted are that the insect populations get regulated by parasitoids, predators and other mortality factors. Also, the insect populations grow rapidly because its female members usually produce a large number of eggs. Therefore, within a short period the population may grow very much. But this does not happen in nature due to various reasons. Some of them die at different stages of their life and so, in nature, we do not observe any type of population explosion.

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The objective of this work is to observe dynamic behavior of a certain insect population, (e.g. larvae), through different stages of their changes. Towards this, here, we have considered some population models and first examined the stability criteria of their steady state solutions and then use appropriate simulation work to find their bifurcation diagram. This leads to understanding their evolutionary properties and also indicates the parameter domain within which the population may evolve regularly or chaotically. For each of these models, we have obtained the plots of their bifurcation diagrams, Lyapunov exponents and also calculated correlation dimensions. Our aim is to establish certain meaningful explanations of evolutionary behavior of the species under various conditions.

We use recursion (iteration) procedure to investigate evolution and bifurcation in discrete maps. A general way to write such recursion is given by

$$x_{n+1} = f(x_n) = f^{(n+1)}(x_0).$$

To study the dynamics of larvae population behavior, the following models have been considered in our present study.

First, we consider the Ricker-type map described by following equation, [11]

$$x_{n+1} = f_0(x_n) = bx_n e^{-cx_n} + (1-\mu)x_n. \tag{1}$$

Here, x_{n+1} represents the density of individuals in a population at census n + 1 for a given density of individuals x_n at census n. The parameter b > 0 stands for the inherent per capita recruitment rate per census interval at small population sizes and e^{-cx_n} represents the fractional reduction of recruitment due to density-dependent effects. The parameter μ , $0 \le \mu \le 1$, represents the fraction of individuals expected to die during one census period.

There are two fixed points for system (1), $x_1^* = 0$ and $x_2^* = \frac{1}{c}\ln(\frac{b}{\mu})$. The point origin, $x_{,1}^* = 0$, is stable in the range $-2 \le b - \mu \le 0$, and within this one can observe the first cycle; outside this range this fixed point is unstable. The other fixed point, $x_2^* = \frac{1}{c}\ln(\frac{b}{\mu})$, is stable when $0 \le \mu \ln(\frac{b}{\mu}) \le 2$ and when $\mu \ln(\frac{b}{\mu}) > 2$ this point is no longer stable. Thus, the orbit originating near this fixed point x_2^* would show attracting behavior as long as the parameters b and μ take values such that the above stability condition holds. The bifurcation diagram would produce a one cycle in such a case. For two cycles, we have to obtain fixed points by solving equation

$$x_2 = f_0(x_1)$$

and proceed to find stability of fixed points by repeating the process of first cycle and in the bifurcation diagram we observe the second cycle. Analytically, one may repeat this for consecutive cycles. After several bifurcations, finally chaos is observed.

If we introduce a certain period –forcing such that the birth rate oscillates with relative amplitude $0 < a < 1$ and average b, then we obtain another model modified from equation (1) as:

$$x_{n+1} = f_a(t, x_n) = b[1 + a(-1)^n]x_n e^{-cx_n} + (1 - \mu)x_n \quad (1)$$

For this map, the fixed points for n even are, 0, $\frac{1}{c}\ln(\frac{b(1+a)}{\mu})$ and when n is odd these are 0, $\frac{1}{c}\ln(\frac{b(1-a)}{\mu})$. One can proceed for stability analysis similar to the first map (1). The parameter values are assigned while computing the bifurcation diagram where one gets the clear ideas of various cycles and then chaos. Emergence of chaos through bifurcations is now quite familiar in literature since Feigenbaum [12].

A two-species population is described by discrete system

$$\begin{aligned} x_{n+1} &= ax_n(1 - x_n - y_n) \\ y_{n+1} &= bx_n y_n . \end{aligned} \quad (3)$$

There are three fixed for this two dimensional map; these are $(0,0), (1-\frac{1}{a}, 0)$ and $(\frac{1}{b}, 1 - \frac{1}{a} - \frac{1}{b})$. Stability of each of these fixed points can be discussed as in the above cases and cycles appearing in the bifurcation diagram can be analyzed. All these are dependent on the parameters *a* and *b*.

