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# Moments of the present value of total dividends and related problems in the risk model with delayed claims

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## Abstract

In this paper, a compound binomial risk model in the presence of a constant dividend barrier is studied. Two types of individual claims, main claims and by-claims, are defined, where every by-claim is induced by the main claim randomly and may be delayed for one time period with a certain probability. In the evaluation of the moments of the present value of dividends, the interest rates are assumed to follow a Markov chain with finite state space. A system of differential equations with certain boundary conditions satisfied by the *n*th moment of the present value of total dividend payments prior to ruin, given the initial environment state, is derived. We arrive at a general formula which allows us to evaluate the moments of the total discounted dividends recursively in terms of the lower order moments. Assuming the claim sizes are $K_h$ distributed or the claim size distributions have finite support, we are able to solve for all necessary components in the general recursive formula and obtain some other results of interest. We present several numerical examples which illustrate the applicabilities of our main results and the effects of the by-claims on the moments of the present value of dividends.

***Keywords:*** Compound binomial risk model; delayed claim; dividend barrier; discounted dividend payments; stochastic interest rate; Markov chain

## 1. Introduction

In reality, insurance claims may be delayed due to various reasons. Since the work by Waters and Paratriandafylou [1], risk models with this special feature have been discussed by many authors in the literature. For example, Yuen and Guo [2] studied a compound binomial model with delayed claims and obtained recursive formulas for the finite time ruin probabilities. Xiao and Guo [3] obtained the recursive formula of the joint distribution of the surplus immediately prior to ruin and deficit at ruin in this model. Xie and Zou [4] derived the exact analytical expressions for the Laplace transforms of the ruin functions in a delayed claims risk model. Yuen et al. [5] studied a risk model with delayed claims, in which the time of delay for the occurrence of a by-claim is assumed to be exponentially distributed. Zou and Xie [6] obtained the expression of non-ruin probability in the Erang(2) risk model with delayed claims. Xie and Zou [7] considered an extension to the compound Poisson risk model for which the occurrence of the claim may be delayed and proved that the ruin probability for this risk model decreases as the probability of the delay of by-claims is increasing.

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A framework of delayed claims is built by introducing two kinds of individual claims, namely main claims and by-claims, and allowing possible delays of the occurrences of by-claims. All risk models described above relied on the assumption that each main claim induce a by-claim to a certainty.

Dividend strategy for insurance risk models was first proposed by De Finetti [8] to reflect more realistically the surplus cash flows in an insurance portfolio, and he found that the optimal strategy must be a barrier strategy. From then on, barrier strategies have been studied in a number of papers and books, including Claramunt et al. [9], Dickson and Waters [10], Zhou [11], Gerber and Shiu [12], Li and Garrido [13], Wu and Li [14], Frosting [15], Albrecher and Hartinger [16], Albrecher and Thonhauser [17], Yang and Hu [18], Gerber et al. [19] and references therein. It is well known that the optimal dividend barrier level which maximizes the expectation of the discounted dividends until the time of ruin is independent of the initial surplus in the classical compound binomial risk model, if the force of interest or the discount factor per period is assumed to be a constant. Based on this assumption, it becomes evident that the discount factor per period embedded into the risk model fails to capture the uncertainty of the (future) risk-free rates of interest. Xie and Zou [20] first considered the risk model with delayed claims and a constant dividend barrier in a financial market driven by a time-homogeneous Markov chain. They also assumed that each main claim induced a by-claim to a certainty and only obtained the expected present value of total dividends. Most recently, the moments of the random variable representing the total discounted dividends paid until ruin occurs have been studied in various related risk models in the actuarial literature. For example, Wan [21] examined a threshold strategy in a compound Poisson model perturbed by diffusion, while Renaud and Zhou [22] determined the moments of discounted dividends for a barrier strategy in a Lévy insurance risk model. Cheung and Drekic [23] computed dividend moments in the dual risk model. Li [24] considered the moments of the present value of total dividends in the compound binomial model under stochastic interest rates. Cheung et al. [25] computed the moments of the total discounted dividends paid until ruin occurs for a threshold strategy in the compound Poisson risk model. Lu and Li [26] studied the moments of the total dividend payments until ruin in the Markovian regime-switching risk model with a threshold dividend strategy. Related works can be found in Li and Lu. [27], Frosting [28] and the references therein.

Motivated by these works, in this paper, we consider the compound binomial model with delayed claims and a dividend barrier, discount factors are defined via the modelization of the one-period interest rates using a time-homogeneous Markov chain with a finite state space. In our model, every by-claim is induced by the associated main claim randomly. We are interested in finding the moments of the present value of total dividends paid until ruin occurs and some other results of interest.

The model proposed in this paper is a generalization of compound binomial risk model with paying dividends and classical risk model with delayed claims. We show that, the moments of the present value of total dividends in this risk model can be obtained. The work of this paper can be seen as a complement to the work of Xie and Zou [20] and Li [24]. This kind of specific dependent risk model may be of practical use: for instance, a serious motor accident may cause different kinds of claim, such as car damage, injury, and death; some can be dealt with immediately while others need a period of time to be settled.

It is obvious that the incorporation of the randomness of delayed claim and dividend payments makes the problem more interesting. It also complicates the evaluation of the moments of the present value of dividends. We use the technique of generating functions to calculate the moments of the present value of total dividends for this risk model. The paper is structured as follows: Section 2 defines the model of interest, describes various payments, including the premiums, claims and dividends, and lists the notation. In Section 3, differential equations with certain boundary conditions are developed for the $n$th moment of the present value of total dividend payments prior to ruin, given the initial environment state. Then a general formula which allows us to evaluate the moments of the total discounted dividends recursively in terms of the lower order moments is derived, using the technique of generating functions. Moreover, closed-form solutions for the $n$th moment of the present value of dividends is obtained for two classes of claim size distributions in Section 4. In particular, when the interest rate is assumed to be a constant and the claim amounts are of constant size, we give the explicit expression for the expected present value of total dividend payments and derive the optimal dividend barrier level. Furthermore, we also prove that the expected present value of the dividend payments up to the time of ruin increases as the probability of the delay of the by-claims is increasing in Section 4. We show that the ruin is certain under constant interest rate in Section 5. Numerical examples are also provided to illustrate the applicabilities of our main results and the effects of the by-claims on the moments of the present value of dividends in Section 5.

## 2. Model description and notation

Here, we consider a discrete time model which involves two types of insurance claims; namely the main claims and the by-claims. Denote the discrete time units by $k = 0, 1, 2, \cdots$. In any time period, the probability of having a main claim is $q$, $0 < q < 1$. The occurrences of main claims in different time periods are independent. It is assumed that each main claim induces a by-claim with probability $\varphi$, $0 < \varphi < 1$, and the main claim doesn't induce a by-claim with probability $1-\varphi$. Moreover, if the main claim induces a by-claim, the by-claim and its associated main claim may occur simultaneously with probability $\theta$, or the occurrence of the by-claim may be delayed to the next time period with probability $1-\theta$. All claim amounts are independent, positive and integer valued. The main claim amounts $X_1, X_2, \cdots$, are independent and identically distributed (i.i.d.). Put $X=X_1$. Then the common probability function (c.p.f.) of the $X_i$ is given by $f_m = \Pr(X= m)$ for $m= 1, 2, \cdots$. The corresponding probability generating function (p.g.f.) and mean are $\tilde{f}(s) = \sum_{m=1}^{\infty} f_m s^m$ and

$\mu_X = \sum_{n=1}^{\infty} nf_n$ , respectively. Let $Y_1$, $Y_2,\cdots$, be i.i.d. by-claim amounts and put $Y=Y_1$. Denote their c.p.f. by $g_n$ = Pr($Y$= $n$) for $n$= 1, 2, $\cdots$, and write the p.g.f. and mean as $\tilde{g}(s) = \sum_{n=1}^{\infty} g_n s^n$ and $\mu_Y$ , respectively.

Let $S_k$ be the total amount of claims up to the end of the $k$th time period, $k\in \mathrm{N}^+$ and $S_0 = 0$ . We define

$$S_k = S_k^X + S_k^Y, \tag{1}$$

where $S_k^X$ and $S_k^Y$ are the total main claims and by-claims, respectively, in the first $k$ time periods.

Assume that premiums are received at the beginning of each time period with a constant premium rate of 1 per period, and all claim payments are made only at the end of each time period. We introduce a dividend policy to the company that a certain amount of dividends will be paid to the policyholder instantly, as long as the surplus of the company at time $k$ is higher than a constant dividend barrier $b$($b$ > 0). It implies that the dividend payments will only possibly occur at the beginning of each period, right after receiving the premium payment. The surplus at the end of the $n$th time period, $U_b(n)$ , is then defined to be, for $n$= 1, 2, $\cdots$,

$$U_b(n) = u + n - S_n - D(n),\ (U_b(0) = u), \tag{2}$$

where $D(n)$ is the sum of the total dividend payments in the first $n$ periods, with the definition

$$D(n) = D_1 + D_2 + \cdots + D_n,\ (D(0) = 0),$$

with $D_n = \max\{U_b(n-1) + 1 - b, 0\}$ being the amount of dividend paid out at the end of period $n$. Here the initial surplus is $u$, $u$ = 0, 1, $\cdots$, $b$.

