IJST (2012) A3: 267-275
Iranian Journal of Science & Technology

http://www.shirazu.ac.ir/en

Capacity and a classification of Finsler spaces

B. Bidabad[1]* and S. Hedayatian[2]

[1]*Faculty of Mathematics and Computer Sciences, Amirkabir University of Technology (Tehran Polytechnic), 424 Hafez Ave 15914 Tehran, Iran*
[2]*Faculty of Mathematics and Computer Sciences, Shahid Chamran University of Ahvaz, P.O. Box: 61355-83151, Ahvaz, Iran*
E-mails: bidabad@aut.ac.ir & hedayatian@scu.ac.ir

Abstract

Using the electrostatic capacity of a condenser, the existence of a distance function on a Finsler space is discussed. This distance function divides Finsler spaces into the two classes, denoted here by I and II. The topology generated by this distance on the Finsler spaces of class II coincides with its intrinsic topology. This work provides a natural extension of mathematical analysis tools needed for developing some prominent features of differential geometry in the large.

***Keywords**:* Conformal; capacity; condenser; Finsler metrics; potential analysis

1. Introduction

Recently, one of the present authors in a joint work has classified Finsler spaces through the conformal transformations, [1]. Here, we give another classification of Finsler spaces by using a distance function determined by the notion of conformal capacity on Finsler spaces, previously introduced by the present authors [2] and [3].

More intuitively, the concept of capacity as a set function arising in potential analysis is analogous to the physical concept of the electrostatic capacity of a condenser, that is, an open set with a relatively compact subset inside.

The capacity of a set as a mathematical concept was introduced first by N.Wiener in 1924 and was subsequently developed by several French mathematicians in connection with the potential theory. The notion of capacity has since been extensively developed for $\mathbb{R}^n$ particularly by M. Vuorinen.

This notion was also used by G. D. Mostow to prove his famous theorem on the rigidity of hyperbolic spaces [4]. As a positive response to the question asked by Vuorinen in [5], J. Ferrand proved that the concept of capacity can be used to define a distance function on Riemannian geometry, [6] and [7]. Her work provides important tools in the studies of global properties of Riemannian spaces. Specifically, she used this theory to give a rigorous and complete proof to the famous "Lichnerowicz conjecture" in conformal transformations groups of Riemannian spaces [8]. Her proof is quite elementary and is based on the behavior of sequences on the whole conformal group of the Riemannian space under consideration.

Defining such a distance function in Finsler spaces seems desirable, since Finsler geometry has come up in many applications particularly in physics. Moreover, Finsler space is a natural generalization of Riemannian and hence Euclidean space. This fact is discovered after the Elie Cartan's Euclidean connection, in 1933, [9] and [10].

The present work has established a distance function which provides an elementary account of some tools in analysis needed for developing Finsler geometry. More precisely, as an extension of Ferrand's work, we show the existence of a distance function determined by notion of a conformal invariant function, called capacity, on a compact subset of a Finsler space. This function, namely μ_M, defined as infimum of capacities of a certain subset of the Finsler space, classifies Finsler metrics into the two classes denoted here by *I* and *II*, for which some examples are provided. The aim of this work is to study properties of Finsler spaces of class *II*.

Theorem A: Let (M, g) be a Finsler space of class *II*, then for every point x_1, x_2 and $x_3 \in M$ we have

*Corresponding author
Received: 20 July 2011 / Accepted: 19 October 2011

(1) $\mu_M(x_1, x_2) = \mu_M(x_2, x_1)$,

(2) $\mu_M(x_1, x_2) = 0 \Leftrightarrow x_1 = x_2$,

(3) $\mu_M(x_1, x_3) \le \mu_M(x_1, x_2) + \mu_M(x_2, x_3)$.

Our goal in this paper is the following theorem.

Theorem B For any Finsler space of class *II*, the topology defined by distance function μ_M coincides with the canonical topology.

2. Notations and Preliminaries

Let M be an n-dimensional C^∞ manifold. For a point $x \in M$, denote by T_xM the tangent space of M at x. The tangent bundle TM on M is the union of tangent spaces T_xM and $TM_0 = TM \setminus \{0\}$. We will denote the elements of TM by (x, y), where $y \in T_xM$. The natural projection $\pi : TM \to M$ is given by $\pi(x, y) := x$.

A Finsler structure on a manifold M is a function $F : TM_0 \to [0, \infty)$ with the following properties: (i) F is C^∞ on TM_0, (ii) F is positively 1-homogeneous on the fibers of tangent bundle TM, that is $\forall \lambda > 0,\ F(x, \lambda y) = \lambda F(x, y)$, (iii) The Hessian of F^2 with elements $g_{ij} = \frac{1}{2}[F^2(x, y)]_{y^i y^j}$ is positive definite on TM_0. The pair (M, g) is called a Finsler space. Throughout this paper, we use *Einstein summation convention* for the expressions with repeated indices. The Finsler structure F is Riemannian if $g_{ij}(x, y)$ are independent of $y \neq 0$.

