IJST (2012) A3: 277-291
Iranian Journal of Science & Technology

http://www.shirazu.ac.ir/en

Optimal boundary control for infinite variables parabolic systems with time lags given in integral form

G. M. Bahaa[1] and M. M. Tharwat[2]*

[1]*Department of Mathematics, Faculty of Science, Taibah University, Al-Madinah Al-Munawarah, Saudi Arabia*
[2]*Department of Mathematics, Faculty of Science, King Abdulaziz University, Jaddah, Saudi Arabia*
[1, 2]*Department of Mathematics, Faculty of Science, Beni-Suef University, Beni-Suef, Egypt*
E-mails: bahaa_gm@hotmail.com & zahraa26@yahoo.com

Abstract

In this paper, the optimal boundary control problem for distributed parabolic systems, involving second order operator with an infinite number of variables, in which constant lags appear in the integral form both in the state equations and in the boundary condition is considered. Some specific properties of the optimal control are discussed.

Keywords*:* Optimal boundary control; parabolic systems; time delays; distributed control problems; Neumann conditions; existence and uniqueness of solutions; operator with an infinite number of variables

1. Introduction

Distributed parameters systems with delays can be used to describe many phenomena in the real world. As is well known, heat conduction, properties of elastic-plastic material, fluid dynamics, diffusion-reaction processes, the transmission of the signals at a certain distance by using electric long lines, etc., all lie within this area. The object that we are studying (temperature, displacement, concentration, velocity, etc.) is usually referred to as the state.

During the last twenty years, equations with deviating argument have been applied not only in applied mathematics, physics and automatic control, but also in some problems of economy and biology. Currently, the theory of equations with deviating arguments constitutes a very important subfield of mathematical control theory.

Consequently, equations with deviating arguments are widely applied in optimal control problems of distributed parameter system with time delays.

Various optimization problems associated with the optimal control of distributed parabolic systems with time delays appearing in the boundary conditions have been studied recently by [1-10].

The necessary and sufficient conditions of optimality for system consists of only one equation and $(n \times n)$ systems governed by different types of partial differential equations defined on spaces of functions of infinitely many variables are discussed in [11-17] in which the argument of [18, 19] were used.

Making use of the Dubovitskii-Milyutin Theorem [20-24], [20], Kotarski et. al. have obtained necessary and sufficient conditions of optimality for similar systems governed by second order operator with an infinite number of variables. The interest in the study of this class of operators is stimulated by problems in quantum field theory. In [21], quadratic Pareto optimal control of parabolic equation with state-control constraints and an infinite number of variables was considered. In [22], time-optimal control problem for parabolic equations with control constraints and infinite number of variables was studied. In [23], time-optimal control problem for infinite order parabolic equation with control constraints was formulated. In [24], optimal control problems of parabolic equations with an infinite number of variables and with equality constraints were investigated. In all these papers the state equations are in evolution equations.

In this paper, we consider the optimal control problem for linear parabolic systems in which constant time lags appear in an integral form both in the state equations and in the Neumann boundary conditions involving second order operator with an infinite number of variables. Such an infinite number of variables parabolic system can be treated as a generalization of the mathematical model for a plasma control process. The optimal control is characterized by the adjoint equations for different type systems. Firstly, we study a system consisting of only one equation, secondly the optimality conditions for (2×2) coupled parabolic systems are formulated. Finally, we extend the discussions

*Corresponding author
Received: 18 October 2011 / Accepted: 12 November 2011

to study the optimal control for $(n \times n)$ coupled parabolic systems with time delay and state-control constraints.

This paper is organized as follows. In section 2, we introduce spaces of functions of infinitely many variables. In section 3, we formulate the mixed Neumann problem for parabolic operator with an infinite number of variables and time lags. In section 4, the boundary optimal control problem for this case is formulated, then we give the necessary and sufficient conditions for the control to be an optimal. In section 5, we generalized the discussion to two cases, the first case: The optimal control for (2×2) coupled parabolic systems with infinite number of variables is studied. The second case: The optimal control for $(n \times n)$ coupled parabolic systems with infinite number of variables was formulated.

2. Sobolev spaces with infinite number of variables

This section covers the basic notations, definitions and properties, which are necessary to present this work, see [11, 12, 13, 15, 25].

Let $(p_k(t))_{k=1}^{\infty}$ be a sequence of weights, fixed in all that follows, such that;

$$0 < p_k(t) \in C^{\infty}(\mathbb{R}^1), \int_{\mathbb{R}^1} p_k(t)dt = 1,$$

with respect to it we introduce on the region $\mathbb{R}^{\infty} = \mathbb{R}^1 \times \mathbb{R}^1 \times \ldots,$ the measure $d\rho(x)$ by setting,

$$d\rho(x) = p_1(x_1)dx_1 \otimes p_2(x_2)dx_2 \otimes \ldots,$$
$$(\mathbb{R}^{\infty} \ni x = (x_k)_{k=1}^{\infty}, x_k \in \mathbb{R}^1).$$

On $\mathbb{R}^{\infty}$ we construct the space $L^2(\mathbb{R}^{\infty}, d\rho(x))$ with respect to this measure i.e., $L^2(\mathbb{R}^{\infty}, d\rho(x))$ is the space of quadratic integrable functions on $\mathbb{R}^{\infty}$. We shall often set $L^2(\mathbb{R}^{\infty}, d\rho(x)) = L^2(\mathbb{R}^{\infty})$.

It is a classical result that $L^2(\mathbb{R}^{\infty})$ is a Hilbert space for the scalar product

$$(\phi,\psi)_{L^2(\mathbb{R}^{\infty})} = \int_{R^{\infty}} \phi(x)\psi(x)d\rho(x).$$

We next consider a Sobolev space in the case of an unbounded region. For functions which are $\ell = 1, 2, \ldots$ times continuously differentiable up to the boundary Γ of $\mathbb{R}^{\infty}$ (Γ is meant to be the boundary of the support of the measure $d\rho(x)$) and which vanish in a neighborhood of ∞, we introduce the scalar product

$$(\phi,\psi)_{W^{\ell}(\mathbb{R}^{\infty})} = \sum_{|\alpha|\le\ell} (D^{\alpha}\phi, D^{\alpha}\psi)_{L^2(\mathbb{R}^{\infty})},$$

where D^{α} is defined by

$$D^{\alpha} = \frac{\partial^{|\alpha|}}{(\partial x_1)^{\alpha_1}(\partial x_2)^{\alpha_2}\cdots}, \quad |\alpha| = \sum_{i=1}^{\infty}\alpha_i,$$

and the differentiation is taken in the sense of generalized functions on $\mathbb{R}^{\infty}$, and after the completion, we obtain the Sobolev space $W^{\ell}(\mathbb{R}^{\infty})$. So in short, Sobolev space $W^1(\mathbb{R}^{\infty})$ is defined by:

$$W^1(\mathbb{R}^{\infty}) = \{\phi \mid \phi, D\phi \in L^2(\mathbb{R}^{\infty})\}.$$

As in the case of a bounded region, the space $W^1(\mathbb{R}^{\infty})$ form the space with positive norm $\|.\|_{W^1(R^{\infty})}$. We can construct the space $W^{-1}(\mathbb{R}^{\infty}) = (W^1(\mathbb{R}^{\infty}))^*$ with negative norm $\|.\|_{W^{-1}(\mathbb{R}^{\infty})}$ with respect to the space $W^0(\mathbb{R}^{\infty}) = L^2(\mathbb{R}^{\infty})$ with zero norm $\|.\|_{L^2(\mathbb{R}^{\infty})}$, then we have the following equipped,

$$W^1(\mathbb{R}^{\infty}) \subseteq L^2(\mathbb{R}^{\infty}) \subseteq W^{-1}(\mathbb{R}^{\infty}),$$

$$\|\phi\|_{W^1(\mathbb{R}^{\infty})} \ge \|\phi\|_{L^2(\mathbb{R}^{\infty})} \ge \|\phi\|_{W^{-1}(\mathbb{R}^{\infty})}.$$

Let $L^2(0,T;W^1(\mathbb{R}^{\infty}))$ be the space of square integrable measurable functions $t \to \phi(t)$ of $]0,T[\to W^1(\mathbb{R}^{\infty})$, where the variable t denotes the time ; $t \in]0,T[$, $T < \infty$. This space is a Hilbert space with respect to the scalar product

$$(\phi,\psi)_{L^2(0,T;W^1(\mathbb{R}^{\infty}))} = \int_0^T (\phi(t),\psi(t))_{W^1(\mathbb{R}^{\infty})} dt,$$

and its dual is the space $L^2(0,T;W^{-1}(\mathbb{R}^{\infty}))$.

Let $\Omega \subset \mathbb{R}^{\infty}$ be a bounded, open set with boundary Γ, which is a C^{∞} manifold of dimension $(n-1)$. Locally, Ω is totally on one side of Γ and denote by $W^1(\Omega, \mathbb{R}^{\infty}, d\rho(x))$

(briefly $W^{1}(\Omega,\mathbb{R}^{\infty})$) the Sobolev space of vector function $y(x)$ defined on Ω.

The construction of the Cartesian product of n-times to the above Hilbert spaces can be construct, for example

$$(W^{1}(\Omega,\mathbb{R}^{\infty}))^{n}=\underbrace{W^{1}(\Omega,\mathbb{R}^{\infty})\times W^{1}(\Omega,\mathbb{R}^{\infty})\times\cdots\times W^{1}(\Omega,\mathbb{R}^{\infty})}_{n-times}$$
$$=\prod_{i=1}^{n}(W^{1}(\Omega,\mathbb{R}^{\infty}))^{i},$$

with norm defined by:

$$\|\phi\|_{(W^{1}(\Omega,\mathbb{R}^{\infty}))^{n}}=\sum_{i=1}^{n}\|\phi_i\|_{W^{1}(\Omega,\mathbb{R}^{\infty})},$$

where $\phi=(\phi_1,\phi_2,\ldots,\phi_n)=(\phi_i)_{i=1}^{n}$ is a vector function and $\phi_i\in W^{1}(\Omega,\mathbb{R}^{\infty})$.