The positive safety loading condition holds if $q(\mu_X + \varphi\mu_Y) < 1$. Define $T_{u,b} = \inf\{n \in \mathrm{N} : U_b(n) < 0\}$ to be the time of ruin, $\Psi(u;b) = \Pr(T_{u,b} < \infty \ \ | U_b(0) = u)$ to be the ruin probability, and $\Phi(u;b) = 1 - \Psi(u;b)$ to be the survival probability.

In this note, we assume that the interest rates $\{R_n, k\in \mathrm{N}\}$ with $R_n$ being the interest rate in the interval ($n$; $n$ + 1] follow a time homogenous Markov chain with finite state space $\{r_1, r_2, \cdots, r_m\}$. The one-period transition probability matrix is given by $\mathrm{P} = (p_{i,j})_{i,j=1}^m$, where $p_{i,j} = \Pr(R_{n+1} = j \mid R_n = i)$, for $n \in \mathrm{N}$. The one-period discount factors are denoted by $v_1, v_2, \cdots$, respectively, where $v_i = 1/(1+r_i)$.

Under the interest rate model described above, the present value of total dividends until time of ruin $T_{u,b}$ given that the initial surplus is $u$ is denoted by

$$D_{u,b} = \sum_{k=1}^{T_{u,b}} D_k \left( \prod_{i=1}^{k} \frac{1}{1+R_i} \right),\ u = 0,1,2,\cdots,b.$$

Define $V_{i,n}(u;\ b)$= $\mathrm{E}[D_{u,b}^n \mid R_0 = i]$, $u = 0,1,\cdots,b, n \in \mathrm{N}^+$, to be the $n$th moment of the present value of dividend payments up to the time of ruin, given that the initial interest rate $R_0 = r_i$.

The aim of this paper is to calculate $V_{i,n}(u; b)$, the $n$th moment of the present value of a sequence of dividend payments until the time of ruin under stochastic interest rates to the dividends, for some special claim-size distributions so to determine whether the optimal dividend barrier level is still independent of the initial surplus and illustrate the effects of the occurrence and delay of by-claims on the moments of the present value of dividends.

## 3. A system of differential equations with boundary conditions

To study the $n$th moment of the present value of the dividend payments, $V_{i,n}(u; b)$, we need to study the claim occurrences in three scenarios.

(I) If a main claim occurs in a certain period and doesn't induce a by-claim, then the surplus process gets renewed except for the initial value;

(II) If a main claim induces a by-claim and they occur concurrently in some period, then there will be no by-claim in the next time period and the surplus process also gets renewed except for the initial value;

(III) If a main claim occurs in some period and induces a by-claim, but the by-claim is delayed to the next period, then the surplus process behaves differently because of the delayed by-claim occurring in the forthcoming period.

Conditional on the third scenario, that is, the main claim occurred in the previous period and induced a by-claim, but the by-claim will occur at the end of the current period, we define the corresponding surplus process as

$$U_b^*(n) = u + n - S_n - D^*(n) - Y,\ \ n = 1,2,\cdots, \tag{3}$$

with $U_b^*(0) = u$ , where $D^*(n)$ is the sum of dividend payments in the first $n$ time periods, and $Y$

is a random variable following the probability function $g_n$, $n$= 1, 2, $\cdots$, and is independent of all other claim amounts random variables $X_i$ and $Y_j$ for all $i$ and $j$. The corresponding $n$th moment of the present value of the dividend payments is denoted by $V^*_{i,n}(u;b)$. Then conditioning on the occurrences of claims at the end of the first time period, we can set up the following differential equations for $V_{i,n}(u;b)$ and $V^*_{i,n}(u;b)$:

$$\begin{aligned}V_{i,n}(u;b) &= v_i^n(1-q)\sum_{j=1}^{m} p_{i,j}V_{j,n}(u+1;b)\\ &+v_i^n q(1-\varphi)\sum_{j=1}^{m} p_{i,j}\sum_{k=1}^{u+1}V_{j,n}(u+1-k;b)f_k\\ &+v_i^n q\varphi\left(\theta\sum_{j=1}^{m} p_{i,j}\sum_{l+k\le u+1}V_{j,n}(u+1-l-k;b)f_k g_l\right.\\ &\left.+(1-\theta)\sum_{j=1}^{m} p_{i,j}\sum_{k=1}^{u+1}V^*_{j,n}(u+1-k;b)f_k\right),\\ u &= 0,1,2,\cdots,b-1. \end{aligned} \quad (4)$$

$$V^*_{i,n}(0;b) = v_i^n(1-q)\sum_{j=1}^{m} p_{i,j}V_{j,n}(0;b)g_1, \quad (5)$$

and for $u = 1,2,\cdots,b-1$,

$$\begin{aligned}V^*_{i,n}(u;b) &= v_i^n(1-q)\sum_{j=1}^{m} p_{i,j}\sum_{h=1}^{u+1}V_{j,n}(u+1-h;b)g_h\\ &+v_i^n q(1-\varphi)\sum_{j=1}^{m} p_{i,j}\sum_{h+k\le u+1}V_{j,n}(u+1-h-k;b)f_k g_h\\ &+v_i^n q\varphi\left(\theta\sum_{j=1}^{m} p_{i,j}\sum_{h+l+k\le u+1}V_{j,n}(u+1-l-k-h;b)f_k g_l g_h\right.\\ &\left.+(1-\theta)\sum_{j=1}^{m} p_{i,j}\sum_{h+k\le u+1}V^*_{j,n}(u+1-k-h;b)f_k g_h\right), \end{aligned} \quad (6)$$

with boundary conditions:

$$V_{i,n}(b;b) = \sum_{k=0}^{n}\binom{n}{k}V_{i,k}(b-1;b), \quad (7)$$

where $i$ = 1, 2, $\cdots$ ,$m$. The boundary condition holds because when the initial surplus is $b$, the premium received at the beginning of the first period will be paid out as a dividend immediately. Except the first dividend payment, the rest of the model is the same as that starting from an initial surplus $b-1$. From (4) and (5) one can rewrite $V^*_{i,n}(u;b)$ as

$$V^*_{i,n}(b;b) = \sum_{h=1}^{u}V_{i,n}(u-h;b)g_h, \; u = 1,2,\cdots,b-1. \quad (8)$$

This result can also be obtained from model (3) as

$$V^*_{i,n}(b;b) = \mathrm{E}[V_{i,n}(u-Y;b)] = \sum_{h=1}^{u}V_{i,n}(u-h;b)g_h$$

substituting (8) into (4) gives

$$\begin{aligned}V_{i,n}(0;b) &= v_i^n(1-q)\sum_{j=1}^{m} p_{i,j}V_{j,n}(1;b)\\ &+v_i^n q(1-\varphi)\sum_{j=1}^{m} p_{i,j}V_{j,n}(0;b)f_1\\ &+v_i^n\varphi(1-q)q(1-\theta)\sum_{j=1}^{m} p_{i,j}v_j^n\sum_{k=1}^{m} p_{j,k}V_{k,n}(0;b)f_1 g_1, \end{aligned} \quad (9)$$

and for $u = 1,2,\cdots,b-1$,

$$\begin{aligned}V_{i,n}(u;b) &= v_i^n(1-q)\sum_{j=1}^{m} p_{i,j}V_{j,n}(u+1;b)\\ &+v_i^n q(1-\varphi)\sum_{j=1}^{m} p_{i,j}\sum_{k=1}^{u+1}V_{j,n}(u+1-k;b)f_k\\ &+v_i^n q\varphi\sum_{j=1}^{m} p_{i,j}\sum_{h+k\le u+1}V_{j,n}(u+1-h-k;b)f_k g_h. \end{aligned} \quad (10)$$

with boundary condition (3.5).

Now let $W_1(u;n)$, $W_2(u;n)$, $\cdots$, $W_m(u;n)$ satisfy the following differential equations:

$$\begin{aligned}W_i(0;n) &= v_i^n(1-q)\sum_{j=1}^{m} p_{i,j}W_j(1;n)\\ &+v_i^n q(1-\varphi)\sum_{j=1}^{m} p_{i,j}W_j(0;n)f_1\\ &+v_i^n\varphi(1-q)q(1-\theta)\sum_{j=1}^{m} p_{i,j}v_j^n\sum_{k=1}^{m} p_{j,k}W_k(0;n)f_1 g_1, \end{aligned} \quad (11)$$

and for $u = 1,2,\cdots$,

$$\begin{aligned}W_i(u;n) &= v_i^n(1-q)\sum_{j=1}^{m} p_{i,j}W_j(u+1;n)\\ &+v_i^n q(1-\varphi)\sum_{j=1}^{m} p_{i,j}\sum_{k=1}^{u+1}W_j(u+1-k;n)f_k\\ &+v_i^n q\varphi\sum_{j=1}^{m} p_{i,j}\sum_{h+k\le u+1}W_j(u+1-h-k;b)f_k g_h. \end{aligned} \quad (12)$$

The solutions of (12) are uniquely determined by the initial conditions $W_i(u;n)$ for $i$ = 1,2$\cdots$,$m$.