Let $\pi : TM \to M$ be the natural projection. Collecting all tangent Spaces T_xM, we form a vector bundle called the *pull-back bundle* or *pull-back tangent space* π^*TM defined by

$$\pi^*TM := \{(x, y, v) \mid y \in T_xM_0, v \in T_xM\}.$$

Both π^*TM and its dual π^*T^*M are n-dimensional vector bundles over TM_0 (see, for instance, [9] and [11]). If we put $VTM = \bigcup_{v \in TM} \ker \pi_*^{\,v}$, then a non-linear connection on TM is a complementary distribution HTM for VTM on TTM. Therefore, we have the following decomposition. $TTM = VTM + HTM.$

We recall that, g_{ij} is a homogeneous tensor of degree zero in y and $g_{ij}(x, y) y^i y^j = g_x(y, y)$, where $g_x(\ ,\ )$ is the local scalar product on the fiber T_xM of π^*TM at any fixed point $x \in M$. Using the induced coordinates (x^i, y^i) on TM, we have the local field of frames $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\}$ on TTM. Let $\{dx^i, dy^i\}$ be the dual of $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\}$, we can choose a local field of frames $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}$ adapted to the above decomposition, namely $\frac{\delta}{\delta x^i} \in \chi(HTM)$ and $\frac{\partial}{\partial y^i} \in \chi(VTM)$. They are sections of horizontal and vertical sub-bundle on HTM and VTM, defined by $\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j}$, where N_i^j are the coefficients of non linear connection. Clearly we have $N_i^j = \gamma^j{}_{ik} y^k - C^j{}_{ik} \gamma^k{}_{rs} y^r y^s$,

where, $\gamma^i{}_{jk} := \frac{1}{2} g^{is} (\frac{\partial g_{sj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^s} + \frac{\partial g_{ks}}{\partial x^j})$, $[g^{ij}]$ is the inverse matrix of $[g_{ij}]$ and $C_{ijk} = \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k}$, is the Cartan torsion tensor.

3. Finslerian Terminology

In Riemannian geometry integral of geometric objects are taken over the underlying manifold M. In Finsler geometry it was first "Lichnerowicz" who proposed that for Finsler case, little is lost by allowing this integrand to live on the projective sphere bundle SM, [12]. The main reason follows from the fact that all geometric quantities constructed from the Finsler structure F are homogeneous of degree zero in *y* and thus naturally live on SM, even though Finsler structure itself does not. With this motivation, in this section we are going to define projective sphere bundle and determine a Riemannian metric on it.

Let S_xM be the set consisting of all rays $[y] := \{\lambda y \mid \lambda > 0\}$, where $y \in T_xM_0$. Here,

$[y]$ is a typical equivalence class that results from the identification $z \sim y \Leftrightarrow z = \lambda y$ for some $\lambda > 0$. Geometrically, each $(x,[y])$, is a ray emanating from the origin of $T_x M$.

Let $SM = \bigcup_{x \in M} S_x M$, then SM has a natural $(2n-1)$ dimensional manifold structure and the total space of a fiber bundle, called projective sphere bundle, or simply sphere bundle over M. We denote the elements of SM by $(x,[y])$ where $y \in T_x M_0$. Given local coordinates (x^i) on M, we can write any $y \in T_x M$ as $y^i \frac{\partial}{\partial x^i}$. This generates local coordinates (x^i, y^j) on SM, where the y^j's are now treated as homogeneous coordinates.
Let $p : SM \to M$ be the natural projection. The pull-back tangent space p^*TM is defined by

$$p^*TM := \{(x,[y],v) \mid y \in T_x M_0, v \in T_x M\}.$$

The pull-back cotangent space p^*T^*M is the dual of p^*TM. Both p^*TM and p^*T^*M are total spaces of vector bundles over SM. Using the coefficients of a non-linear connection on TM one can define a non-linear connection on SM by using the objects which are invariant under positive re-scaling $y \to \lambda y$.

In this way the coefficients of non-linear connection N_i^j are defined on the sphere bundle SM. Our preference for remaining on SM forces us to work with $\frac{N_i^j}{F} = \gamma^j{}_{ik} l^k - C^j{}_{ik} \gamma^k{}_{rs} l^r l^s$, where $l^i = \frac{y^i}{F}$. We also prefer to work with the local field of frames $\{\frac{\delta}{\delta x^i}, F\frac{\partial}{\partial y^j}\}$ and $\{dx^i, \frac{\delta y^j}{F}\}$ which are invariant under the positive re-scaling of y, and over SM they can be used as a local field of frames for p^*TM and p^*T^*M respectively. The pull-back tangent bundle p^*TM over SM has a canonical section l defined by $l_{(x,[y])} = (x,[y], \frac{y}{F(x,y)})$.

Let $\{\partial_i := (x,[y], \frac{\partial}{\partial x^i})\}$ be a natural local field of frames for p^*TM. The natural dual co-frame for p^*T^*M is $\{dx^i\}$. We use the following lemma from classical differential geometry to replace the C^∞ functions on TM_0 by those on SM.