There can be various discrete models but, here we have selected only these few models. Our objective is to see the change in behavioral dynamics of the population with regard to the change of control parameters by observing the phenomena of bifurcation. Besides drawing bifurcations for the above models, we have numerically evaluated the Lyapunov exponents and correlation dimensions to justify regular and chaotic evolutions. Correlation dimension gives the measure of complexity whenever a system evolves chaotically.

2. Bifurcation Phenomena

Bifurcation in an ordinary sense is splitting into two. In dynamical system it is a sudden change in behavior due to the sudden change of a set of parameter values according to a certain rule. During changing process of the parameters a critical set of values is obtained where we observe a sudden change in the behavior of the system. Such a point is known as the bifurcation point. We witness bifurcation of systems at certain bifurcation points within the parameter domain.

During numerical exploration, we observed bifurcations of the above described maps and explained the dynamic behavior through various graphics. Emergence of chaos can be easily visualized by observing these diagrams and obtaining dynamic behavior. Certain periodic windows appearing in Figs. have very specific significance for nonlinear systems emerging to chaos.

Bifurcation Analysis

a) Bifurcation in one dimensional model represented by eqn. (1):

(i) For the model described by equation (1), with $c = 1$, $\mu = 0.93$ fixed and b varying from $b = 0$ to $b = 250$, we observe the appearance of one cycle up to the value $b = 7.988$. Then, one cycle suddenly changes into two cycles when b exceeds 7.99 up to some higher value. The system again bifurcates and shows four cycles around the value $b = 110$ and the process of bifurcation continues. Then, it finally, becomes chaotic when b takes the value $b = 195$ and onwards. The bifurcation scenario of this system, when we vary b is shown in Fig. 1.

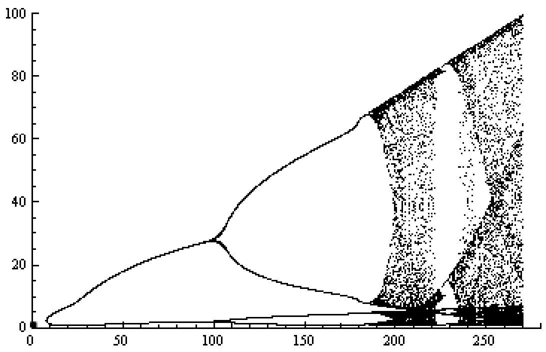

Fig. 1. Bifurcation of model described by eqn. (1). Here, $c = 1,\ \mu = 0.93$ and $0 \le b \le 280$.

(ii) Then, we fixed $c = 1,\ b = 60$ and varied μ from $\mu = 0$ to $\mu = 1$. We see that up to the values $\mu = 0.44$, only one cycle appears. One cycle bifurcates into two cycles at about the parameter values $\mu = 0.45$. Then the system sees four cycles around the value $\mu = 0.6$. Bifurcation phenomena continue until the parameter μ reaches the value μ = 0.7 where it is highly chaotic. We can see this result by the following bifurcation diagram shown in Fig. 2.

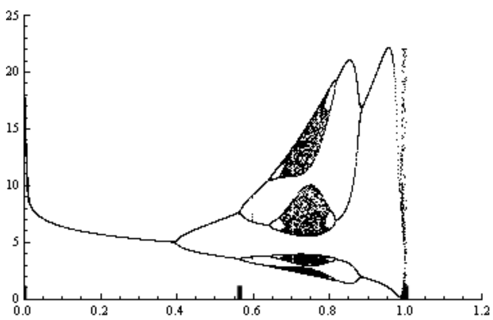

Fig. 2. Bifurcation of model described by eqn. (1). Here, $c = 1,\ b = 60$ and μ is varied from $0 \le \mu \le 1.2$.

b) Bifurcation in one dimensional model represented by eqn. (2):

To explain the bifurcation scenario of the model described by equation (2), which is obtained when the relative amplitude is increased slightly from zero, (e.g., $a = 0.01$), we have considered two cases for n even and for n odd. Fixing $c = 1, \mu = 0.93$ and changing b from $b = 0$ to $b = 250$, we have obtained the bifurcation diagram shown in Fig. 3a when n is even and in Fig. 3b when n is odd. For the first case, we see from Fig. 3, up to the value $b = 7.90$, only one cycle appears. From one cycle the system suddenly emerges into two cycles at about the parameter value $b = 7.91$. Then to four cycles around $b = 107$ and so on. The bifurcation shows chaos after the value $b = 195$.