Moreover, apart from a multiplicative constant, the solution of (11) and (12) is unique (see Xie and Zou [20], Wu and Li [14], Landriault [29]). Therefore, for an integer value $1 \le j \le m$, we can set $W_{1,j}(u;n)$, $W_{2,j}(u;n), \cdots, W_{m,j}(u;n)$ be the particular solutions of (11) and (12) with the initial conditions $W_{i,j}(0;n)=I(i=j)$, where $I(\cdot)$ is the indicator function. Then the general solution of (12) is of the form:

$$W_i(u;n) = \sum_{j=1}^{m} W_j(0;n) V_{i,j}(u;n),\ u \in \mathbb{N},\ i=1,2\cdots,m.$$

It follows that the solutions to (9) and (10) with boundary conditions (7) can be expressed as

$$V_{i,n}(u;b) = \sum_{j=1}^{m} V_{j,n}(0;b) W_{i,j}(u;n),\ u=0,1\cdots,b.$$

or in matrix notation

$$\vec{\mathbf{V}}_n(u;b) = \mathbf{W}_n(u)\vec{\mathbf{V}}_n(0;b),\ u=0,1\cdots,b. \quad (13)$$

where $\vec{\mathbf{V}}_n(u;b) = (V_{1,n}(u;b), V_{2,n}(u;b), \cdots, V_{m,n}(u;b))^T$ is an $m \times 1$ column vector and $\mathbf{W}_n(u) = (W_{i,j}(u;n))_{i,j=1}^{m}$ is an $m \times m$ matrix. The value of vector $\vec{\mathbf{V}}_n(0;b)$ can be obtained from the boundary conditions (7),

$$V_{1,n}(b;b) = \sum_{k=0}^{n} \binom{n}{k} V_{i,k}(b-1;b),$$

for $i = 1,2\cdots, m$, which can be rewritten in the following matrix form

$$\vec{\mathbf{V}}_n(b;b) - \vec{\mathbf{V}}_n(b-1;b) = \sum_{k=0}^{n-1} \binom{n}{k} \vec{\mathbf{V}}_k(b-1;b). \quad (14)$$

Setting $u = b$ and $b-1$ in (13) and plugging them into (14) gives

$$\vec{\mathbf{V}}_n(0;b) = [\mathbf{W}_n(b) - \mathbf{W}_n(b-1)]^{-1} \times \sum_{k=0}^{n-1} \binom{n}{k} \vec{\mathbf{V}}_k(b-1;b), \quad (15)$$

where $\vec{\mathbf{V}}_0(b-1;b) = (1,1,\cdots,1)^T$ is an $m \times 1$ column vector and $\vec{\mathbf{V}}_k(b-1;b)$ for $k = 1,2\cdots, n-1$ can be calculated repeatedly using (13) by setting $u = b-1$ and $n = k$.

Then we have the following matrix factorization formula for $\vec{\mathbf{V}}_n(u;b)$:

$$\vec{\mathbf{V}}_n(u;b) = \mathbf{W}_n(u)[\mathbf{W}_n(b) - \mathbf{W}_n(b-1)]^{-1} \times \sum_{k=0}^{n-1} \binom{n}{k} \vec{\mathbf{V}}_k(b-1;b), \quad (16)$$

in particular, when $n = 1$, we have the following matrix factorization formula for expected present value of total dividends $\vec{\mathbf{V}}_1(u;b)$:

$$\vec{\mathbf{V}}_1(u;b) = \mathbf{W}_1(u)[\mathbf{W}_1(b) - \mathbf{W}_1(b-1)]^{-1} \vec{\mathbf{I}}_1, \quad u = 0,1,\cdots,b, \quad (17)$$

where $\vec{\mathbf{I}}_1 = (1,1,\cdots,1)^T$ is the $m \times 1$ column vector. We know that $W_{1,j}(u;n)$, $W_{2,j}(u;n), \cdots, W_{m,j}(u;n)$ be the particular solutions of (11) and (12) with the initial conditions $W_{i,j}(0;n)=I(i=j)$, for $j = 1,2\cdots,m$, where $I(\cdot)$ is the indicator function. Then

$$W_{i,j}(0;n) = v_i^n(1-q)\sum_{k=1}^{m} p_{i,k} W_{k,j}(1;n) + v_i^n q(1-\varphi)\sum_{k=1}^{m} p_{i,k} W_{k,j}(0;n) f_1 + v_i^n \varphi(1-q)q(1-\theta)\sum_{k=1}^{m} p_{i,k} v_k^n \sum_{l=1}^{m} p_{k,l} W_{l,j}(0;n) f_1 g_1, \quad (18)$$

and for $u = 1,2,\cdots,$

$$W_{i,j}(u;n) = v_i^n(1-q)\sum_{k=1}^{m} p_{i,k} W_{k,j}(u+1;n) + v_i^n q(1-\varphi)\sum_{k=1}^{m} p_{i,k} \sum_{h=1}^{u+1} W_{k,j}(u+1-h;n) f_h + v_i^n q\varphi \sum_{k=1}^{m} p_{i,k} \sum_{h+l \le u+1} W_{k,j}(u+1-h-l;b) f_h g_l. \quad (19)$$

where $i, j = 1,2\cdots,m$. The matrix form of (18) is

$$\mathbf{I} = (1-q)\mathbf{v}_n \mathbf{P}(W_{i,j}(1;n))_{i,j=1}^{m} + q(1-\varphi) f_1 \mathbf{v}_n \mathbf{P} + \varphi q(1-q)(1-\theta) f_1 g_1 \mathbf{v}_n \mathbf{P} \mathbf{v}_n \mathbf{P}, \quad (20)$$

where $\mathbf{I}$ is the $m \times m$ identity matrix, $\mathbf{v}_n = \mathrm{diag}(v_1^n, v_2^n, \cdots, v_m^n)$.

We now apply the tool of generating functions to find the particular solutions $W_{i,j}(u;n)$ to the particular Eqs.(19).

Let $\tilde{W}_{i,j}(s;n)=\sum_{u=0}^{\infty}s^u W_{i,j}(u;n)$ be the generating function of $W_{i,j}(u;n)$. Taking generating functions on both sides of Eqs.(19) yields

$$\tilde{W}_{i,j}(s;n)-W_{i,j}(0;n)=v_i^n(1-q)s^{-1}$$
$$\times\sum_{k=1}^{m}p_{i,k}[\tilde{W}_{k,j}(s;n)-W_{k,j}(1;n)s-W_{k,j}(0;n)]$$
$$+v_i^n q(1-\varphi)s^{-1}\sum_{k=1}^{m}p_{i,k}\tilde{W}_{k,j}(s;n)\tilde{f}(s)$$
$$+v_i^n q\varphi s^{-1}\sum_{k=1}^{m}p_{i,k}\tilde{W}_{k,j}(s;n)\tilde{f}(s)\tilde{g}(s), \qquad (21)$$

where $W_{i,j}(0;n)=I(i=j)$ and $i,j = 1,2,\cdots,m$ . In matrix form,

$$(\mathbf{I}-(1-q)s^{-1}\mathbf{v}_n\mathbf{P}-q(1-\varphi)\tilde{f}(s)s^{-1}\mathbf{v}_n\mathbf{P}-q\varphi s^{-1}\tilde{f}(s)\tilde{g}(s)\mathbf{v}_n\mathbf{P})$$
$$\times\tilde{\mathbf{W}}_n(\mathrm{s})=\mathrm{I}-(1-q)\mathbf{v}_n\mathbf{P}(W_{i,j}(1;n))_{i,j=1}^m-(1-q)s^{-1}\mathbf{v}_n\mathbf{P}, \quad (22)$$

where $\tilde{\mathbf{W}}_n(\mathrm{s})=(W_{i,j}(s;n))_{i,j=1}^m$. Plunging (22) into (20) yields

$$\tilde{\mathbf{W}}_n(\mathrm{s})=[\mathbf{A}_n(\mathrm{s})]^{-1}\{(q(1-\varphi)f_1\mathbf{I}s+\varphi q(1-q)(1-\theta)f_1\mathrm{g}_1\mathbf{v}_n\mathbf{P}s$$
$$-(1-q)\mathbf{I})\mathbf{v}_n\mathbf{P}\}$$
$$=\frac{\mathbf{A}_n^*(\mathrm{s})}{\det[\mathbf{A}_n(\mathrm{s})]}\{(q(1-\varphi)f_1\mathbf{I}s+\varphi q(1-q)(1-\theta)f_1\mathrm{g}_1\mathbf{v}_n\mathbf{P}s$$
$$-(1-q)\mathbf{I})\mathbf{v}_n\mathbf{P}\}, \qquad (23)$$

where
$\mathbf{A}_n(\mathrm{s})=s\mathbf{I}-((1-q)+q(1-\varphi)\tilde{f}(s)+q\varphi\tilde{f}(s)\tilde{g}(s))\mathbf{v}_n\mathbf{P}$,
and $\mathbf{A}_n^*(\mathrm{s})$ is the adjoint matrix of $\mathbf{A}_n(\mathrm{s})$.