Finally, we have the following chain:

$$(L^2(0,T;W^{1}(\Omega,\mathbb{R}^{\infty})))^{n}\subseteq(L^2(Q))^{n}\subseteq(L^2(0,T;W^{-1}(\Omega,\mathbb{R}^{\infty})))^{n},$$

where $(L^2(0,T;W^{-1}(\Omega,\mathbb{R}^{\infty})))^{n}$ are the dual spaces of $(L^2(0,T;W^{1}(\Omega,\mathbb{R}^{\infty})))^{n}$. The spaces considered in this paper are assumed to be real.

3. Mixed Neumann problem for parabolic system with time lags

The object of this section is to formulate the following mixed initial boundary value problem for the parabolic system with time lag which defines the state of the system model, see [2-9, 13, 21-28]:

$$\frac{\partial y}{\partial t}+\mathcal{A}(t)y+\int_a^b c(x,t)y(x,t-h)dh=u,\qquad(1)$$
$$x\in\Omega,t\in(0,T),h\in(a,b),$$

$$y(x,t')=\Phi_0(x,t'),\qquad(2)$$
$$x\in\Omega,t'\in[-b,0),$$

$$y(x,0)=y_0(x),\quad x\in\Omega,\qquad(3)$$

$$\frac{\partial y(x,t)}{\partial\nu_{\mathcal{A}}}=\int_a^b d(x,t)y(x,t-h)dh+v,\qquad(4)$$
$$x\in\Gamma,t\in(0,T),h\in(a,b),$$

$$y(x,t')=\Psi_0(x,t'),\qquad(5)$$
$$x\in\Gamma,t'\in[-b,0),$$

where Ω and Γ have the same properties as in Section 2. We have

$$y\equiv y(x,t;v),\quad u\equiv u(x,t),\quad v\equiv v(x,t),$$
$$Q\equiv\Omega\times(0,T),\quad\overline{Q}\equiv\overline{\Omega}\times[0,T],\quad Q_0\equiv\Omega\times[-b,0)$$
$$\Sigma\equiv\Gamma\times(0,T),\quad\Sigma_0\equiv\Gamma\times[-b,0),$$

- T is a specified positive number representing a time horizon,
- y is a function defined on Q such that $\Omega\times(0,T)\ni(x,t)\to y(x,t)\in\mathbb{R}$,
- u,v are functions defined on Q and Σ such that $\Omega\times(0,T)\ni(x,t)\to u(x,t)\in\mathbb{R}$ and $\Gamma\times(0,T)\ni(x,t)\to v(x,t)\in\mathbb{R}$, respectively,
- c is a given real C^{∞} function defined on $\overline{Q}$ ($\overline{Q}$ closure of Q),
- d is a given real C^{∞} function defined on Σ,
- h is a time delay such that $h\in(a,b)$ and $a>0$,
- Φ_0,Ψ_0 are initial functions defined on Q_0 and Σ_0 such that $\Omega\times[-b,0)\ni(x,t')\to\Phi_0(x,t')\in\mathbb{R}$, and $\Gamma\times[-b,0)\ni(x,t')\to\Psi_0(x,t')\in\mathbb{R}$, respectively.

The parabolic operator $\frac{\partial}{\partial t}+\mathcal{A}(t)$ in the state equation (1) is a second order parabolic operator with infinite number of variables and $\mathcal{A}(t)$, see [11, 12, 20, 25], is given by:

$$\mathcal{A}(t)y(x)=\left(-\sum_{k=1}^{\infty}\frac{1}{\sqrt{p_k(x_k,t)}}\frac{\partial^2}{\partial x_k^2}\sqrt{p_k(x_k,t)}+q(x,t)\right)y(x)$$
$$=-\sum_{k=1}^{\infty}D_k^2y(x)+q(x,t)y(x),$$

(6)

where

$$D_ky(x)=\frac{1}{\sqrt{p_k(x_k,t)}}\frac{\partial}{\partial x_k}\sqrt{p_k(x_k,t)}y(x),\qquad(7)$$

and $q(x,t)$ is a real-valued function in x which is a bounded and measurable on $\Omega\subset\mathbb{R}^{\infty}$, such that $q(x,t)\geq c_0>1$, c_0 is a constant. The operator $\mathcal{A}(t)$ is a bounded second order self-adjoint elliptic partial differential operator with an infinite number of variables maps $W^{1}(\Omega,\mathbb{R}^{\infty})$ onto $W^{-1}(\Omega,\mathbb{R}^{\infty})$.

For this operator we define the bilinear form as follows:

Definition 3.1 For each $t \in (0,T)$, we define a family of bilinear forms on $W^1(\Omega, \mathbb{R}^\infty)$ by:

$$\pi(t;y,\phi) = (\mathcal{A}(t)y,\phi)_{L^2(\Omega,\mathbb{R}^\infty)}, \quad y,\phi \in W^1(\Omega,\mathbb{R}^\infty), \tag{8}$$

where $\mathcal{A}(t)$ maps $W^1(\Omega,\mathbb{R}^\infty)$ onto $W^{-1}(\Omega,\mathbb{R}^\infty)$ and takes the above form. Then

$$\begin{aligned}\pi(t;y,\phi) &= (\mathcal{A}(t)y,\phi)_{L^2(\Omega,\mathbb{R}^\infty)}\\ &= \left(-\sum_{k=1}^{\infty} D_k^2 y(x) + q(x,t)y(x), \phi(x)\right)_{L^2(\Omega,\mathbb{R}^\infty)}\\ &= \int_\Omega \sum_{k=1}^{\infty} D_k y(x) D_k \phi(x) d\rho(x) + \int_\Omega q(x,t) y(x)\phi(x) d\rho(x).\end{aligned}$$

Lemma 3.1. The bilinear form $\pi(t;y,\phi)$ is coercive on $W^1(\Omega,\mathbb{R}^\infty)$, that is

$$\pi(t;y,y) \geq \lambda \| y\|^2_{W^1(\Omega,\mathbb{R}^\infty)}, \quad \lambda > 0. \tag{9}$$

Proof: It is well known that the ellipticity of $\mathcal{A}(t)$ is sufficient for the coerciveness of $\pi(t;y,\phi)$ on $W^1(\Omega,\mathbb{R}^\infty)$.

$$\pi(t;\phi,\psi) = \int_\Omega \sum_{k=1}^{\infty} D_k \phi(x) D_k \psi(x) d\rho + \int_\Omega q(x,t)\phi(x)\psi(x) d\rho.$$

Then

$$\begin{aligned}\pi(t;y,y) &= \int_\Omega \sum_{k=1}^{\infty} |D_k y(x)|^2 d\rho(x) + \int_\Omega q(x,t)|y(x)|^2 d\rho(x)\\ &\geq \sum_{k=1}^{\infty} \| D_k y(x)\|^2_{L^2(\Omega,R^\infty)} + c_0 \| y(x)\|^2_{L^2(\Omega,\mathbb{R}^\infty)}\\ &= \| y(x)\|^2_{W^1(\Omega,R^\infty)} + c_0 \| y(x)\|^2_{L^2(\Omega,\mathbb{R}^\infty)}\\ &\geq \| y(x)\|^2_{W^1(\Omega,\mathbb{R}^\infty)} = \lambda \| y\|^2_{W^1(\Omega,\mathbb{R}^\infty)}, \quad \lambda > 0.\end{aligned}$$

Also we have:

$$\left.\begin{array}{l}\forall y,\phi \in W^1(\Omega,\mathbb{R}^\infty) \text{ the function } t \to \pi(t;y,\phi) \text{ is}\\ \text{continuously differentiable in } (0,T) \textit{ and}\\ \pi(t;y,\phi) = \pi(t;\phi,y).\end{array}\right\} \tag{10}$$

Equations (1)-(5) constitute a Neumann problem. Then the left-hand side of the boundary condition (4) may be written in the following form:

$$\frac{\partial y(v)}{\partial \nu_{\mathcal{A}}} = \sum_{k=1}^{\infty} (D_k y(v)) \cos(n,x_k) = g(x,t), \tag{11}$$

where $\frac{\partial}{\partial \nu_{\mathcal{A}}}$ is a normal derivative at Γ, directed towards the exterior of Ω, $\cos(n,x_k)$ is the $k-th$ direction cosine of n, with n being the normal at Γ exterior to Ω, and

$$\begin{aligned}&g(x,t) = \int_a^b d(x,t) y(x,t-h) dh + v(x,t),\\ &x \in \Gamma, t \in (0,T), h \in (a,b).\end{aligned} \tag{12}$$

Remark 3.2. We shall apply the indication $g(x,t)$ appearing in (12) to prove the existence of a unique solution for (1)-(5).
We shall formulate sufficient conditions for the existence of a unique solution of the mixed initial-boundary value problem (1)-(5) for the case where the boundary control v is an element of the space $L^2(\Sigma)$ i.e., $v \in L^2(0,T;W^0(\Sigma,\mathbb{R}^\infty)) = L^2(\Sigma)$.

For this purpose, for any pair of real numbers $r,s \geq 0$, we introduce the Sobolev space $W^{r,s}(Q)$, see [19, p. 6], defined by

$$W^{r,s}(Q) = L^2\left(0,T;W^r(\Omega,\mathbb{R}^\infty)\right) \cap W^s\left(0,T;L^2(\Omega,\mathbb{R}^\infty)\right) \tag{13}$$

which is a Hilbert space normed by

$$\left(\int_0^T \| y(t)\|^2_{W^r(\Omega,\mathbb{R}^\infty)} dt + \| y\|^2_{W^s\left(0,T;L^2(\Omega,\mathbb{R}^\infty)\right)}\right)^{1/2}, \tag{14}$$

where $W^s\left(0,T;L^2(\Omega,\mathbb{R}^\infty)\right)$ denotes the Sobolev space of order s of functions defined on $(0,T)$ and taking values in $L^2(\Omega,\mathbb{R}^\infty)$.