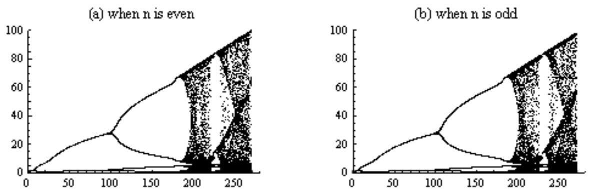

Fig. 3. Bifurcation of model described by eqn. (2) (a) when n is even and (b) when n is odd. Here, $a = 0.01,\ c = 1,\ \mu = 0.93$ and b is varied $0 \le b \le 270$.

Fixing $c = 1,\ b = 60$ and varying the death rate μ from $\mu = 0$ to $\mu = 1$, we have again drawn the bifurcation diagrams for both cases. We observe only one cycle up to value $\mu = 0.39$. From one cycle the system suddenly emerges into two cycles at the parameter value $\mu = 0.4$. The system emerges to four cycles around the value $\mu = 0.6$. Then, it becomes highly chaotic after the value $\mu = 0.7$. The results are demonstrated by the following bifurcation diagram for equation (2).

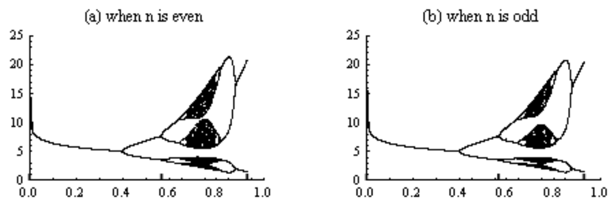

Fig. 4. Bifurcation diagram of model described by eqn. (2), (a) when n is even and (b) when n is odd. Here, $a = 0.01,\ c = 1,\ b = 60$ and μ is varied $0 \le \mu \le 1$.

c) Bifurcation in one dimensional model represented by eqn. (3):

In the case of model (3), keeping $b = 0.5$ fixed and changing the other parameter a from 0 to 4.0, we have obtained the bifurcation diagram. We find that, Fig. 5, up to the value $a = 2.98$, only one cycle appears. As we proceed further at $a = 2.99$ the system suddenly changes into two cycles and around $a = 3.46$ it bifurcates and changes into four cycles. Proceeding further in similar fashion we can observe that the system become chaotic around the value $a = 3.6$.

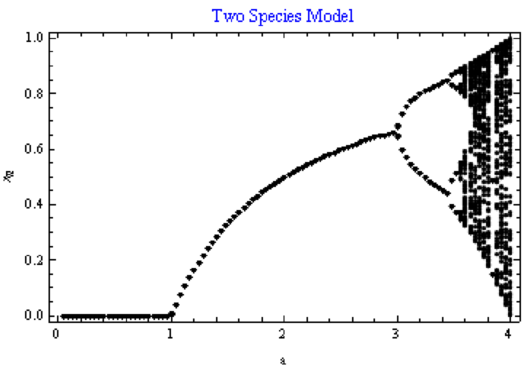

Fig. 5. Bifurcation of model described by eqn. (3). Here, $b = 0.5$ and a is varied $0 \le a \le 4$.