In particular, let $m = 1$, that is to say, $r_1 = r_2 = \cdots = r_m = r$, that is to say, the interest rate is constant $r$ in each period, then the expected present value of total dividend payments up to ruin, $V_1(u;b)$, with the definition

$$V_1(u;b)=\mathrm{E}\left[\sum_{k=1}^{T_u}\frac{D_k}{(1+r)^k}\middle|U_b(0)=u\right], u=0,1,\cdots,b.$$

can be expressed as

$$V_1(u;b)=W(u)[W(b)-W(b-1)]^{-1}, \qquad (24)$$

where $W(0)=1$. Let $\tilde{W}(\mathrm{s})$ be the generating function of $W(u)$, then $\tilde{W}(\mathrm{s})$ satisfies the following expression:

$$\tilde{W}(\mathrm{s})=\frac{(q(1-\varphi)f_1+\varphi q(1-q)(1-\theta)f_1\mathrm{g}_1v-(1-q)s^{-1})v}{1-((1-q)+q(1-\varphi)\tilde{f}(s)+q\varphi\tilde{f}(s)\tilde{g}(s))vs^{-1}}, \quad (25)$$

where $v = 1/(1 + r)$.

Similar to the method of Yuen and Guo [2], we construct two new generating functions $\tilde{h}(s,1)=1-q+q(1-\varphi)\tilde{f}(s)+q\varphi\tilde{f}(s)\tilde{g}(s)$ and $\tilde{h}(s,k)=[\tilde{h}(z,1)]^k$, $-1 < \mathbf{R}(s) < 1$. We denote the probability function of $\tilde{h}(s,k)$ by $h(i;k)$. Note that the denominator on the right-hand side of (25) is $1-v\tilde{h}(s,1)s^{-1}$. Rewriting $[1-v\tilde{h}(s,1)s^{-1}]^{-1}$ in terms of a power series in $s$, we have

$$\tilde{W}(s)=(q(1-\varphi)f_1+\varphi q(1-q)(1-\theta)f_1\mathrm{g}_1v)v\sum_{k=0}^{\infty}v^k\tilde{h}(s,k)s^{-k}$$
$$-(1-q)v\sum_{k=0}^{\infty}v^k\tilde{h}(s,k)s^{-(k+1)}.$$

Comparing the coeffcients of $s^u$ in both sides gives, for $u = 1, 2, \cdots, b$,

$$W(u)=(q(1-\varphi)f_1+\varphi q(1-q)(1-\theta)f_1\mathrm{g}_1v)v\sum_{i=u+1}^{\infty}v^{i-u}h(i,i-u)$$
$$-(1-q)v\sum_{i=u+2}^{\infty}v^{i-u-1}h(i,i-u-1)$$
$$=\sum_{i=1}^{\infty}v^{i+1}[(q(1-\varphi)f_1+\varphi q(1-q)(1-\theta)f_1\mathrm{g}_1v)h(i+u,i)$$
$$-(1-q)h(i+u+1,i)]. \qquad (26)$$

The above result together with (24) gives us the explicit expression for $V_1(u;b)$.

When $m > 1$, $\tilde{\mathbf{W}}_n(\mathrm{s})$ can be inverted if each element is a rational function, while each element is a rational function if and only if claim size distributions have rational p.g.f. or the claim size distributions have finite support so that the p.g.f. are polynomials, as will be seen in the following Section.

**Remarks:** (I) When $\varphi = 1$, that is to say, each main claim induces a by-claim to a certainty. Xie and Zou [20] studied the expected present value of total dividends $\vec{\mathbf{V}}_1(u;b)$ in this case. When $\theta = 1$ and $\varphi = 1$, that is to say, in any time period, each main claim induces a by-claim, the main claim and its associated by-claim occur simultaneously. Actually, this case is very similar to the one proposed by Li [24]. In our set-up, there is a by-claim, $Y$, occuring simultaneously with the main claim $X$. Hence, the only difference is that we use $X + Y$ as our claim amount random variable while Li simply considers $X$. Hence our results in this paper include the corresponding results of Xie and Zou [20] and Li [24].

(II) When $m = 1$, from the expressions for $V_1(u; b)$ in (24) and $W(u)$ in (25), we can see that, for a fixed initial surplus $u$, the optimal dividend barrier $b^*$ which maximizes $V_1(u; b)$ or minimizes the denominator of (24) is independent of $u$, since the denominator of (24) is a function independent of $u$.

(III) If $m > 1$, $V_{i;1}(u; b) = \vec{e}_i^T \vec{\mathbf{V}}_1(u;b)$, where $\vec{e}_i$ is an $m \times 1$ column vector with the $i$-th element being 1 and all other elements being 0. It follows from (17) that, for a fixed $i$ and $u$, $V_{i;1}(u; b)$ cannot be written as product of a function $u$ and a function $b$, therefore the optimal dividend barrier level $b^*$ which maximizes $V_{i;1}(u; b)$ depends on the initial surplus $u$.

## 4. Two classes of claim size distributions

In this section, we consider two special cases for the claim size distributions such that $\tilde{\mathbf{W}}_n(s)$ has a rational generating function which can be easily inverted. One case is that the probability functions of claim sizes have finite support such that their p.g.f. are polynomials, and the other case is that claim sizes have discrete $Kn$ distributions, i.e., the p.g.f. of $X$ and $Y$ are the ratio of two polynomials with certain conditions.

### *4.1. Claim amount distributions with finite support*

Now assume that the distributions of $X_1$ and $X_1 + Y_1$ have finite support, e.g., for $N = 2, 3, \cdots$,

$$f_x = \Pr(X_1 = x) = \kappa_x,\ x = 1,2,\cdots, N-1,$$

and

$$(f * g)_x = \Pr(X_1 + Y_1 = y) = \zeta_y,\ y = 2,3,\cdots, N,$$

where * denotes convolution. Then

$$\tilde{f}(s) = \sum_{x=1}^{N-1} s^x \kappa_x \text{ and } \tilde{f}(s)\tilde{g}(s) = \sum_{x=2}^{N} s^x \zeta_x,$$

$-1 < \mathbf{R}(s) < 1$, are the polynomials of degree $N-1$ and $N$, respectively, so $\det[\mathbf{A}_n(s)]$ is a polynomial of degree $N \cdot m$ and each element of $\mathbf{A}^*_n(s)$ is a polynomial of degree $N(m-1)$. Let $\rho_1,\ \rho_2, \cdots, \rho_{N\cdot m}$ be the roots of equation $\det[\mathbf{A}_n(s)]=0$, then $\det[\mathbf{A}_n(s)] = a_{N\cdot m}^{(n)} \prod_{i=1}^{N\cdot m}(s-\rho_i)$, where $a_{N\cdot m}^{(n)}$ is the leading coefficient of the polynomial $\det[\mathbf{A}_n(s)]$. For simplicity, we assume that $\rho_1,\ \rho_2, \cdots,\ \rho_{N\cdot m}$ are distinct. It follows from partial fractions that (23) can be rewritten as

$$\tilde{\mathbf{W}}_n(\mathrm{s}) = \left[\sum_{i=1}^{N\cdot m} \frac{\mathbf{M}_i^{(n)}}{(\rho_i - s)}\right] \mathbf{v}_n \mathbf{P}, \qquad (27)$$

where

$$\mathbf{M}_i^{(n)} = \frac{\mathbf{A}_n^*(\rho_i)}{a_{N\cdot m}^{(n)} \prod_{j=1, j\neq i}^{N\cdot m}(\rho_i - \rho_j)} \{(1-q)\mathbf{I} - q(1-\varphi)\kappa_1 \mathbf{I}\rho_i$$
$$-\varphi q(1-q)(1-\theta)\zeta_2 \mathbf{v}_n \mathbf{P} \rho_i\}$$

$i = 1,2,\cdots, N\,.m$, is an $m\times m$ matrix. Inverting (27), yields

$$\mathbf{W}_n(u) = \left[\sum_{i=1}^{N\cdot m} \mathbf{M}_i^{(n)} \rho_i^{-(u+1)}\right] \mathbf{v}_n \mathbf{P},\ n \in \mathbf{N}^+.$$

Once $\mathbf{W}_n(u)$ is obtained, $\vec{\mathbf{V}}_n(u;b)$ can be calculated using (15) and (16), then

$$\vec{\mathbf{V}}_n(0;b) = \left[\left[\sum_{i=1}^{N\cdot m} \mathbf{M}_i^{(n)} \left(\rho_i^{-(b+1)} - \rho_i^{-b}\right)\right] \mathbf{v}_n \mathbf{P}\right]^{-1}$$
$$\times \sum_{k=0}^{n-1} \binom{n}{k} \vec{\mathbf{V}}_k(b-1;b), \qquad (28)$$

and for $u = 1, 2, \cdots, b$,

$$\vec{\mathbf{V}}_n(u;b) = \left[\sum_{i=1}^{N\cdot m} \mathbf{M}_i^{(n)} \rho_i^{-(u+1)}\right] \left[\sum_{i=1}^{N\cdot m} \mathbf{M}_i^{(n)} \left(\rho_i^{-(b+1)} - \rho_i^{-b}\right)\right]^{-1}$$
$$\times \sum_{k=0}^{n-1} \binom{n}{k} \vec{\mathbf{V}}_k(b-1;b), \qquad (29)$$

where $\vec{\mathbf{V}}_k(b-1;b)$ for $k = 1, 2, \cdots, n-1$ can be calculated repeatedly using $\vec{\mathbf{V}}_n(u;b) = \mathbf{W}_n(u)\vec{\mathbf{V}}_n(b-1;b)$ by setting $u = b-1$ and $n = k$.