The existence of a unique solution for the mixed initial-boundary value problem (1)-(5) on the cylinder Q can be proved using a constructive method, i.e., first, solving (1)-(5) on the sub-cylinder Q_1 and in turn on Q_2, and so on, until the procedure covers the whole cylinder Q. In this way, the solution in the previous step determines the next one.

For simplicity, we introduce the following notation:

$$\begin{aligned}&E_j \triangleq ((j-1)a, ja), \quad Q_j = \Omega \times E_j,\\ &\Sigma_j = \Gamma \times E_j, \quad j = 1,2,\ldots.\end{aligned} \tag{15}$$

Using Theorem 15.2 of [19, p. 81], we can prove the following lemma.

Lemma 3.3. Let

$$u \in W^{-\frac{1}{2},-\frac{1}{4}}(Q), \quad v \in L^2(\Sigma) \tag{16}$$

$$f_j \in W^{-\frac{1}{2},-\frac{1}{4}}(Q_j), \tag{17}$$

where

$$f_j = u(x,t) - \int_a^b c(x,t) y_{j-1}(x,t-h)dh,$$

$$y_{j-1}(\cdot,(j-1)a) \in W^{\frac{1}{2}}(\Omega,\mathbb{R}^\infty), \tag{18}$$

$$g_j \in L^2(\Sigma_j), \tag{19}$$

where

$$g_j(x,t) = \int_a^b d(x,t) y_{j-1}(x,t-h)dh + v(x,t).$$

Then, there exists a unique solution $y_j \in W^{\frac{3}{2},\frac{3}{4}}(Q_j)$ for the mixed initial-boundary value problem (1), (4) and (18).

Proof: For $j=1$, the assumptions (17)-(19) are fulfilled if we assume that $\Phi_0 \in W^{\frac{3}{2},\frac{3}{4}}(Q_0)$, $y_0 \in W^{\frac{1}{2}}(\Omega,\mathbb{R}^\infty)$ and $\Psi_0 \in L^2(\Sigma_0)$. These assumptions are sufficient to ensure the existence of a unique solution $y_1 \in W^{\frac{3}{2},\frac{3}{4}}(Q_1)$. In order to extend the result to Q_2, we have to prove that $y_1(\cdot,a) \in W^{\frac{1}{2}}(\Omega,\mathbb{R}^\infty)$, $y_1|_{\Sigma_1} \in L^2(\Sigma_1)$ and $f_2 \in W^{-\frac{1}{2},-\frac{1}{4}}(Q_2)$. First using Theorem 3.1 of [19, p. 19] we can prove that $y_1 \in W^{\frac{3}{2},\frac{3}{4}}(Q_1)$ implies that the mapping $t \to y_1(\cdot,t)$ is continuous from $[0,a] \to W^{\frac{3}{4}}(\Omega,\mathbb{R}^\infty) \subset W^{\frac{1}{2}}(\Omega,\mathbb{R}^\infty)$. Hence $y_1(\cdot,a) \in W^{\frac{1}{2}}(\Omega,\mathbb{R}^\infty)$. Again, from Trace Theorem of [19, p. 9], we can verify that $y_1 \in W^{\frac{3}{2},\frac{3}{4}}(Q_1)$ implies that $y_1 \to y_1|_{\Sigma_1}$ is a linear, continuous mapping of $W^{\frac{3}{2},\frac{3}{4}}(Q_1) \to W^{1,\frac{1}{2}}(\Sigma_1)$. Thus $y_1|_{\Sigma_1} \in L^2(\Sigma_1)$. Moreover, it is worth mentioning that the assumption (17) follows from the fact that $y_1 \in W^{\frac{3}{2},\frac{3}{4}}(Q_1)$ and $u \in W^{-\frac{1}{2},-\frac{1}{4}}(Q)$. Then, there exists a unique solution $y_2 \in W^{\frac{3}{2},\frac{3}{4}}(Q_2)$. Finally, we can extend our result to any Q_j, $j = 3,4\ldots$.

Theorem 3.4. Let y_0, Φ_0, Ψ_0, v and u be given with $y_0 \in W^{\frac{1}{2}}(\Omega,\mathbb{R}^\infty)$, $\Phi_0 \in W^{\frac{3}{2},\frac{3}{4}}(Q_0)$, $\Psi_0 \in L^2(\Sigma_0)$, $v \in L^2(\Sigma)$ and $u \in W^{-\frac{1}{2},-\frac{1}{4}}(Q)$. Then, there exists a unique solution $y \in W^{\frac{3}{2},\frac{3}{4}}(Q)$ for the mixed initial-boundary value problem (1)-(5). Moreover, $y(\cdot,ja) \in W^{\frac{1}{2}}(\Omega,\mathbb{R}^\infty)$ for $j = 1,2,\ldots$.

4. Problem formulation-optimization theorems

Now, we formulate the optimal control problem for (1)-(5) in the context of the Theorem 3.4, that is $v \in L^2(\Sigma)$.

Let us denote by $U = L^2(\Sigma)$ the space of controls. The time horizon T is fixed in our problem.

The performance functional is given by

$$I(v) = \lambda_1 \int_Q [y(x,t;v) - z_d]^2 d\rho dt + \lambda_2 \int_\Sigma (Nv)v\, d\Gamma dt \tag{20}$$

where $\lambda_i \geq 0$, and $\lambda_1 + \lambda_2 > 0$, z_d is a given element in $L^2(Q)$; N is a positive linear operator on $L^2(\Sigma)$ into $L^2(\Sigma)$.

Control constraints: We define the set of admissible controls U_{ad} such that

$$U_{ad} \text{ is closed, convex subset of } U = L^2(\Sigma). \tag{21}$$

Let $y(x,t;v)$ denote the solution of the mixed initial-boundary value problem (1)-(5) at (x,t) corresponding to a given control $v \in U_{ad}$. We note from Theorem 3.4 that for any $v \in U_{ad}$ the performance functional (20) is well-defined since $y(v) \in W^{\frac{3}{2},\frac{3}{4}}(Q) \subset L^2(Q)$.

Making use of the Loins's scheme we shall derive the necessary and sufficient conditions of optimality for the optimization problem (1)-(5), (20) and (21). The solving of the formulated optimal control problem is equivalent to seeking a $v^* \in U_{ad}$ such that

$$I(v^*) \le I(v), \quad \forall\, v \in U_{ad}.$$

From the Lion's scheme, Theorem 1.3 of [18, p. 10], it follows that for $\lambda_2 > 0$ a unique optimal control v^* exists. Moreover, v^* is characterized by the following condition

$$I'(v^*)(v - v^*) \ge 0 \quad \forall v \in U_{ad}. \tag{22}$$

For the performance functional of form (20) the relation (22) can be expressed as

$$\begin{aligned}&\lambda_1 \int_Q (y(v^*) - z_d)[y(v) - y(v^*)]d\rho dt \\ &+\lambda_2 \int_\Sigma N v^*(v - v^*) d\Gamma dt \ge 0, \quad \forall v \in U_{ad}.\end{aligned} \tag{23}$$

To simplify (23), we introduce the adjoint equation, and for every $v \in U_{ad}$, we define the adjoint variable $p = p(v) = p(x,t;v)$ as the solution of the following system

$$\begin{aligned}&-\frac{\partial p(v)}{\partial t} + \mathcal{A}^*(t)p(v) + \int_a^b c(x,t+h)p(x,t+h;v)dh \\ &= \lambda_1(y(v) - z_d), \quad x \in \Omega, t \in (0, T-b),\end{aligned} \tag{24}$$

$$\begin{aligned}&-\frac{\partial p(v)}{\partial t} + \mathcal{A}^*(t)p(v) + \int_a^{T-t} c(x,t+h)p(x,t+h;v)dh \\ &= \lambda_1(y(v) - z_d), \; x \in \Omega, t \in (T-b, T-a),\end{aligned} \tag{25}$$

$$\begin{aligned}&-\frac{\partial p(v)}{\partial t} + \mathcal{A}^*(t)p(v) = \lambda_1(y(v) - z_d), \\ &x \in \Omega, t \in (T-a, T),\end{aligned} \tag{26}$$

$$p(x,T;v) = 0, \quad x \in \Omega, \tag{27}$$

$$\begin{aligned}&\frac{\partial p(v)}{\partial \nu_{\mathcal{A}^*}}(x,t) = \int_a^b d(x,t+h)p(x,t+h;v)dh, \\ &x \in \Gamma, t \in (0, T-b),\end{aligned} \tag{28}$$

$$\begin{aligned}&\frac{\partial p(v)}{\partial \nu_{\mathcal{A}^*}}(x,t) = \int_a^{T-t} d(x,t+h)p(x,t+h;v)dh, \\ &x \in \Gamma, t \in (T-b, T-a),\end{aligned} \tag{29}$$

$$\frac{\partial p(v)}{\partial \nu_{\mathcal{A}^*}}(x,t) = 0, \quad x \in \Gamma, t \in (T-a, T), \tag{30}$$

where

$$\frac{\partial p(v)}{\partial \nu_{\mathcal{A}^*}}(x,t) = \sum_{k=1}^{\infty}(D_k p(v))\cos(n, x_k), \tag{31}$$

$$\mathcal{A}^*(t)p(v) = (-\sum_{k=1}^{\infty} D_k^2 + q(x,t))p(v). \tag{32}$$

As in the above section with a change of variables, i.e. with reversed sense of time. i.e., $t' = T - t$, for given $z_d \in L^2(Q)$ and any $v \in L^2(\Sigma)$, there exists a unique solution $p(v) \in W^{\frac{3}{2},\frac{3}{4}}(Q)$ for problem (24)-(30).