3. Calculations of LCE and Lyapunov numbers

The Lyapunov Characteristic Exponent [LCE] gives the rate of exponential divergence from perturbed initial conditions. To examine the behavior of an orbit around any point, $X^*(n)$, we use perturbation method for the system explained by Chirikov [13], [10], [14 - 15]. We proceed with equation

$$X^*(n) = X^*(n) + U(n) \tag{4}$$

where, $U(n)$ is the average deviation of the unperturbed trajectory at time *n*. In a chaotic region the LCE, σ is independent of $X^*(0)$ and is given by the Oseledec theorem, (1965), which states that

$$\sigma = \lim_{n\to\infty} \frac{1}{n} \ln \hat{u}\, U(n)\, \hat{u} . \tag{5}$$

For n-dimensional mapping the Lyapunov characteristic exponent is given by

$$\sigma_i = \lim_{n\to\infty} \ln \hat{u}\, \lambda_i(N)\, \hat{u} \tag{6}$$

where, $i = 1, 2, ..., n$ and λ_i is the Lyapunov characteristic numbers.

We have plotted the LCE curve of our above models, discussed in Section 1, by using Mathematica software, [16] LCE curves corresponding to these models are shown through Figs. 6-12.

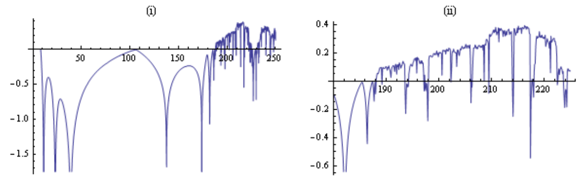

Fig. 6. LCE curve of Equation (1) for the parameter values $c = 1$, $\mu = 0.93$ and b is varied (i) $7.8 \le b \le 250$ (ii) $180 \le b \le 225$.

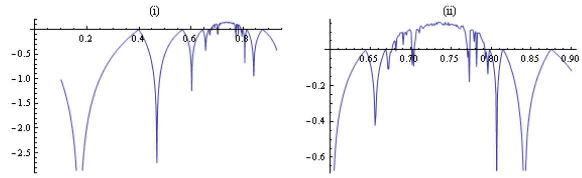

Fig. 7. LCE curve of Equation (1) for the parameter values $c = 1$, $b = 60$ and μ is varied (i) $0.1 \le \mu \le 1$. (ii) $0.6 \le \mu \le 0.9$.

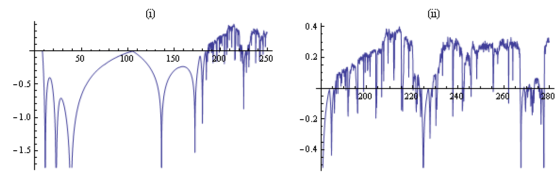

Fig. 8. LCE curve of Equation (2) for the parameter values $c = 1$, $\mu = 0.93$, $a = 0.01$ and b is varied (i) $7.8 \le b \le 250$ (ii) $180 \le b \le 280$ and n is even.

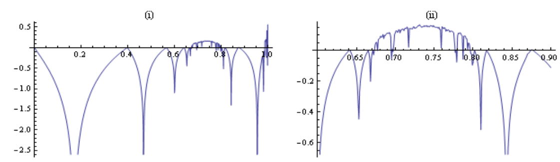

Fig. 9. LCE curve of Equation (2) for the parameter values $c = 1$, $b = 60$, $a = 0.01$ and μ is varied (i) $0 \le \mu \le 1$ (ii) $0.6 \le \mu \le 0.9$ and n is even.

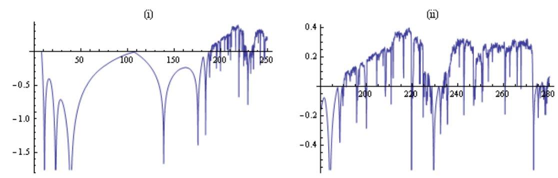

Fig. 10. LCE curve of Equation (2) for the parameter values $c = 1$, $\mu = 0.93$, $a = 0.01$ and b is varied (i) $8 \le b \le 250$ (ii) $180 \le b \le 280$ and n is odd.