**Example 1.** In this example, we assume $f_1 = g_1 = 1$. Then $S_k - S_{k-1}$ can only take three possible values: 1, 0, or $-1$. This generalizes De Finetti's [8] original model where periodic gains are +1 or -1. Then $N = 2$, $\mathbf{A}_n(s) = s\mathbf{I} - ((1-q) + q(1-\varphi)s$ $+q\varphi s^2)\mathbf{v}_n \mathbf{P}$ and

$$\mathbf{M}_i^{(n)} = \frac{\mathbf{A}_n^*(\rho_i)}{a_{2m}^{(n)}\prod_{j=1,j\neq i}^{2m}(\rho_i-\rho_j)}\{(1-q)\mathbf{I}-q(1-\varphi)\mathbf{I}\rho_i$$

$$-\varphi q(1-q)(1-\theta)\mathbf{v}_n\mathbf{P}\rho_i\}\text{, } i=1, 2, \cdots, 2m.$$

From (28) and (29), we can get

$$\vec{\mathbf{V}}_n(0;b) = \left[\left[\sum_{i=1}^{2m}\mathbf{M}_i^{(n)}\left(\rho_i^{-(b+1)}-\rho_i^{-b}\right)\right]\mathbf{v}_n\mathbf{P}\right]^{-1}$$

$$\times\sum_{k=0}^{n-1}\binom{n}{k}\vec{\mathbf{V}}_k(b-1;b)\text{,} \tag{30}$$

and for $u = 1, 2, \cdots, b$,

$$\vec{\mathbf{V}}_n(u;b) = \left[\sum_{i=1}^{2m}\mathbf{M}_i^{(n)}\rho_i^{-(u+1)}\right]\left[\sum_{i=1}^{2m}\mathbf{M}_i^{(n)}\left(\rho_i^{-(b+1)}-\rho_i^{-b}\right)\right]^{-1}$$

$$\times\sum_{k=0}^{n-1}\binom{n}{k}\vec{\mathbf{V}}_k(b-1;b)\text{,} \tag{31}$$

In particular, let $m = 1$, the interest rate is constant $r$ in each period. The generating function of $W(u)$ in (25), has a simplified expression

$$\tilde{W}(s) = \frac{v(1-q)-[q(1-\varphi)v+\varphi q(1-q)(1-\theta)v^2]s}{vq\varphi s^2-[1-vq(1-\varphi)]s+v(1-q)}. \tag{32}$$

Let $R_1$ and $R_2$ be the solutions of the equation $vq\varphi s^2 - [1-vq(1-\varphi)]s+v(1-q)=0$, then

$$R_1 = \frac{1-qv(1-\varphi)-\sqrt{(1-qv(1-\varphi))^2-4(1-q)q\varphi v^2}}{2qv\varphi},$$

and

$$R_2 = \frac{1-qv(1-\varphi)+\sqrt{(1-qv(1-\varphi))^2-4(1-q)q\varphi v^2}}{2qv\varphi},$$

where $0 < q < 1$, $0 < \varphi < 1$, and $0 < v < 1$. From the expansion formulae of $(1-qv-qv\varphi)^2$ and $(1-qv(1-\varphi))^2-4(1-q)q\varphi v^2$, it follows that $(1-qv-qv\varphi)^2 < (1-qv(1-\varphi))^2-4(1-q)q\varphi v^2$, then we can know that $R1 < 1$. Moreover, it is easy to see that $R_1R_2 = (1-q)/(q\varphi) > 0$ and $R1 + R2 = [1-vq(1-\varphi)]/(vq\varphi)$. In this case, the positive safety loading condition, $q(1+\varphi) < 1$, implies that

$$R_1+R_2 = 1+\frac{1-vq}{vq\varphi} > 1+\frac{v-vq}{vq\varphi} > 2.$$

From these discussions, it follows that $0<R1<1<R2$. By partial fractions, (32) can be rewritten as

$$\tilde{W}(s) = \frac{1}{q\varphi}\left(\frac{a_1}{R_1-s}+\frac{a_2}{R_2-s}\right),$$

where

$$a_1 = \frac{[q(1-\varphi)+\varphi q(1-q)(1-\theta)v]R_1-(1-q)}{R_1-R_2},$$

$$a_2 = \frac{[q(1-\varphi)+\varphi q(1-q)(1-\theta)v]R_2-(1-q)}{R_2-R_1},$$

inverting the generating function $\tilde{W}(s)$ yields

$$W(u) = \frac{1}{q\varphi}\left(a_1R_1^{-(u+1)}+a_2R_2^{-(u+1)}\right). \tag{33}$$

Substituting (33) into (24) gives, for $u = 0, 1, \cdots, b$,

$$V_1(u,b) = \frac{a_1R_1^{-(u+1)}+a_2R_2^{-(u+1)}}{a_1(1-R_1)R_1^{-(b+1)}+a_2(1-R_2)R_2^{-(b+1)}}. \tag{34}$$

Another value of interest in this case is the optimal dividend barrier $b^*$, which is the optimal value of $b$ that maximizes $V_1(u; b)$ for a given $u$. From (34) we know that $b^*$ is the solution of equation

$$\frac{\mathrm{d}}{\mathrm{d}b}\left(a_1(1-R_1)R_1^{-(b+1)}+a_2(1-R_2)R_2^{-(b+1)}\right)=0$$

we have

$$b^* = \frac{\ln\dfrac{-a_1R_2(1-R_1)\ln(R_1)}{a_2R_1(1-R_2)\ln(R_2)}}{\ln(R_1)-\ln(R_2)},$$

which does not depend on the initial surplus $u$. Practically, we round $b^*$ to the closest integral value. Furthermore, we can prove the following result.

**Theorem 4.1.** If the interest rate is constant $r$ in each period, then the expected present value of the dividend payments up to the time of ruin in the risk model considered in Example 1, $V_1(u; b)$, increases as the probability of a delay of the by-claims is increasing as well.

**Proof:** The theorem can be proved by the following fact:

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{a_1R_1^{-(u+1)}+a_2R_2^{-(u+1)}}{a_1(1-R_1)R_1^{-(b+1)}+a_2(1-R_2)R_2^{-(b+1)}}\right)$$
$$=\frac{-(R_1-R_2)(R_1R_2)^{-1-b-u}(1-q)^2 qv\varphi}{\left(a_1(1-R_1)R_1^{-(b+1)}+a_2(1-R_2)R_2^{-(b+1)}\right)^2}$$
$$\times\left(R_1^uR_2^b(-1+R_1)+R_1^bR_2^u(1-R_2)\right)<0,$$

since $0 < R_1 < 1 < R_2$.

In Example 1, we assume that $q = 0.3$ and the interest rates have three possible values: $r_1 = 0.02$, $r_2 =$ $0.05$, $r_3 = 0.08$. The transition probability matrix $\mathbf{P}$ is

$$\mathbf{P}=\begin{pmatrix}0.75 & 0.15 & 0.1\\ 0.2 & 0.3 & 0.5\\ 0.05 & 0.8 & 0.15\end{pmatrix}.$$

Table 1 summarizes the results for $V_{i,1}(u;b)(i = 1, 2, 3)$ with fixed $\theta = 0.4$ and $\varphi = 0.6$ for $u = 0,1,2,3,4,6,8,10$ and $b = 1,2,3,4,5$. From Table 1, we can see that for $u = 0$, the optimal dividend barrier $b^*$ which maximizes $V_i(u; b)$ is 1 for $i = 1,2,3$. For $u\geq 1$, the optimal dividend barrier $b^*$ is equal to the initial surplus $u$ for $i = 1,2,3$. Furthermore, as expected, $V_i(u; b)(i = 1; 2; 3)$ is increasing in $u$ for fixed $b$ and $i$.

In Table 2, the 2th moments of the present values of dividend payments up to the time of ruin $V_{i,2}(u; b)(i = 1,2,3)$ with fixed $\theta = 0.4$ and $\varphi = 0.6$ for $u = 0,1,2, 3, 4, 6,8,10$ and $b = 1,2,3,4,5$ are provided. The numbers show that the higher the initial surplus of the insurance company, the higher the 2th moment of the present value of dividend payments up to the time of ruin for fixed $b$.

### *4.2. $K_h$ claim amount distributions*

Li [30] studied a class of discrete Sparre Andersen risk models in which the claims inter-arrival times are $K_h$ distributed. This class of distributions includes geometric, negative binomial, discrete phase-type, as well as linear combinations (including mixtures) of these.

For the two independent claim amount random variables $X_1$ and $Y_1$, if they have $K_h$ distributions, so does their sum. Therefore, in this Section, we assume that $f_x = \Pr(X_1 = x)$ and $g_x = \Pr(Y_1 = x)$ are $K_h$ distributed and $K_l$ distributed, respectively, for $x = 1,2,\cdots$, and $h,l = 1,2,\cdots$, i.e., the p.g.f. of $f$ and $g$ are given by

$$\tilde{f}(s)=\frac{E_h(s)}{\prod_{i=1}^h(1-s\varsigma_i)},$$
$$\tilde{g}(s)=\frac{F_l(s)}{\prod_{j=1}^l(1-s\psi_j)},$$
$$\mathbf{R}(s)<\min\left(\frac{1}{\varsigma_i},\frac{1}{\psi_j}:1\le i\le h,1\le j\le l\right),$$

where $0 < \varsigma_i < 1$, $0 < \psi_j < 1$, for $i = 1, 2, \cdots, h$, $j = 1, 2, \cdots, l$, $E_h(s)$ is a polynomial of degree $h$ with $E_h(0) = 0$, $E_h(1) = \prod_{i=1}^h(1-\varsigma_i)$ and $F_l(s)$ is a polynomial of degree $l$ with $F_l(0) = 0$, $F_l(1) = \prod_{j=1}^l(1-\psi_j)$.