Remark 4.1. If $T < b$, then we consider (25) and (29) on $\Omega \times (0, T-a)$ and $\Gamma \times (0, T-a)$, respectively.

The existence of a unique solution for the problem (24)-(30) on the cylinder $\Omega \times (0,T)$ can be proved using a constructive method. It is easy to notice that for given z_d and u, the problem (24)-(30) can be solved backwards in time starting from $t = T$, i.e. first solving (24)-(30) on the sub-cylinder Q_K and in turn on Q_{K-1}, etc. until the procedure covers the whole cylinder $\Omega \times (0,T)$. For this purpose, we may apply Theorem 3.4 (with an obvious change of variables).

Hence, using Theorem 3.4, the following result can be proved.

Lemma 4.2. Let the hypothesis of Theorem 3.4 be satisfied. Then for given $z_d \in L^2(\Omega, \mathbb{R}^\infty)$ and any $v \in L^2(\Sigma)$, there exists a unique solution $p(v) \in W^{\frac{3}{2},\frac{3}{4}}(Q)$ for the adjoint problem (24)-(30).

Now, we have the main result.

Theorem 4.3. For the problem (1)-(5) with the performance functional (20), $z_d \in L^2(Q)$, $\lambda_2 > 0$ and with constraints on controls (21), then the optimal control v^* exists and is characterized by the following condition

$$\int_0^T\int_\Gamma (p(v^*)+\lambda_2 Nv^*)(v-v^*)\,d\Gamma dt \geq 0, \quad \forall v \in U_{ad}, \tag{33}$$

where $p(v^*)$ is the solution of the adjoint system (24)-(30).

Proof: We simplify (23) using the adjoint equation (24)-(30). For this purpose setting $v=v^*$ in (24)-(30), multiplying both sides of (24), (25) and (26) by $y(v)-y(v^*)$, then integrating over $\Omega\times(0,T-b)$, $\Omega\times(T-b,T-a)$ and $\Omega\times(T-a,T)$, respectively, and then adding both sides of (24)-(26), we get

$$\begin{aligned}
\lambda_1\int_Q (y(T;v^*)-z_d)[y(T;v)-y(T;v^*)]\,d\rho dt &= \int_0^T\int_\Omega \Big(-\frac{\partial p(v^*)}{\partial t}+\mathcal{A}^*(t)p(v^*)\Big)[y(v)-y(v^*)]\,d\rho dt \\
&\quad +\int_0^{T-b}\int_\Omega \Big(\int_a^b c(x,t+h)p(x,t+h;v^*)dh\Big)[y(x,t;v)-y(x,t;v^*)]\,d\rho dt \\
&\quad +\int_{T-b}^{T-a}\int_\Omega \Big(\int_a^{T-t} c(x,t+h)p(x,t+h;v^*)dh\Big)[y(x,t;v)-y(x,t;v^*)]\,d\rho dt \\
&= -\int_\Omega p(x,T;v^*)[y(x,T;v)-y(x,T;v^*)]\,d\rho \\
&\quad +\int_0^T\int_\Omega p(v^*)\frac{\partial}{\partial t}[y(v)-y(v^*)]\,d\rho dt+\int_0^T\int_\Omega \mathcal{A}^*(t)p(v^*)[y(v)-y(v^*)]\,d\rho dt \\
&\quad +\int_0^{T-b}\int_\Omega\int_a^b c(x,t+h)p(x,t+h;v^*)[y(x,t;v)-y(x,t;v^*)]\,dh\,d\rho dt \\
&\quad +\int_{T-b}^{T-a}\int_\Omega\int_a^{T-t} c(x,t+h)p(x,t+h;v^*)[y(x,t;v)-y(x,t;v^*)]\,dh\,d\rho dt.
\end{aligned} \tag{34}$$

Then applying (27), the formula (34) can be expressed as

$$\begin{aligned}
&\lambda_1\int_Q (y(T;v^*)-z_d)[y(T;v)-y(T;v^*)]\,d\rho dt \\
&= \int_0^T\int_\Omega p(v^*)\frac{\partial}{\partial t}[y(v)-y(v^*)]\,d\rho dt+\int_0^T\int_\Omega \mathcal{A}^*(t)p(v^*)[y(v)-y(v^*)]\,d\rho dt \\
&\quad +\int_0^{T-b}\int_\Omega\int_a^b c(x,t+h)p(x,t+h;v^*)[y(x,t;v)-y(x,t;v^*)]\,dh\,d\rho dt \\
&\quad +\int_{T-b}^{T-a}\int_\Omega\int_a^{T-t} c(x,t+h)p(x,t+h;v^*)[y(x,t;v)-y(x,t;v^*)]\,dh\,d\rho dt.
\end{aligned} \tag{35}$$

Using (1), the first integral on the right-hand side of (35) can be rewritten as

$$\begin{aligned}
\int_0^T\int_\Omega p(v^*)\frac{\partial}{\partial t}[y(v)-y(v^*)]\,d\rho dt &= -\int_0^T\int_\Omega p(v^*)\mathcal{A}(t)[y(v)-y(v^*)]\,d\rho dt \\
&\quad -\int_0^T\int_\Omega p(x,t;v^*)\Big(\int_a^b c(x,t)[y(x,t-h;v)-y(x,t-h;v^*)]dh\Big)d\rho dt \\
&= -\int_0^T\int_\Omega p(v^*)\mathcal{A}(t)[y(v)-y(v^*)]\,d\rho dt \\
&\quad -\int_0^T\int_\Omega\int_a^b p(x,t;v^*)c(x,t)[y(x,t-h;v)-y(x,t-h;v^*)]\,dh\,d\rho dt \\
&= -\int_0^T\int_\Omega p(v^*)\mathcal{A}(t)[y(v)-y(v^*)]\,d\rho dt \\
&\quad -\int_a^b\int_0^T\int_\Omega p(x,t;v^*)c(x,t)[y(x,t-h;v)-y(x,t-h;v^*)]\,dt\,d\rho dh \\
&= -\int_0^T\int_\Omega p(v^*)\mathcal{A}(t)[y(v)-y(v^*)]\,d\rho dt \\
&\quad -\int_a^b\int_\Omega\int_{-h}^{T-h} p(x,t'+h;v^*)c(x,t'+h)[y(x,t';v)-y(x,t';v^*)]\,dt'\,d\rho dh \\
&= -\int_0^T\int_\Omega p(v^*)\mathcal{A}(t)[y(v)-y(v^*)]\,d\rho dt \\
&\quad -\int_a^b\int_\Omega\int_{-h}^{0} p(x,t'+h;v^*)c(x,t'+h)[y(x,t';v)-y(x,t';v^*)]\,dt'\,d\rho dh \\
&\quad -\int_a^b\int_\Omega\int_{0}^{T-b} p(x,t'+h;v^*)c(x,t'+h)[y(x,t';v)-y(x,t';v^*)]\,dt'\,d\rho dh \\
&\quad -\int_a^b\int_\Omega\int_{T-b}^{T-h} p(x,t'+h;v^*)c(x,t'+h)[y(x,t';v)-y(x,t';v^*)]\,dt'\,d\rho dh \\
&= -\int_0^T\int_\Omega p(v^*)\mathcal{A}(t)[y(v)-y(v^*)]\,d\rho dt \\
&\quad -\int_a^b\int_\Omega\int_{-h}^{0} p(x,t'+h;v^*)c(x,t'+h)[y(x,t';v)-y(x,t';v^*)]\,dt'\,d\rho dh \\
&\quad -\int_a^b\int_\Omega\int_{0}^{T-b} p(x,t'+h;v^*)c(x,t'+h)[y(x,t';v)-y(x,t';v^*)]\,dt'\,d\rho dh \\
&\quad -\int_a^{T-t}\int_\Omega\int_{T-b}^{T-a} p(x,t'+h;v^*)c(x,t'+h)[y(x,t';v)-y(x,t';v^*)]\,dt'\,d\rho dh.
\end{aligned} \tag{36}$$

The second integral on the right-hand side of (35), in view of Green's formula, can be expressed as

$$
\begin{aligned}
\int_0^T\int_\Omega \mathcal{A}^*(t)p(v^*)[y(v)-y(v^*)]d\rho dt &= \int_0^T\int_\Omega p(v^*)\mathcal{A}(t)[y(v)-y(v^*)]d\rho dt \\
&+\int_0^T\int_\Gamma p(v^*)\left(\frac{\partial y(v)}{\partial \nu_{\mathcal{A}}}-\frac{\partial y(v^*)}{\partial \nu_{\mathcal{A}}}\right)d\Gamma dt \\
&-\int_0^T\int_\Gamma \frac{\partial p(v^*)}{\partial \nu_{\mathcal{A}^*}}[y(v)-y(v^*)]d\Gamma dt.
\end{aligned} \tag{37}
$$