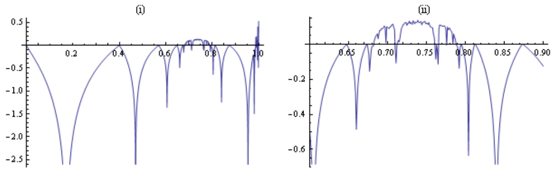

Fig. 11. LCE curve of Equation (2) for the parameter values $c = 1, b = 60, a = 0.01$ and μ is varied (i) $0 \le \mu \le 1$ (ii) $0.6 \le \mu \le 0.9$ and n is odd.

For the two dimensional map of two species problem model (3) we have the following Lyapunov characteristic exponent map.

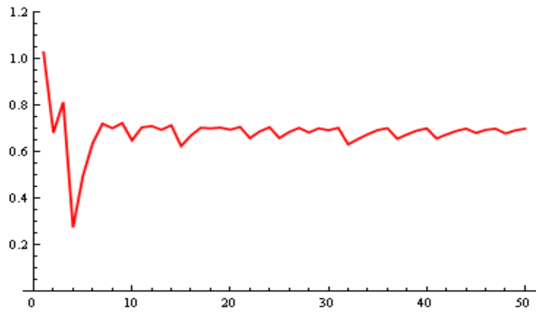

Fig. 12. LCE curve of Equation (3) for the parameter values $b = 0.5$ and a is varying from $0 \le a \le 4$.

4. Correlation dimensions

A statistical measure provides us more authenticity in analysis of the behavior of the models discussed in the previous sections [10]. Towards this end, we wish to introduce the correlation dimension which helps to explain the fractal nature of the chaotic system that possibly underlies a strange attractor [17]. Correlation dimension describes the measure of dimensionality of the space occupied by chaotic attractor of any system having presence of complexity.

During numerical simulation, correlation dimension can be calculated using the distances between each pair of points in the set of N number of points, $s_{ij} = |X_i - X_j|$. To calculate the correlation dimension the following steps must be followed [18]:

For an orbit $O(X_1) = \{X_1, X_2, X_3, X_4,\}$, of a map $f: U \to U$, where U is an open bounded set in p_n and for a given positive real number r, first we obtain the correlation integral [16],

$$C(r) = \lim_{n\to\infty} \frac{1}{n(n-1)} \sum_{i \ne j}^{n} H(r - \|X_i - X_j\|),$$

where

$$H(X) = \begin{cases} 0, & X < 0 \\ 1, & X \ge 0 \end{cases},$$

is the unit-step function, (Heaviside function). The summation indicates that the number of pairs of vectors closer to r for $1 \le i, j \le n$ and $i \ne j$. The $C(r)$ measures the density of a pair of distinct vectors X_i and X_j that are closer to r.

Then, the correlation dimension D_c of $O(X_1)$ is defined as

$$\mathrm{D_c} = \lim_{r\to 0} \frac{\log \mathrm{C(r)}}{\log \mathrm{(r)}}.$$

To obtain D_c , $\log C(r)$ is plotted against $\log r$ and then we find a straight line fitted to this curve. The y- intercept of this straight line provides the value of the correlation dimension D_c .

We have calculated the correlation for models (1), (2) and (3) and demonstrated through graphics Figs. 13-17 below.

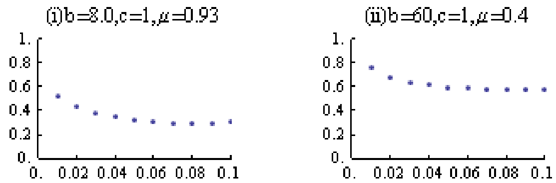

Fig. 13. Correlation dimension curves of the model (1) for regular case.(i) when μ is constant and b is varying & (ii) when b is constant and μ is varying

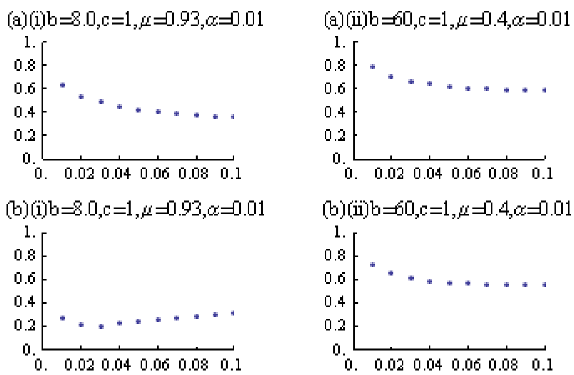

Fig. 14. Correlation dimension curves of the model (2) for regular case (a) when n is odd, (i) μ is constant and b is varying (ii) b is constant and μ is varying (b) when n is even, (i) μ is constant and b is varying (ii) b is constant and μ is varying.