Then $\tilde{\mathbf{W}}_n(s)$ can be transformed to the following rational function

$$\tilde{\mathbf{W}}_n(s)=\frac{\mathbf{B}_n^*(s)\prod_{i=1}^h(1-s\varsigma_i)\prod_{j=1}^l(1-s\psi_j)}{\det[\mathbf{B}_n(s)]}\{(q(1-\varphi)f_1\mathbf{I}s$$
$$+\varphi q(1-q)(1-\theta)f_1g_1\mathbf{v}_n\mathbf{P}s-(1-q)\mathbf{I})\mathbf{v}_n\mathbf{P}\}, \qquad (35)$$

where

$$\mathbf{B}_n(s)=s\prod_{i=1}^h(1-s\varsigma_i)\prod_{j=1}^l(1-s\psi_j)\mathbf{I}-\{(1-q)$$
$$\times\prod_{i=1}^h(1-s\varsigma_i)\prod_{j=1}^l(1-s\psi_j)+q(1-\varphi)E_h(s)$$
$$\times\prod_{j=1}^l(1-s\psi_j)+q\varphi E_h(s)F_l(s)\}\mathbf{v}_n\mathbf{P}.$$

Then $\det[\mathbf{B}_n(s)]$ is a polynomial of degree $m(h+l+1)$ and each element of $\mathbf{B}_n^*(s)$ is a polynomial of degree $(m-1)(h+l+1)$.

**Table 1.** Values of $V_{i,1}(u;b)(i=1,2,3)$ when $f_1 = g_1 = 1$, $\theta = 0.4$ ,and $\varphi = 0.6$

| $u \backslash b$ | 1 | 2 | 3 | 4 | 5 | $R_0$ |
|---|---|---|---|---|---|---|
| $u=0$ | 1.92093379 | 1.09186542 | 0.67173379 | 0.41593782 | 0.25594156 | $r_1 = 0.02$ |
| | 0.27576488 | 0.05729057 | 0.01199763 | 0.00251272 | 0.00052625 | $r_2 = 0.05$ |
| | 0.11200906 | 0.01108302 | 0.00109854 | 0.00010889 | 0.00001079 | $r_3 = 0.08$ |
| $u=1$ | 2.9209337 | 1.66026889 | 1.02142507 | 0.63246679 | 0.38917966 | $r_1 = 0.02$ |
| | 1.27576488 | 0.26504209 | 0.05550435 | 0.01162455 | 0.00243457 | $r_2 = 0.05$ |
| | 1.11200906 | 0.11003057 | 0.01090616 | 0.00108102 | 0.00010715 | $r_3 = 0.08$ |
| $u=2$ | | 2.66026889 | 1.63664173 | 1.01340918 | 0.62358727 | $r_1 = 0.02$ |
| | | 1.26504209 | 0.26492146 | 0.05548381 | 0.01162014 | $r_2 = 0.05$ |
| | | 1.11003057 | 0.11002555 | 0.01090571 | 0.00108097 | $r_3 = 0.08$ |
| $u=3$ | | | 2.63664173 | 1.63260956 | 1.00460363 | $r_1 = 0.02$ |
| | | | 1.26492146 | 0.26491874 | 0.05548274 | $r_2 = 0.05$ |
| | | | 1.11002555 | 0.11002552 | 0.01090571 | $r_3 = 0.08$ |
| $u=4$ | | | | 2.63260956 | 1.61993975 | $r_1 = 0.02$ |
| | | | | 1.26491874 | 0.26491581 | $r_2 = 0.05$ |
| | | | | 1.11002552 | 0.11002551 | $r_3 = 0.08$ |
| $u=5$ | | | | | 2.61993975 | $r_1 = 0.02$ |
| | | | | | 1.26491581 | $r_2 = 0.05$ |
| | | | | | 1.11002551 | $r_3 = 0.08$ |

**Table 2.** Values of $V_{i,2}(u;b)(i=1,2,3)$ when $f_1 = g_1 = 1$, $\theta = 0.4$ , and $\varphi = 0.6$

| $u \backslash b$ | 1 | 2 | 3 | 4 | 5 | $R_0$ |
|---|---|---|---|---|---|---|
| $u=0$ | 8.63081524 | 4.28771261 | 2.54467326 | 1.53733911 | 0.93821331 | $r_1 = 0.02$ |
| | 0.40011079 | 0.07791821 | 0.01549515 | 0.00308215 | 0.00061308 | $r_2 = 0.05$ |
| | 0.12551688 | 0.01144682 | 0.00104884 | 0.00009610 | 0.00000881 | $r_3 = 0.08$ |
| $u=1$ | 13.4726828 | 6.69310957 | 3.97222914 | 2.39978283 | 1.46454883 | $r_1 = 0.02$ |
| | 1.95164054 | 0.38006559 | 0.07558145 | 0.01503397 | 0.00299047 | $r_2 = 0.05$ |
| | 1.34953501 | 0.12307413 | 0.01127703 | 0.00103330 | 0.00009468 | $r_3 = 0.08$ |
| $u=2$ | | 11.0136473 | 6.53638349 | 3.94889125 | 2.40994475 | $r_1 = 0.02$ |
| | | 1.91014978 | 0.37986048 | 0.07555836 | 0.01502962 | $r_2 = 0.05$ |
| | | 1.34313528 | 0.12306873 | 0.01127666 | 0.00103327 | $r_3 = 0.08$ |
| $u=3$ | | | 10.8096669 | 6.53055307 | 3.98549139 | $r_1 = 0.02$ |
| | | | 1.90970342 | 0.37986066 | 0.07555962 | $r_2 = 0.05$ |
| | | | 1.34311983 | 0.12306872 | 0.01127666 | $r_3 = 0.08$ |
| $u=4$ | | | | 10.7957722 | 6.58848596 | $r_1 = 0.02$ |
| | | | | 1.90969814 | 0.37986578 | $r_2 = 0.05$ |
| | | | | 1.34311976 | 0.12306872 | $r_3 = 0.08$ |
| $u=5$ | | | | | 10.8283655 | $r_1 = 0.02$ |
| | | | | | 1.90969741 | $r_2 = 0.05$ |
| | | | | | 1.34311974 | $r_3 = 0.08$ |

$$+\varphi q(1-q)(1-\theta) f_1 g_1 \mathbf{v}_n \mathbf{P} s-(1-q)\mathbf{I})\mathbf{v}_n\mathbf{P}\}, \quad (35)$$

where

$$\mathbf{B}_n(s)=s\prod_{i=1}^{h}(1-s\varsigma_i)\prod_{j=1}^{l}(1-s\psi_j)\mathbf{I}-\{(1-q)\times\prod_{i=1}^{h}(1-s\varsigma_i)\prod_{j=1}^{l}(1-s\psi_j)+q(1-\varphi)E_h(s)\times\prod_{j=1}^{l}(1-s\psi_j)+q\varphi E_h(s)F_l(s)\}\mathbf{v}_n\mathbf{P}.$$

Then $\det[\mathbf{B}_n(s)]$ is a polynomial of degree $m(h+l+1)$ and each element of $\mathbf{B}_n^*(s)$ is a polynomial of degree $(m-1)(h+l+1)$.

Let $\sigma_1, \sigma_2, \cdots, \sigma_{m(h+l+1)}$ be the roots of equation $\det[\mathbf{B}_n(s)]=0$. Then

$$\det[\mathbf{B}_n(s)]=b_{m(h+l+1)}^{(n)}\prod_{i=1}^{m(h+l+1)}(s-\sigma_i),$$

where $b_{m(h+l+1)}^{(n)}$ is the leading coefficient of the polynomial $\det[\mathbf{B}_n(s)]$. For simplicity, we assume that $\sigma_1, \sigma_2, \cdots, \sigma_{m(h+l+1)}$ are distinct. It follows from partial fractions that (35) can be rewritten as

$$\tilde{\mathbf{W}}_n(s)=\left[\sum_{t=1}^{m(h+l+1)}\frac{\mathbf{L}_t^{(n)}}{(\sigma_t-s)}\right]\{((1-q)\mathbf{I}-q(1-\varphi)f_1\mathbf{I}s-\varphi q(1-q)(1-\theta)f_1g_1\mathbf{v}_n\mathbf{P}s)\mathbf{v}_n\mathbf{P}\}, \quad (36)$$

where

$$\mathbf{L}_t^{(n)}=\frac{\mathbf{B}_n^*(\sigma_t)\prod_{i=1}^{h}(1-\sigma_t\varsigma_i)\prod_{j=1}^{l}(1-\sigma_t\psi_j)}{b_{m(h+l+1)}^{(n)}\prod_{k=1,k\neq t}^{m(h+l+1)}(\sigma_t-\sigma_k)},$$