Using the boundary condition (4), the second component on the right-hand side of (37) can be written as

$$
\begin{aligned}
\int_0^T\int_\Gamma p(v^*)\left(\frac{\partial y(v)}{\partial \nu_{\mathcal{A}}}-\frac{\partial y(v^*)}{\partial \nu_{\mathcal{A}}}\right)d\Gamma dt &= \int_0^T\int_\Gamma p(x,t;v^*)\left(\int_a^b d(x,t)[y(x,t-h;v)-y(x,t-h;v^*)]dh\right)d\Gamma dt \\
&+\int_0^T\int_\Gamma p(x,t;v^*)(v-v^*)d\Gamma dt \\
&=\int_0^T\int_\Gamma\int_a^b p(x,t;v^*)d(x,t)[y(x,t-h;v)-y(x,t-h;v^*)]dh\, d\Gamma dt \\
&+\int_0^T\int_\Gamma p(x,t;v^*)(v-v^*)d\Gamma dt \\
&=\int_a^b\int_\Gamma\int_0^T p(x,t;v^*)d(x,t)[y(x,t-h;v)-y(x,t-h;v^*)]dt\, d\Gamma dh \\
&+\int_0^T\int_\Gamma p(x,t;v^*)(v-v^*)d\Gamma dt \\
&=\int_a^b\int_\Gamma\int_{-h}^{T-h} p(x,t'+h;v^*)d(x,t'+h)[y(x,t';v)-y(x,t';v^*)]dt'd\Gamma dh \\
&+\int_0^T\int_\Gamma p(x,t;v^*)(v-v^*)d\Gamma dt \\
&=\int_a^b\int_\Gamma\int_{-h}^{0} p(x,t'+h;v^*)d(x,t'+h)[y(x,t';v)-y(x,t';v^*)]dt'd\Gamma dh \\
&+\int_a^b\int_\Gamma\int_{0}^{T-b} p(x,t'+h;v^*)d(x,t'+h)[y(x,t';v)-y(x,t';v^*)]dt'd\Gamma dh \\
&+\int_a^b\int_\Gamma\int_{T-b}^{T-h} p(x,t'+h;v^*)d(x,t'+h)[y(x,t';v)-y(x,t';v^*)]dt'd\Gamma dh \\
&+\int_0^T\int_\Gamma p(x,t;v^*)(v-v^*)d\Gamma dt \\
&=\int_a^b\int_\Gamma\int_{-h}^{0} p(x,t'+h;v^*)d(x,t'+h)[y(x,t';v)-y(x,t';v^*)]dt'd\Gamma dh \\
&+\int_a^b\int_\Gamma\int_{0}^{T-b} p(x,t'+h;v^*)d(x,t'+h)[y(x,t';v)-y(x,t';v^*)]dt'd\Gamma dh \\
&+\int_a^{T-t}\int_\Gamma\int_{T-b}^{T-h} p(x,t'+h;v^*)d(x,t'+h)[y(x,t';v)-y(x,t';v^*)]dt'd\Gamma dh \\
&+\int_0^T\int_\Gamma p(x,t;v^*)(v-v^*)d\Gamma dt.
\end{aligned} \tag{38}
$$

The last component in (37) can be rewritten as

$$
\begin{aligned}
\int_0^T\int_\Gamma \frac{\partial p(v^*)}{\partial \nu_{\mathcal{A}^*}}[y(v)-y(v^*)]d\Gamma dt &= \int_0^{T-b}\int_\Gamma \frac{\partial p(v^*)}{\partial \nu_{\mathcal{A}^*}}[y(v)-y(v^*)]d\Gamma dt \\
&+\int_{T-b}^{T-a}\int_\Gamma \frac{\partial p(v^*)}{\partial \nu_{\mathcal{A}^*}}[y(v)-y(v^*)]d\Gamma dt+\int_{T-a}^{T}\int_\Gamma \frac{\partial p(v^*)}{\partial \nu_{\mathcal{A}^*}}[y(v)-y(v^*)]d\Gamma dt.
\end{aligned} \tag{39}
$$

Substituting (38) and (39) into (37) and then (37) into (35), we obtain

$$\begin{aligned}
\lambda_1\int_Q (y(T;v^*)-z_d)[y(T;v)-y(T;v^*)]d\rho\, dt \quad &= -\int_0^T\int_\Omega p(v^*)A(t)[y(v)-y(v^*)]d\rho dt \\
&-\int_{-h}^0\int_\Omega\int_a^b c(x,t+h)p(x,t+h;v^*)[y(x,t;v)-y(x,t;v^*)]dh\,d\rho dt \\
&-\int_0^{T-b}\int_\Omega\int_a^b c(x,t+h)p(x,t+h;v^*)[y(x,t;v)-y(x,t;v^*)]dh\,d\rho dt \\
&-\int_{T-b}^{T-a}\int_\Omega\int_a^{T-t} c(x,t+h)p(x,t+h;v^*)[y(x,t;v)-y(x,t;v^*)]dh\,d\rho dt \\
&+\int_0^T\int_\Omega p(v^*)\mathcal{A}(t)[y(v)-y(v^*)]d\rho dt \\
&+\int_{-h}^0\int_\Gamma\int_a^b d(x,t+h)p(x,t+h;v^*)[y(x,t;v)-y(x,t;v^*)]dh\,d\Gamma dt \\
&+\int_0^{T-b}\int_\Gamma\int_a^b d(x,t+h)p(x,t+h;v^*)[y(x,t;v)-y(x,t;v^*)]dh\,d\Gamma\, dt \\
&+\int_{T-b}^{T-a}\int_\Gamma\int_a^{T-t} d(x,t+h)p(x,t+h;v^*)[y(x,t;v)-y(x,t;v^*)]dh\,d\Gamma dt \\
&+\int_0^T\int_\Gamma p(x,t;v^*)(v-v^*)d\Gamma dt \\
&-\int_0^{T-b}\int_\Gamma \frac{\partial p(v^*)}{\partial \nu_{\mathcal{A}^*}}[y(x,t;v)-y(x,t;v^*)]d\Gamma dt \\
&-\int_{T-b}^{T-a}\int_\Gamma \frac{\partial p(v^*)}{\partial \nu_{\mathcal{A}^*}}[y(x,t;v)-y(x,t;v^*)]d\Gamma dt \\
&-\int_{T-a}^{T}\int_\Gamma \frac{\partial p(v^*)}{\partial \nu_{\mathcal{A}^*}}[y(x,t;v)-y(x,t;v^*)]d\Gamma dt \qquad (40)\\
&+\int_0^{T-b}\int_\Omega\int_a^b c(x,t+h)p(x,t+h;v^*)[y(x,t;v)-y(x,t;v^*)]dh\,d\rho dt \\
&+\int_{T-b}^{T-a}\int_\Omega\int_a^{T-t} c(x,t+h)p(x,t+h;v^*)[y(x,t;v)-y(x,t;v^*)]dh\,d\rho dt.
\end{aligned}$$

Afterwards, using the fact that $y(x,t;v)=y(x,t;v^*)=\Phi_0(x,t)$ for $x\in\Omega$ and $t\in[-b,0)$ and $y(x,t';v)=y(x,t';v^*)=\Psi_0(x,t')$ for $x\in\Gamma$ and $t'\in[-b,0)$, we obtain

$$\lambda_1\int_Q (y(T;v^*)-z_d)[y(T;v)-y(T;v^*)]d\rho\, dt = \int_0^T\int_\Gamma p(v^*)(v-v^*)d\Gamma dt. \qquad (41)$$

Substituting (41) into (23) gives (33).

Mathematical examples

Example 4.1 Let $U_{ad}=U=L^2(\Sigma)$, the case where there are no constraints on the control. Thus the maximum condition (33) is satisfied when $v^*=-\lambda_2^{-1}N^{-1}p(v^*)$.

If N is the identity operator on $L^2(\Sigma)$, then from lemma 4.2 it follows that $v^*\in W^{\frac{3}{2},\frac{3}{4}}(Q)$.

Example 4.2. We can also consider an analogous optimal control problem where the performance functional is given by

$$\hat{I}(v)=\lambda_1\int_\Sigma [y(x,t;v)-z_d]^2 d\Gamma dt+\lambda_2\int_\Sigma (Nv)v\, d\Gamma dt \qquad (42)$$

where $z_d\in L^2(\Sigma)$.

From Theorem 3.4 and the Trace Theorem of [19, p. 9], for each $v\in L^2(\Sigma)$, there exists a unique solution $y(v)\in W^{\frac{3}{2},\frac{3}{4}}(Q)$ with $y|_\Sigma\in L^2(\Sigma)$. Thus, $\hat{I}$ is well defined. Then, the optimal control v^* is characterized by

$$\begin{aligned}
&\lambda_1\int_\Sigma (y(v^*)-z_d)[y(v)-y(v^*)]d\Gamma dt \\
&+\lambda_2\int_\Sigma Nv^*(v-v^*)d\Gamma dt\ge 0,\quad \forall v\in U_{ad}.
\end{aligned} \qquad (43)$$

We define the adjoint variable $p=p(v^*)=p(x,t;v^*)$ as the solution of the equations

$$\begin{aligned}
&-\frac{\partial p(v^*)}{\partial t}+\mathcal{A}^*(t)p(v^*) \\
&+\int_a^b c(x,t+h)p(x,t+h;v^*)dh=0, \qquad (44)\\
&x\in\Omega, t\in(0,T-b),
\end{aligned}$$

$$-\frac{\partial p(v^*)}{\partial t}+\mathcal{A}^*(t)p(v^*)$$
$$+\int_a^{T-t}c(x,t+h)p(x,t+h;v^*)dh=0, \quad (45)$$
$$x\in\Omega, t\in(T-b,T-a),$$

$$-\frac{\partial p(v^*)}{\partial t}+\mathcal{A}^*(t)p(v^*)=0, \quad (46)$$
$$x\in\Omega, t\in(T-a,T),$$

$$p(x,T;v^*)=0, \quad x\in\Omega,$$
(47)

$$\frac{\partial p(v^*)}{\partial \nu_{\mathcal{A}^*}}(x,t)=\int_a^b d(x,t+h)p(x,t+h;v^*)dh \quad (48)$$
$$+\lambda_1(y(v^*)-z_d), \quad x\in\Gamma, t\in(0,T-b),$$

$$\frac{\partial p(v^*)}{\partial \nu_{\mathcal{A}^*}}(x,t)=\int_a^{T-t} d(x,t+h)p(x,t+h;v^*)dh \quad (49)$$
$$+\lambda_1(y(v^*)-z_d), \quad x\in\Gamma, t\in(T-b,T-a),$$

$$\frac{\partial p(v^*)}{\partial \nu_{\mathcal{A}^*}}(x,t)=\lambda_1(y(v^*)-z_d) \quad (50)$$
$$x\in\Gamma, t\in(T-a,T).$$

As in the above section, we have the following result.