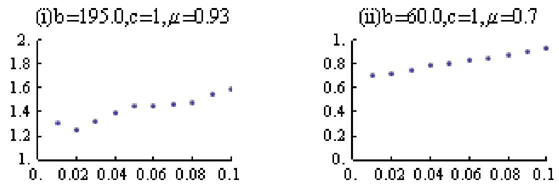

Fig. 15. Correlation dimension curves of the model (1) for chaotic case.(i) when μ is constant and b is varying & (ii) when b is constant and μ is varying.

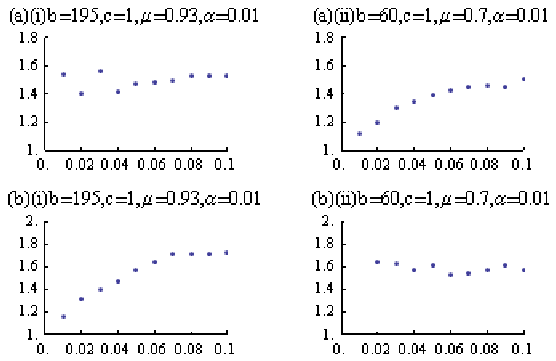

Fig. 16. Correlation dimension curves of the model (2) for chaotic case (a) when n is odd, (i) μ is constant and b is varying (ii) b is constant and μ is varying (b) when n is even, (i) μ is constant and b is varying (ii) b is constant and μ is varying

Correlation dimensions of the model (3) for parameters $a = 4, b = 0.5$ are calculated and plotted. From the graph, Fig. 17 and by linear fitting the data obtained for the graph correlation dimension was calculated as 1.15427. Similar procedure is also applied for other models and their correlation dimensions are obtained as per the following discussion.

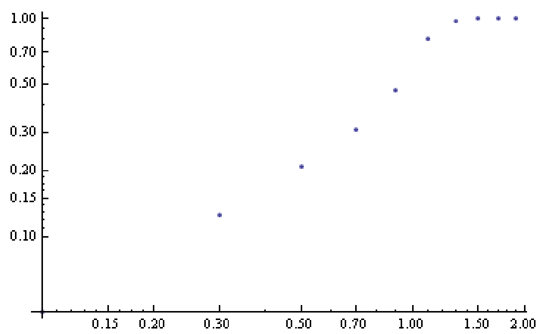

Fig. 17. Correlation dimension curve of the model (3)

5. Discussions

Evolutions of certain insects represented through models (1), (2) and (3) are explained by drawing bifurcation diagrams of these models for variation of certain parameter for their certain ranges. Significant revelation obtained through bifurcation graphics indicates how the stable solutions change into chaotic when the parameters varied. Also, the bifurcation scenarios are different in many ways for different models. For model (1), one observes bifurcation Figs. 1 and 2 are completely different when b and μ are made to vary. The case is the same for model (2), which was obtained after certain modification of an earlier model, Figs. 3 and 4. We observe clear period doubling type bifurcation phenomena for model (3) in Fig. 5 when b is varied. Lyapunov characteristic exponents (LCE) have been calculated for the above explained models to see the parameter ranges showing regularity and chaotic evolution. The positive LCE indicates the chaotic regions, whereas its negative value indicates the regular regions of evolution. Figures 6-12, are drawn for LCE for models (1)-(3). We have also used meaningful statistical measures to justify the results obtained through this study. The correlation dimensions for each model are shown through Figs. 13-17. We conclude that the environmental situations and certain social conditions for coexistence have a significant role for regular and chaotic evolution of the insect population.

Acknowledgments

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