$t=1, 2, \cdots, m(h+l+1)$, is an $m\times m$ matrix. Inverting $\tilde{\mathbf{W}}_n(s)$ yields

$$\mathbf{W}_n(0)=\left[\sum_{t=1}^{m(h+l+1)}\mathbf{L}_t^{(n)}\sigma_t^{-1}\right](1-q)\mathbf{v}_n\mathbf{P}=\mathbf{I}, \quad (37)$$

and

$$\mathbf{W}_n(u)=\left[\sum_{t=1}^{m(h+l+1)}\mathbf{L}_t^{(n)}\sigma_t^{-1}((1-q)\mathbf{I}-q(1-\varphi)f_1\mathbf{I}\sigma_t-\varphi q(1-q)(1-\theta)f_1g_1\mathbf{v}_n\mathbf{P}\sigma_t)\right]\mathbf{v}_n\mathbf{P}, \ u=1, 2, \cdots. \quad (38)$$

Then finally we have

$$\vec{\mathbf{V}}_n(0;b)=\left[\left[\sum_{t=1}^{m(h+l+1)}\mathbf{L}_t^{(n)}(\sigma_t^{-(b+1)}-\sigma_t^{-b})((1-q)\mathbf{I}-q(1-\varphi)f_1\mathbf{I}\sigma_t-\{(1-q)\varphi q(1-q)(1-\theta)f_1g_1\mathbf{v}_n\mathbf{P}\sigma_t)\right]\mathbf{v}_n\mathbf{P}\right]^{-1}\times\sum_{k=0}^{n-1}\binom{n}{k}\vec{\mathbf{V}}_k(b-1;b), \quad (39)$$

and

$$\vec{\mathbf{V}}_n(u;b)=\left[\sum_{t=1}^{m(h+l+1)}\mathbf{L}_t^{(n)}\sigma_t^{-(u+1)}((1-q)\mathbf{I}-q(1-\varphi)f_1\mathbf{I}\sigma_t-\varphi q(1-q)(1-\theta)f_1g_1\mathbf{v}_n\mathbf{P}\sigma_t)\right]\left[\sum_{t=1}^{m(h+l+1)}\mathbf{L}_t^{(n)}(\sigma_t^{-(b+1)}-\sigma_t^{-b})\times((1-q)\mathbf{I}-q(1-\varphi)f_1\mathbf{I}\sigma_t-\varphi q(1-q)(1-\theta)f_1g_1\mathbf{v}_n\mathbf{P}\sigma_t)\right]^{-1}\times\sum_{k=0}^{n-1}\binom{n}{k}\vec{\mathbf{V}}_k(b-1;b), \ u=1, 2, \cdots, b. \quad (40)$$

**Example 2.** In this example, we assume that the main claim $X_1$ follows a geometric distribution with $f_x=\alpha^{x-1}(1-\alpha)$, $0<\alpha<1$, $x=1, 2, \cdots$, and the by-claim $Y_1$ follows a geometric distribution with $g_x=\beta^{x-1}(1-\beta)$, $0<\beta<1$, $x=1, 2, \cdots$, so that $\tilde{f}(s)=\dfrac{(1-\alpha)s}{1-\alpha s}$, and $\tilde{g}(s)=\dfrac{(1-\beta)s}{1-\beta s}$.

Here $h=l=1$, $\varsigma_i=\alpha$, $\psi_1=\beta$ and $E_h(s)=(1-\alpha)s$, $F_l(s)=(1-\beta)s$.

From (39) and (40), we can get

$$\vec{\mathbf{V}}_n(0;b)=\left[\left[\sum_{t=1}^{3m}\mathbf{L}_t^{(n)}(\sigma_t^{-(b+1)}-\sigma_t^{-b})((1-q)\mathbf{I}-q(1-\varphi)(1-\alpha)\mathbf{I}\sigma_t-\varphi q(1-q)(1-\theta)(1-\alpha)(1-\beta)\mathbf{v}_n\mathbf{P}\sigma_t)\right]\mathbf{v}_n\mathbf{P}\right]^{-1}\times\sum_{k=0}^{n-1}\binom{n}{k}\vec{\mathbf{V}}_k(b-1;b), \quad (41)$$

and

$$\vec{\mathbf{V}}_n(u;b)=\left[\sum_{t=1}^{3m}\mathbf{L}_t^{(n)}\sigma_t^{-(u+1)}((1-q)\mathbf{I}-q(1-\varphi)(1-\alpha)\mathbf{I}\sigma_t\right.$$
$$\left.-\varphi q(1-q)(1-\theta)(1-\alpha)(1-\beta)\mathbf{v}_n\mathbf{P}\sigma_t)\right]\left[\sum_{t=1}^{3m}\mathbf{L}_t^{(n)}(\sigma_t^{-(b+1)}-\sigma_t^{-b})\right.$$
$$\left.\times((1-q)\mathbf{I}-q(1-\varphi)f_1\mathbf{I}\sigma_t-\varphi q(1-q)(1-\theta)(1-\alpha)(1-\beta)\mathbf{v}_n\mathbf{P}\sigma_t)\right]^{-1}$$
$$\times\sum_{k=0}^{n-1}\binom{n}{k}\vec{\mathbf{V}}_k(b-1;b),\ u=1,2,\cdots,b. \qquad (42)$$

where

$$\mathbf{L}_t^{(n)}=\frac{\mathbf{B}_n^*(\sigma_t)(1-\sigma_t\alpha)(1-\sigma_t\beta)}{b_{m(h+l+1)}^{(n)}\prod_{j=1,j\neq t}^{3m}(\sigma_t-\sigma_j)},\ t=1,2,\cdots,3m,$$

Where

$$\mathbf{B}_n(s)=s(1-s\alpha)(1-s\beta)\mathbf{I}-\{(1-q)(1-s\alpha)(1-s\beta)$$
$$+q(1-\varphi)(1-\alpha)s(1-s\beta)+q\varphi(1-\alpha)(1-\beta)s^2\}\mathbf{v}_n\mathbf{P}.$$

As an example, let $q$ = 0.4, $\alpha$ = 0.6, $\beta$ = 0.3 and the interest rates have two possible values: $r_1$ =0.02, $r_2$ = 0.07. The transition probability matrix **P** is

$$\mathbf{P}=\begin{pmatrix}0.75 & 0.25\\ 0.3 & 0.7\end{pmatrix}.$$

The values of $V_{i,1}(u;b)(i=1,2)$ with fixed $\varphi$= 0.5 for $\theta$= 0, 0.25, 0.5, 0.75, 1 and $u$ = 0, 1, 2, 3, 4, 6, 8, 10 are listed in Table 3. We observe that $V_{i,1}(u;b)(i=1,2)$ is an increasing function with respect to $u$, and a decreasing function over $\theta$ for fixed $\varphi$. Also, if the main claim once induce a by-claim, the impact of the delay of by-claims on the expected present value of dividends is reduced for a higher initial surplus of the company.

In Table 4, the expected present values of dividend payments $V_{i,1}(u;b)$ ($i$ = 1; 2) with fixed $\theta$ = 0.4 for $\varphi$= 0.1, 0.3, 0.5, 0.7, 1 and $u$ = 0, 1, 2, 3, 4, 6, 8, 10 are provided. We observe that $V_{i,1}(u;b)$ ($i$ = 1; 2) is decreasing in $\varphi$ for fixed $\theta$. Also, the impact of the occurrence of by-claim on the expected present value of dividends is reduced for a higher initial surplus of the company.

Let $\sigma_1$, $\sigma_2,\cdots$, $\sigma_{m(h+l+1)}$ be the roots of equation det [$\mathbf{B}_n(s)$] =0. Then

$$\det[\mathbf{B}_n(s)]=b_{m(h+l+1)}^{(n)}\prod_{i=1}^{m(h+l+1)}(s-\sigma_i),$$

where $b_{m(h+l+1)}^{(n)}$ is the leading coefficient of the polynomial det[$\mathbf{B}_n(s)$]. For simplicity, we assume that $\sigma_1$, $\sigma_2,\cdots,\sigma_{m(h+l+1)}$ are distinct. It follows from partial fractions that (35) can be rewritten as

$$\tilde{\mathbf{W}}_n(s)=\left[\sum_{t=1}^{m(h+l+1)}\frac{\mathbf{L}_t^{(n)}}{(\sigma_t-s)}\right]\{((1-q)\mathbf{I}-q(1-\varphi)f_1\mathbf{I}s$$
$$-\varphi q(1-q)(1-\theta)f_1g_1\mathbf{v}_n\mathbf{P}s)\mathbf{v}_n\mathbf{P}\}, \qquad (36)$$

## 5. Ruin probabilities under constant interest rate

In this section, we show that the ruin is certain in the risk model described in (2) when $m$ = 1. In this case, the interest rate is constant in each period. For $b$ = 1, since $\Phi_1(0;1)=(1-q)\Phi(0;1)g_1$ and $0<q<1$, then $\Phi_1(0;1)\le\Phi(0;1)$. Moreover, $0\le\Phi(0;1)\le\Phi(1;1)$ $=(1-q)\Phi(1;1)+q(1-\varphi)\Phi(0;1)f_1+q\varphi(1-\theta)(1-q)\times\Phi(0;1)f_1g_1\le[(1-q)+q(1-\varphi)f_1+q\varphi(1-\theta)(1-q)\times f_1g_1]\,\Phi(1;1)$. Since $0<(1-q)+q(1-\varphi)f_1+q\varphi(1-\theta)\times(1-q)f_1g_1<1$, then the inequality gives $\Phi(0;1)=\Phi(1;1)$=0. The following theorem shows that ruin is certain for $b\geqslant 2$ under certain conditions.