Lemma 4.4. Let the hypothesis of Theorem 2.1 be satisfied. Then, for given $z_d\in L^2(\Sigma)$ and any $v\in L^2(\Sigma)$, there exists a unique solution $p(v^*)\in W^{\frac{3}{2},\frac{3}{4}}(Q)$ to the adjoint problem (44)-(50).

Using the adjoint equations (44)-(50) in this case, the condition (43) can also be written in the following form

$$\int_0^T\int_\Gamma(p(v^*)+\lambda_2 Nv^*)(v-v^*)d\Gamma dt\geq 0, \quad (51)$$
$$\forall v\in U_{ad}.$$

The following result is now summarized.

Theorem 4.5. For the problem (1)-(5) with the performance function (42) with $z_d\in L^2(\Sigma)$ and $\lambda_2>0$, and with constraint (21), and with adjoint equations (44)-(50), there exists a unique optimal control v^* which satisfies the maximum condition (51).

Example 4.3. If $v\in L^2(Q)$, we can also consider an analogous optimal control problem where the performance functional is given by

$$\hat{I}(v)=\lambda_1\int_Q[y(x,t;v)-z_d]^2 d\rho dt+\lambda_2\int_Q(Nv)v\,d\rho dt \quad (52)$$

where $z_d\in L^2(Q)$.

From Theorem 3.4 and the Trace Theorem of [19, p. 9], for each $v\in L^2(Q)$, there exists a unique solution $y(v)\in W^{\frac{3}{2},\frac{3}{4}}(Q)$. Thus, $\hat{\hat{I}}$ is well defined. Then, the optimal control v^* is characterized by

$$\lambda_1\int_Q(y(v^*)-z_d)[y(v)-y(v^*)]d\rho dt$$
$$+\lambda_2\int_Q Nv^*(v-v^*)d\rho dt\geq 0, \quad \forall v\in U_{ad}. \quad (53)$$

We define the adjoint variable $p=p(v^*)=p(x,t;v^*)$ as the solution of the equations

$$-\frac{\partial p(v^*)}{\partial t}+\mathcal{A}^*(t)p(v^*)+\int_a^b c(x,t+h)p(x,t+h;v^*)dh \quad (54)$$
$$=\lambda_1(y(v^*)-z_d), \quad x\in\Omega, t\in(0,T-b),$$

$$-\frac{\partial p(v^*)}{\partial t}+\mathcal{A}^*(t)p(v^*)+\int_a^{T-t} c(x,t+h)p(x,t+h;v^*)dh \quad (55)$$
$$=\lambda_1(y(v^*)-z_d), \quad x\in\Omega, t\in(T-b,T-a),$$

$$-\frac{\partial p(v^*)}{\partial t}+\mathcal{A}^*(t)p(v^*)=\lambda_1(y(v^*)-z_d), \quad (56)$$
$$x\in\Omega, t\in(T-a,T),$$

$$p(x,T;v^*)=0, \quad x\in\Omega,$$
(57)

$$\frac{\partial p(v^*)}{\partial \nu_{\mathcal{A}^*}}(x,t)=\int_a^b d(x,t+h)p(x,t+h;v^*)dh, \quad (58)$$
$$x\in\Gamma, t\in(0,T-b),$$

$$\frac{\partial p(v^*)}{\partial \nu_{\mathcal{A}^*}}(x,t)=\int_a^{T-t} d(x,t+h)p(x,t+h;v^*)dh, \quad (59)$$
$$x\in\Gamma, t\in(T-b,T-a),$$

$$\frac{\partial p(v^*)}{\partial \nu_{\mathcal{A}^*}}(x,t)=0, \quad x\in\Gamma, t\in(T-a,T). \quad (60)$$

As in the above section, we have the following result.

Lemma 4.6. Let the hypothesis of Theorem 3.4 be satisfied. Then, for given $z_d \in L^2(Q)$ and any $v \in L^2(Q)$, there exists a unique solution $p(v^*) \in W^{\frac{3}{2},\frac{3}{4}}(Q)$ to the adjoint problem (54)-(60).

Using the adjoint equations (54)-(60) in this case, the condition (53) can also be written in the following form

$$\int_0^T \int_\Omega (p(v^*) + \lambda_2 N v^*)(v - v^*) d\rho dt \geq 0, \quad \forall v \in U_{ad}. \tag{61}$$

The following result is now summarized.

Theorem 4.7. For the problem (1)-(5) with the performance function (52) with $z_d \in L^2(Q)$ and $\lambda_2 > 0$, and with constraint (21), and with adjoint equations (54)-(60), there exists a unique optimal control v^* which satisfies the maximum condition (61).

5. Generalization

The optimal control problems presented here can be extended to certain different cases. For example, case 1: Optimal control for (2×2) coupled parabolic systems with time lags and infinite number of variables and for case 2: Optimal control for $(n\times n)$ coupled parabolic systems with time lags and infinite number of variables. Such extension can be applied to solving many control problems in mechanical engineering.

5.1. Case 1: Optimal control for (2×2) coupled parabolic systems with time lags and infinite number of variables

We can extend the discussions to study the optimal control for (2×2) coupled parabolic systems with time lags and infinite number of variables. We consider the case where $v = (v_1, v_2) \in L^2(\Sigma)\times L^2(\Sigma)$, the performance functional is given by

$$I(v) = \sum_{i=1}^{2} (\lambda_1 \int_Q [y_i(x,t;v) - z_{id}]^2 d\rho dt + \lambda_2 \int_\Sigma (N_i v_i) v_i d\Gamma dt), \tag{62}$$

where $z_d = (z_{1d}, z_{2d}) \in (L^2(Q))^2$ and N_i is a positive linear operator on $L^2(\Sigma)$ into $L^2(\Sigma)$, $i = 1,2$.

The following results can now proved.

Theorem 5.1. Let y_0, Φ_0, Ψ_0, v and u be given with $y_0 = (y_{0,1}, y_{0,2}) \in (W^{\frac{1}{2}}(\Omega, \mathbb{R}^\infty))^2$,

$$\Phi_0 = (\Phi_{0,1}, \Phi_{0,2}) \in (W^{\frac{3}{2},\frac{3}{4}}(Q_0))^2, \Psi_0 = (\Psi_{0,1}, \Psi_{0,2}) \in (L^2(\Sigma_0))^2,$$

$v = (v_1, v_2) \in (L^2(\Sigma))^2$ and $u = (u_1, u_2) \in (W^{-\frac{1}{2},-\frac{1}{4}}(Q))^2$. Then, there exists a unique solution $y = (y_1, y_2) \in (W^{\frac{3}{2},\frac{3}{4}}(Q))^2$ for the following mixed initial-boundary value problem:

$$\frac{\partial y_1}{\partial t} + (-\sum_{k=1}^{\infty} D_k^2 + q(x,t) + 1) y_1 + \int_a^b c_1(x,t) y_1(x,t-h) dh - y_2 = u_1, \text{ in } Q, \tag{63}$$

$$\frac{\partial y_2}{\partial t} + (-\sum_{k=1}^{\infty} D_k^2 + q(x,t) + 1) y_2 + \int_a^b c_2(x,t) y_2(x,t-h) dh + y_1 = u_2, \quad \text{in } Q, \tag{64}$$

$$\begin{aligned} &y_1(x,t';u) = \Phi_{0,1}(x,t'),\\ &y_2(x,t';u) = \Phi_{0,2}(x,t'),\\ &x \in \Omega, t' \in [-b,0), \end{aligned} \tag{65}$$

$$\begin{aligned} &y_1(x,0) = y_{0,1}(x),\\ &y_2(x,0) = y_{0,2}(x),\\ &x \in \Omega, \end{aligned} \tag{66}$$

$$\begin{aligned} &\frac{\partial y_1}{\partial \nu_A} = \int_a^b d_1(x,t) y_1(x,t-h) dh + v_1,\\ &\frac{\partial y_2}{\partial \nu_A} = \int_a^b d_2(x,t) y_2(x,t-h) dh + v_2, \quad \text{on } \Sigma, \end{aligned} \tag{67}$$

$$y_1(x,t';u)=\Psi_{0,1}(x,t'),$$
$$y_2(x,t';u)=\Psi_{0,2}(x,t'), \quad (68)$$
$$x\in\Gamma, t'\in[-b,0),$$

where

$y\equiv y(x,t;v)=(y_1(x,t;v),\, y_2(x,t;v))\in(W^{\frac{3}{2},\frac{3}{4}}(Q))^2$,

$u\equiv u(x,t)=(u_1(x,t),u_2(x,t))\in(W^{-\frac{1}{2},-\frac{1}{4}}(Q))^2$,

$v\equiv v(x,t)=(v_1(x,t),v_2(x,t))\in(L^2(\Sigma))^2$,

c_i and d_i, $i=1,2$, are real C^∞ functions defined on $\overline{Q}$ and Σ, respectively, $\Phi_{i,0}$ and $\Psi_{i,0}$, $i=1,2$, are initial functions defined on Q_0 and Σ_0, respectively.

Lemma 5.2. Let the hypothesis of Theorem 5.1 be satisfied. Then for given $z_d=(z_{1d},z_{2d})\in(L^2(Q))^2$ and any $v=(v_1,v_2)\in(L^2(\Sigma))^2$, there exists a unique solution $p(v)=(p_1(v),p_2(v))\in(W^{\frac{3}{2},\frac{3}{4}}(Q))^2$ for the adjoint problem:

$$-\frac{\partial p_1(v)}{\partial t}+(-\sum_{k=1}^{\infty}D_k^2+q(x,t)+1)p_1(v)$$
$$+\int_a^b c_1(x,t+h)p_1(x,t+h:v)dh \quad (69)$$
$$+p_2(v)=\lambda_1(y_1(v)-z_{1d}),$$
$$x\in\Omega, t\in(0,T-b),$$

$$-\frac{\partial p_2(v)}{\partial t}+(-\sum_{k=1}^{\infty}D_k^2+q(x,t)+1)p_2(v)$$
$$+\int_a^b c_2(x,t+h)p_2(x,t+h:v)dh-p_1(v) \quad (70)$$
$$=\lambda_1(y_2(v)-z_{2d}),$$
$$x\in\Omega, t\in(0,T-b),$$

$$-\frac{\partial p_1(v)}{\partial t}+(-\sum_{k=1}^{\infty}D_k^2+q(x,t)+1)p_1(v)$$
$$+\int_a^{T-t} c_1(x,t+h)p_1(x,t+h:v)dh \quad (71)$$
$$+p_2(v)=\lambda_1(y_1(v)-z_{1d}),$$
$$x\in\Omega, t\in(T-b,T-a),$$

$$-\frac{\partial p_2(v)}{\partial t}+(-\sum_{k=1}^{\infty}D_k^2+q(x,t)+1)p_2(v)$$
$$+\int_a^{T-t} c_2(x,t+h)p_2(x,t+h:v)dh \quad (72)$$
$$-p_1(v)=\lambda_1(y_2(v)-z_{2d}),$$
$$x\in\Omega, t\in(T-b,T-a),$$

$$-\frac{\partial p_1(v)}{\partial t}+(-\sum_{k=1}^{\infty}D_k^2+q(x,t)+1)p_1(v)+p_2(v) \quad (73)$$
$$=\lambda_1(y_1(v)-z_{1d}), \quad x\in\Omega, t\in(T-a,T),$$

$$-\frac{\partial p_2(v)}{\partial t}+(-\sum_{k=1}^{\infty}D_k^2+q(x,t)+1)p_2(v)-p_1(v) \quad (74)$$
$$=\lambda_1(y_2(v)-z_{2d}), \quad x\in\Omega, t\in(T-a,T),$$

$$p_1(x,T;v)=0, \quad p_2(x,T;v)=0, \quad x\in\Omega, \quad (75)$$

$$\frac{\partial p_1(x,t)}{\partial v_{\mathcal{A}^*}}=\int_a^b d_1(x,t+h)p_1(x,t+h;v)dh,$$
$$\frac{\partial p_2(x,t)}{\partial v_{\mathcal{A}^*}}=\int_a^b d_2(x,t+h)p_2(x,t+h;v)dh, \quad (76)$$
$$x\in\Gamma, t\in(0,T-b),$$

$$\frac{\partial p_1(x,t)}{\partial v_{\mathcal{A}^*}}=\int_a^{T-t} d_1(x,t+h)p_1(x,t+h;v)dh, \quad \frac{\partial p_2(x,t)}{\partial v_{\mathcal{A}^*}}$$
$$=\int_a^{T-t} d_2(x,t+h)p_2(x,t+h;v)dh \quad (77)$$
$$x\in\Gamma, t\in(T-b,T-a),$$

$$\frac{\partial p_1(x,t)}{\partial v_{\mathcal{A}^*}}=0, \quad \frac{\partial p_2(x,t)}{\partial v_{\mathcal{A}^*}}=0, \quad (78)$$
$$x\in\Gamma, t\in(T-a,T).$$

Theorem 5.3. The optimal control $v^*\equiv v^*(x,t)=(v_1^*(x,t),v_2^*(x,t))\in(L^2(\Sigma))^2$ is characterized by the following maximum condition

$$\int_0^T\int_\Gamma([p_1(v^*)+\lambda_2 N_1 v_1^*](v_1-v_1^*)$$
$$+[p_2(v^*)+\lambda_2 N_2 v_2^*](v_2-v_2^*))d\Gamma dt\geq 0, \quad (79)$$
$$\forall v=(v_1,v_2)\in(\mathcal{U}_{ad})^2,$$

where $p\equiv p(x,t;v)=(p_1(x,t;v),p_2(x,t;v))\in(W^{\frac{3}{2},\frac{3}{4}}(Q))^2$ is the adjoint state.

Theorem 5.4. For the problem (63)-(68) with the performance function (62), with

$z_d = (z_{1d}, z_{2d}) \in (L^2(Q))^2$ and $\lambda_2 > 0$, and with constraint: $(\mathcal{U}_{ad})^2$ is closed, convex subset of $(L^2(\Sigma))^2$, and with adjoint equations (69)-(79), then there exists a unique optimal control $v^* \equiv v^*(x,t) = (v_1^*(x,t), v_2^*(x,t)) \in (L^2(\Sigma))^2$ which satisfies the maximum condition (79).

5.2. Case 2: Optimal control for $(\mathbf{n} \times \mathbf{n})$ coupled parabolic systems with time lags and infinite number of variables

We can extend the discussion to $(n \times n)$ coupled parabolic systems. We consider the case where $v = (v_1, v_2, ..., v_n) \in (L^2(\Sigma))^n$, the performance functional is given by

$$I(v) = \sum_{i=1}^{n}(\lambda_1 \int_Q [y_i(x,t;v) - z_{id}]^2 d\rho dt \\ + \lambda_2 \int_\Sigma (N_i v_i) v_i \, d\Gamma dt), \tag{80}$$

where $z_d = (z_{1d}, z_{2d}, ..., z_{nd}) \in (L^2(Q))^n$ and N_i is a positive linear operator on $L^2(\Sigma)$ into $L^2(\Sigma)$, $i = 1, 2, \ldots, n$.
The following results can now proved.

Theorem 5.5. Let y_0, Φ_0, Ψ_0, v and u be given with $y_0 = (y_{0,1}, y_{0,2}, \ldots, y_{0,n}) \in (W^{\frac{1}{2}}(\Omega, \mathbb{R}^\infty))^n$, $\Phi_0 = (\Phi_{0,1}, \Phi_{0,2}, \ldots, \Phi_{0,n}) \in (W^{32,34}(Q_0))^n$, $\Psi_0 = (\Psi_{0,1}, \Psi_{0,2}, \ldots, \Psi_{0,n}) \in (L^2(\Sigma_0))^n$, $v = (v_1, v_2, \ldots, v_n) \in (L^2(\Sigma))^n$ and $u = (u_1, u_2, \ldots, u_n) \in (W^{-\frac{1}{2}, -\frac{1}{4}}(Q))^n$. Then, there exists a unique solution $y = (y_1, y_2, \ldots, y_n) \in (W^{\frac{3}{2},\frac{3}{4}}(Q))^n$ for the following mixed initial-boundary value problem: for all $i = 1, 2, \ldots, n$ we have

$$\frac{\partial y_i}{\partial t} + \mathcal{S}(t) y_i + \int_a^b c_i(x,t) y(x,t-h) dh = u_i, \tag{81}$$
$x \in \Omega, t \in (0,T), h \in (a,b)$,

$$y_i(x,t') = \Phi_{i,0}(x,t'), \quad x \in \Omega, t' \in [-b, 0), \tag{82}$$

$$y_i(x,0) = y_{i,0}(x), \quad x \in \Omega, \tag{83}$$

$$\frac{\partial y_i(x,t)}{\partial \nu_{\mathcal{S}}} = \int_a^b d_i(x,t) y_i(x,t-h) dh + v_i, \tag{84}$$
$x \in \Gamma, t \in (0,T), h \in (a,b)$,

$$y_i(x,t') = \Psi_{i,0}(x,t'), \quad x \in \Gamma, t' \in [-b, 0), \tag{85}$$

where

$$\mathcal{S}(t) y_i(x) = \left(-\sum_{k=1}^{\infty} D_k^2 + q(x,t)\right) y_i(x) \\ + \sum_{j=1}^{n} a_{ij} y_j(x) \quad \forall i = 1, 2, \ldots, n, \tag{86}$$

$$a_{ij} = \begin{cases} 1, & i \geq j; \\ -1, & i < j. \end{cases} \tag{87}$$

The operator $\mathcal{S}(t)$ is $(n \times n)$ matrix which takes the form

$$\mathcal{S}(t) = \begin{pmatrix} -\sum_{k=1}^{\infty} D_k^2 + q + 1 & -1 & \cdots & -1 \\ 1 & -\sum_{k=1}^{\infty} D_k^2 + q + 1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & -\sum_{k=1}^{\infty} D_k^2 + q + 1 \end{pmatrix}_{n \times n}. \tag{88}$$

Also, we have

$$\begin{aligned} y_i &\equiv y_i(x,t;v), \\ u_i &\equiv u_i(x,t), \\ v_i &\equiv v_i(x,t), \\ u &\equiv (u_1, u_2, \ldots, u_n), \end{aligned}$$

c_i and d_i, $i = 1, 2, \ldots, n$, are real C^∞ functions defined on $\overline{Q}$ and Σ, respectively, $\Phi_{i,0}$ and $\Psi_{i,0}$, $i = 1, 2, \ldots, n$, are initial functions defined on Q_0 and Σ_0, respectively.