**Theorem 5.1.** The ruin probability in a compound binomial risk model with delayed claims and a constant dividend barrier is one, i.e., $\Psi(u;b)=1$, for $u$ = 0, 1, $\cdots$,$b$, provided that $\sum_{k=1}^{b}f_k<1$, for $b\ge 2$.

**Proof:** Since $\Psi(u;b)\ge\Psi(b;b)$ for $u$ = 0, 1, $\cdots$,$b$, then it is sufficient to prove that $\Psi(b;b)$ = 1 or $\Phi(b;b)$= 0 for $b\ge 2$.

Conditioning on the occurrences of claims at the end of the first time period gives

$$\Phi(b;b)=(1-q)\Phi(b;b)+q(1-\varphi)\sum_{k=1}^{b}\Phi(b-k;b)f_k$$
$$+q\varphi\left(\theta\sum_{k+h\le b}\Phi(b-k-h;b)f_kg_h\right.$$
$$\left.+(1-\theta)\sum_{k=1}^{b}\Phi_1(b-k;b)f_k\right), \qquad (43)$$

**Table 3.** Values of $V_{i,1}(u;10)(i=1,2,3)$ for geometric distributed claims when $\varphi = 0.5$

| $V_{i,1}(u;10)$ | $\theta = 0$ | $\theta = 0.25$ | $\theta = 0.5$ | $\theta = 0.75$ | $\theta = 1$ | $R_0$ |
|---|---|---|---|---|---|---|
| $u=0$ | 0.00179411 | 0.00179281 | 0.00179153 | 0.00179026 | 0.00178901 | $r_1 = 0.02$ |
| | 0.00039267 | 0.00039249 | 0.00039233 | 0.00039216 | 0.00039199 | $r_2 = 0.07$ |
| $u=1$ | 0.00380748 | 0.00380599 | 0.00380453 | 0.00380308 | 0.00380165 | $r_1 = 0.02$ |
| | 0.00094427 | 0.00094407 | 0.00094388 | 0.00094368 | 0.00094349 | $r_2 = 0.07$ |
| $u=2$ | 0.00778571 | 0.00778444 | 0.00778318 | 0.00778194 | 0.00778071 | $r_1 = 0.02$ |
| | 0.00220681 | 0.00220665 | 0.00220648 | 0.00220632 | 0.00220616 | $r_2 = 0.07$ |
| $u=3$ | 0.01569298 | 0.01569196 | 0.01569096 | 0.01568997 | 0.01568899 | $r_1 = 0.02$ |
| | 0.00510867 | 0.00510854 | 0.00510841 | 0.00510829 | 0.00510816 | $r_2 = 0.07$ |
| $u=4$ | 0.03145598 | 0.03145518 | 0.03145440 | 0.03145363 | 0.03145286 | $r_1 = 0.02$ |
| | 0.01178977 | 0.01178968 | 0.01178958 | 0.01178949 | 0.01178939 | $r_2 = 0.07$ |
| $u=6$ | 0.12574307 | 0.12574259 | 0.12574212 | 0.12574165 | 0.12574119 | $r_1 = 0.02$ |
| | 0.06264570 | 0.06264565 | 0.06264559 | 0.06264554 | 0.06264549 | $r_2 = 0.07$ |
| $u=8$ | 0.50185430 | 0.50185399 | 0.50185369 | 0.50185339 | 0.50185310 | $r_1 = 0.02$ |
| | 0.33268421 | 0.33268418 | 0.33268415 | 0.33268412 | 0.33268408 | $r_2 = 0.07$ |
| $u=10$ | 2.00248358 | 2.00248331 | 2.00248305 | 2.00248305 | 2.00248253 | $r_1 = 0.02$ |
| | 1.76663865 | 1.76663862 | 1.76663859 | 1.76663857 | 1.76663854 | $r_2 = 0.07$ |

**Table 4.** Values of $V_{i,1}(u;10)(i=1,2,3)$ for geometric distributed claims when $\theta = 0.4$

| $V_{i,1}(u;10)$ | $\varphi = 0.1$ | $\varphi = 0.3$ | $\varphi = 0.5$ | $\varphi = 0.7$ | $\varphi = 1$ | $R_0$ |
|---|---|---|---|---|---|---|
| $u=0$ | 0.00319672 | 0.00237088 | 0.00179204 | 0.00135708 | 0.00090657 | $r_1 = 0.02$ |
| | 0.00064217 | 0.00050042 | 0.00039239 | 0.00030945 | 0.00021897 | $r_2 = 0.07$ |
| $u=1$ | 0.00638572 | 0.00491201 | 0.00380512 | 0.00296171 | 0.00205488 | $r_1 = 0.02$ |
| | 0.00146526 | 0.00117352 | 0.00094396 | 0.00076249 | 0.00055785 | $r_2 = 0.07$ |
| $u=2$ | 0.01235870 | 0.00978404 | 0.00778368 | 0.00622089 | 0.00448474 | $r_1 = 0.02$ |
| | 0.00326553 | 0.00267940 | 0.00220655 | 0.00182373 | 0.00137959 | $r_2 = 0.07$ |
| $u=3$ | 0.02363833 | 0.01921910 | 0.01569136 | 0.01286461 | 0.00962613 | $r_1 = 0.02$ |
| | 0.00722384 | 0.00606493 | 0.00510847 | 0.00431667 | 0.00337311 | $r_2 = 0.07$ |
| $u=4$ | 0.04500964 | 0.03755649 | 0.03145471 | 0.02644319 | 0.02052789 | $r_1 = 0.02$ |
| | 0.01594222 | 0.01368983 | 0.01178962 | 0.01018219 | 0.00821596 | $r_2 = 0.07$ |
| $u=6$ | 0.16251169 | 0.14273297 | 0.12574231 | 0.11110739 | 0.09280539 | $r_1 = 0.02$ |
| | 0.07750633 | 0.06960326 | 0.06264561 | 0.05650726 | 0.04860482 | $r_2 = 0.07$ |
| $u=8$ | 0.58600609 | 0.54165427 | 0.50185382 | 0.46604879 | 0.41882187 | $r_1 = 0.02$ |
| | 0.37665024 | 0.35370333 | 0.33268416 | 0.31339754 | 0.28734616 | $r_2 = 0.07$ |
| $u=10$ | 2.11268371 | 2.05505952 | 2.00248315 | 1.95438885 | 1.88962393 | $r_1 = 0.02$ |
| | 1.83028964 | 1.79732073 | 1.76663861 | 1.73803697 | 1.69864159 | $r_2 = 0.07$ |

$$\Phi_1(u;b)=\sum_{n=1}^{u}\Phi(u-n;b)g_n\,,\; u=0,\,1,\,\cdots,b. \quad (44)$$

Substituting (44) into (43) yields

$$\Phi(b;b)=(1-q)\Phi(b;b)+q(1-\varphi)\sum_{k=1}^{b}\Phi(b-k;b)\,f_k$$
$$+q\varphi\sum_{k+h\le b}\Phi(b-k-h;b)\,f_k g_h\,.$$

It follows from the inequality $\Phi(b-n;b)\le\Phi(b;b)$ for $1\le n\le b$ that

$$\Phi(b;b)\le(1-q)\Phi(b;b)+q(1-\varphi)\Phi(b;b)\sum_{k=1}^{b}f_k$$
$$+q\varphi\Phi(b;b)\sum_{k+h\le b}f_k g_h\,. \quad (45)$$

Since $\sum_{k=1}^{b} f_k$ < 1, then $\sum_{k+h\le b} f_k g_h$ < 1. From these discussions, it follows that 0 < $(1-q)$ + $q(1-\varphi)\sum_{k=1}^{b} f_k+q\varphi\sum_{k+h\le b} f_k g_h$ <1. Since $0\le\Phi(b;b)\le1$, then inequality (45) gives $\Phi(b;b)$ = 0, this implies that $\Psi(b;b)$ = 1. This completes the proof.

## 6. Concluding Remarks

In this paper, we study a compound binomial risk model with a constant dividend barrier in a financial market driven by a time-homogeneous Markov chain. In this risk model, there are two types of individual claims: main claims and by-claims, each main claim induces a by-claim with probability $\varphi$, and the main claim doesn't induce a by-claim with probability $1-\varphi$. Moreover, if the main claim induces a by-claim, the by-claim and its associated main claim may occur simultaneously with probability $\theta$, or the occurrence of the by-claim may be delayed to the next time period with probability $1-\theta$. The interest rates are assumed to follow a Markov chain with finite state space. We study how to compute the *n*th moment of the discounted dividend payments prior to ruin in this risk model. The results show that, unlike the constant interest rate case, the optimal dividend barrier level depends on the initial surplus, the initial interest rate, the probabilities of occurrence and delay of the by-claim. The results also illustrate the impact of the occurrence and delay of by-claim on the *n*th moment of the present value of dividends. The formulae are readily programmable in practice and they can be used to approximate the corresponding results in the compound Poisson risk model with delayed claims under stochastic interest rates and a barrier dividend strategy. We also prove that the ruin probability in this delayed claims risk model under constant interest rate is one. In particular, when the claim amounts are of constant size, we give the optimal dividend barrier *b** and prove the expected present value of the dividend payments up to the time of ruin increases as the probability of the delay of the by-claims is increasing.

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