Lemma 5.6. Let the hypothesis of Theorem 5.5 be satisfied. Then for given $z_d = (z_{1d}, z_{2d}, \ldots, z_{nd}) \in (L^2(Q))^n$ and any $v(x,t) = (v_1(x,t), v_2(x,t), \ldots, v_n(x,t)) \in (L^2(\Sigma))^n$, there exists a unique solution

$$p(v)\equiv p(x,t;v)=(p_1(x,t;v),$$
$$p_1(x,t;v),\ldots,p_n(x,t;v))\in(W^{\frac{3}{2},\frac{3}{4}}(Q))^n,$$

for the adjoint problem: for all $i=1,2,\ldots,n$, we have

$$\begin{cases}
-\dfrac{\partial p_i(v)}{\partial t}+\mathcal{S}^*(t)p_i(v)+\int_a^b c_i(x,t+h)p_i(x,t+h;v)dh \\
=\lambda_1(y_i(v)-z_{id}), \quad x\in\Omega, t\in(0,T-b), \\
-\dfrac{\partial p_i(v)}{\partial t}+\mathcal{S}^*(t)p_i(v)+\int_a^{T-t} c_i(x,t+h)p_i(x,t+h;v)dh \\
=\lambda_1(y_i(v)-z_{id}), \quad x\in\Omega, t\in(T-b,T-a), \\
-\dfrac{\partial p_i(v)}{\partial t}+\mathcal{S}^*(t)p_i(v)=\lambda_1(y_i(v)-z_{id}), \\
x\in\Omega, t\in(T-a,T), \\
p_i(x,T,v)=0, \quad x\in\Omega, \\
\dfrac{\partial p_i(v)}{\partial \nu_{\mathcal{S}^*}}(x,t)=\int_a^b d_i(x,t+h)p_i(x,t+h;v)dh, \\
\quad x\in\Gamma, t\in(0,T-b), \\
\dfrac{\partial p_i(v)}{\partial \nu_{\mathcal{S}^*}}(x,t)=\int_a^{T-t} d_i(x,t+h)p_i(x,t+h;v)dh, \\
x\in\Gamma, t\in(T-b,T-a), \\
\dfrac{\partial p_i(v)}{\partial \nu_{\mathcal{S}^*}}(x,t)=0, \quad x\in\Gamma, t\in(T-a,T).
\end{cases} \tag{89}$$

Theorem 5.7. The optimal control $v^*\equiv v^*(x,t)=(v_1^*(x,t),v_2^*(x,t),\ldots,v_n^*(x,t))\in(L^2(\Sigma))^n$ is characterized by the following maximum condition

$$\sum_{i=1}^{n}\int_0^T\int_\Gamma([p_i(v^*)+\lambda_2 N_i v_i^*](v_i-v_i^*))d\Gamma dt\geq 0 \tag{90}$$
$$\forall v=(v_1,v_2,\ldots,v_n)\in(\mathcal{U}_{ad})^n,$$

where

$$v^*=(v_1^*,v_2^*,\ldots,v_n^*)\in(L^2(\Sigma))^n,$$

is the optimal control and

$$p(v^*)\equiv p(x,t;v^*)=(p_1(x,t;v^*),$$
$$p_1(x,t;v^*),\ldots,p_n(x,t;v^*))\in(W^{\frac{3}{2},\frac{3}{4}}(Q))^n,$$

is the adjoint state.

Theorem 5.8. For the problem (81)-(85) with the performance function (80) with $z_d=(z_{1d},z_{2d},\ldots,z_{nd})\in(L^2(Q))^n$ and $\lambda_2>0$, and with constraint: $(\mathcal{U}_{ad})^n$ is closed, convex subset of $(L^2(\Sigma))^n$, and with adjoint equations (89), then there exists a unique optimal control $v^*\equiv v^*(x,t)=(v_1^*(x,t),v_2^*(x,t),\ldots,v_n^*(x,t))\in(L^2(\Sigma))^n$ which satisfies the maximum condition (90).

Remark 5.9. In the case of performance functionals ((20), (42), (52), (62), (80)) with $\lambda_1>0$ and $\lambda_2=0$, the optimal control problem (with the initial state given by a known function) reduces to minimization of the functional on a closed and convex subset in a Hilbert space. Then, the optimization problem is equivalent to a quadratic programming one, which can be solved by the use of the well-known Gilbert algorithm.

6. Conclusions

The optimization problem presented in the paper constitutes a generalization of the optimal boundary control problem of a parabolic system with Neumann boundary condition involving constant time lag appearing both in the state equations and in the boundary conditions considered in [2-7, 13, 14, 16, 17, 20, 28].

Also, the main result of the paper contains necessary and sufficient conditions of optimality for $(n\times n)$ parabolic systems involving second order operator with infinite number of variables that give characterization of optimal control (Theorem 5.8). But it is easily seen that obtaining analytical formulas for optimal control is very difficult. This results from the fact that state equations (81)--(85), adjoint equations (89) and minimum condition (90) are mutually connected, which cause the usage of derived conditions to be difficult. Therefore, we must resign from the exact determination of the optimal control and we are forced to use approximation methods. It is also evident that by modifying:
• the boundary conditions,
• the nature of the control (distributed, boundary),
• the nature of the observation,
• the initial differential system,
an infinity of variations on the above problem is possible to study with the help of [18] and Dubovitskii-Milyutin formalisms [21-24]. Those problems need further investigations and form tasks for future research. These ideas mentioned above will be developed in forthcoming papers.

References

[1] Knowles, G. (1978). Time-optimal control of parabolic systems with boundary conditions involving time delays. *J. Optim. Theor. Appl.*, *25*(4), 563-574.

[2] Kowalewski, A. (1988). Boundary control of distributed parabolic system with boundary condition involving a time varying lag. *International Journal of Control*, *48*(6), 2233-2248.

[3] Kowalewski, A. (1990). Feedback control for distributed parabolic system with boundary condition involving a time varying lag. *IMA J. Math. Control Inform.*, *7*(2), 143-157.

[4] Kowalewski, A. (1990). Optimal control of distributed parabolic system involving time lags. *IMA J. Math. Control Inform.*, *7*(4), 375-393.

[5] Kowalewski, A. (1993). Optimal control of parabolic systems with time-varying lags. *IMA J. Math. Control Inform.*, *10*(2), 113-129.

[6] Kowalewski, A. (1998). Optimal control of a distributed parabolic systems with multiple time varying lags. *Int. J. Control*, *69*(3), 361-381.

[7] Kowalewski, A. (1999). Optimization of parabolic systems with deviating arguments. *Int. J. Control*, *72*(11), 947-959.

[8] Kowalewski, A. (2001). *Optimal Control of Infinite Dimensional Distributed Parameter systems with Delays*. Cracow, University of Mining and Metallurgy Press.

[9] Kowalewski, A. & Duda, J. (1992). On some optimal control problem for a parabolic system with boundary condition involving a time-varying lag. *IMA J. Math. Control Inform.*, *9*(2), 131-146.

[10] Wang, P. K. C. (1975). Optimal control of parabolic systems with boundary conditions involving time delays. *SIAM J. Control*, *13*(2), 274-293.

[11] Gali, I. M. & El-Saify, H. A. (1982). Optimal control of a system governed by hyperbolic operator with an infinite number of variables. *J. Math. Anal. Appl.*, *85*, 24-30.

[12] Gali, I. M. & El-Saify, H. A. (1983). Distributed control of a system governed by Dirichlet and Neumann problems for a self-adjoint elliptic operator with an infinite number of variables. *J. Optim. Theory Appl.*, *39*, 293-298.

[13] El-Saify, H. A. & Bahaa, G. M. (2001). Optimal control for $(n \times n)$ systems of hyperbolic types. *Revista de Matemáticas Aplicadas*, *22*, 41-58.

[14] El-Saify, H. A. & Bahaa, G. M. (2003). Optimal control for $(n \times n)$ coupled systems of Petrowsky type with an infinite number of variables. *Mathematica Slovaca*, *53*, 291-311.

[15] El-Saify, H. A., Serag, H. M, & Bahaa, G. M. (2000). On optimal control for $(n \times n)$ elliptic system involving operators with an infinite number of variables. *Advances in Modelling & Analysis*, *37*, 47-61.

[16] El-Saify, H. A. (2005). Optimal control for $(n \times n)$ parabolic system involving time lag. *IMA J. Math. Control and Inform.*, *22*(3), 240-250.

[17] El-Saify H. A. (2006). Optimal boundary control problem for $(n \times n)$ infinite order parabolic lag system. *IMA J. Math. Control and Inform.*, *23*(4), 433-445.

[18] Lions, J. L. (1971). Optimal Control of Systems Governed by Partial Differential Equations. *Springer-Verlag, Band*, *170*.

[19] Lions, J. L. & Magenes, E. (1972). *Non-Homogeneous Boundary Value Problem and Applications*. I, II, New York, Springer-Verlag.

[20] Kotarski, W., El-Saify, H. A. & Bahaa, G. M. (2002). Optimal control of parabolic equation with an infinite number of variables for non-standard functional and time delay. *IMA J. Math. Control and Inform.*, *19*, 461-476.

[21] Bahaa, G. M. (2003). Quadratic Pareto optimal control of parabolic equation with state-control constraints and an infinite number of variables. *IMA J. Math. Control and Inform.*, *20*, 167-178.

[22] Bahaa, G. M. (2005). Time-optimal control problem for parabolic equations with control constraints and infinite number of variables. *IMA J. Math. Control and Inform.*, *22*, 364-375.

[23] Bahaa, G. M. (2005). Time-optimal control problem for infinite order parabolic equation with control constraints. *Differ. Equ. Control Process. Electron. J.*, *4*, 64-81.

[24] Bahaa, G. M. (2008). Optimal control problems of parabolic equations with an infinite number of variables and with equality constraints. *IMA J. Math. Control and Inform.*, *25*, 37-48.

[25] Berezanskii, Ju. M. (1975). Self-adjointness of elliptic operator with an infinite number of variables. *Ukrain. Math. Z.*, *27*, 729-742.

[26] Kowalewski, A. & Krakowiak, A. (1994). Time-optimal control of parabolic time lag system. *Applied Mathematics and Computer Science*, *4*(1), 19-28.

[27] Kowalewski, A. & Krakowiak, A. (2000). Time-optimal control of parabolic system with time lags given in the integral form. *IMA Journal of Mathematical Control and Information*, *17*(3), 209-225.

[28] Kowalewski, A. & Krakowiak, A. (2006). Time-optimal boundary control of a parabolic system with time lags given in the integral form. *International Journal of Applied Mathematics and Computer Science*, *16*(3), 287-295.