Lemma 3.1. [13] Let η be the function, $\eta : TM_0 \to SM$ where $\eta(x,y) = (x,[y])$ and $f \in C^\infty(TM_0)$. Then there exists a function $g \in C^\infty(SM)$ satisfying $\eta^* g = f$ if and only if $f(x,y) = f(x, \lambda y)$, where $y \in T_x M_0, \lambda > 0$ and η^* is the pull-back of η.

Let f be a C^∞ function on M and $f^v \in C^\infty(TM_0)$ the vertical lift of f defined by $f^v : TM \to \mathbb{R}$, where $f^v(x,y) := f o \pi(x,y) = f(x)$. The vertical lift f^v is independent of y and according to Lemma 3.1 there is a function g on $C^\infty(SM)$ related to f^v, determined by $\eta^* g = f^v$. In the sequel, g is denoted by f^v for simplicity.

Clearly, if the differentiable manifold M is compact then the sphere bundle SM is compact too, and hence SM is orientable whether M is orientable or not.

It turns out that the manifold TM_0 has a natural Riemannian metric, known in the literature as Sasakian metric (see for instance [9], [11] or [14]); $\tilde{g} = g_{ij}(x,y) dx^i dx^j + g_{ij}(x,y) \frac{\delta y^i}{F} \frac{\delta y^j}{F}$, where, $g_{ij}(x,y)$ are the Hessian of Finsler structure F^2. They are functions on TM_0 and invariant under positive rescaling of y, therefore they can be considered as functions on SM. With respect to this metric, the horizontal subspace spanned by $\frac{\delta}{\delta x^i}$ is orthogonal to the vertical subspace spanned by $F\frac{\partial}{\partial y^j}$. The metric $\tilde{g}$ is invariant under the positive rescaling of y and can be considered as a Riemannian metric on SM. Therefore, SM is a natural $(2n-1)$-dimensional Riemannian manifold with its Sasaki type metric induced by the fundamental tensor $g_{ij}(x,y)$.

4. Capacity of a compact subset on Finsler spaces

In what follows (M,g) denotes a connected Finsler space of class C^∞ and dimension $n \geq 2$. Let $(SM,\tilde{g})$ be its Riemannian sphere bundle. The Finsler structure $F(x,y)$ induces a canonical 1-form on SM defined by $\omega := l_i dx^i$, where $l_i = g_{ij} l^j$, called Hilbert form of F. The gradient vector field ∇f of a function $f \in C^\infty(SM)$ is given by $\tilde{g}(\nabla f, \tilde{X}) = df(\tilde{X}), \ \forall \tilde{X} \in \Gamma(SM, TTM)$, where $\Gamma(SM,TTM)$ is the set of all differentiable sections on SM. Using the local coordinate system $(x^i,[y^i])$ on SM, the section $\tilde{X} \in \Gamma(SM,TTM)$ is given by $\tilde{X} = X^i(x,y)\frac{\delta}{\delta x^i} + Y^j(x,y)F\frac{\partial}{\partial y^j}$, where $X^i(x,y)$ and $Y^j(x,y)$ are C^∞ functions on SM.

A simple calculation shows that the gradient vector field ∇f can be written locally as $\nabla f = g^{ij}\frac{\delta f}{\delta x^i}\frac{\delta}{\delta x^j} + F^2 g^{ij}\frac{\partial f}{\partial y^i}\frac{\partial}{\partial y^j}$.

The norm of ∇f with respect to the Riemannian metric $\tilde{g}$ is given by $|\nabla f|^2 = \tilde{g}(\nabla f, \nabla f) = g^{ij}\frac{\delta f}{\delta x^i}\frac{\delta f}{\delta x^j} + F^2 g^{ij}\frac{\partial f}{\partial y^i}\frac{\partial f}{\partial y^j}$.

We consider the volume element $\eta(g)$ on SM defined in [9] as follows

$$\eta(g) := \frac{(-1)^N}{(n-1)!}\omega \wedge (d\omega)^{(n-1)},$$

where $N = \frac{n(n-1)}{2}$ and ω is the Hilbert form of F. Let $\mathcal{C}(M)$ be the linear space of continuous real valued functions on M, $u \in \mathcal{C}(M)$ and u^v its vertical lift on SM. For M compact or not we denote by $H(M)$ the set of all functions in $\mathcal{C}(M)$ admitting a generalized L^n-integrable gradient ∇u^v satisfying

$$I(u,M) = \int_{SM} |\nabla u^v|^n \eta(g) < \infty. \tag{1}$$

It should be remarked later in the next section that the function $I(u,M)$ and the corresponding notion are finite.

If M is non-compact we denote by $H_0(M)$ the subspace of functions $u \in H(M)$ for which the vertical lift u^v has a compact support in SM.

A relatively compact subset is a subset whose closure is compact. A function $u \in \mathcal{C}(M)$ will be called monotone if for any relatively compact domain D of M

$\sup_{x\in\partial D} u(x) = \sup_{x\in D} u(x)$,

$\inf_{x\in\partial D} u(x) = \inf_{x\in D} u(x)$.

Let us denote by $H^*(M)$ the set of monotone functions $u \in H(M)$. We define notion of capacity for Finsler space as follows.

Definition 4.1. The capacity of a compact subset C of a non-compact Finsler space (M,g) is defined by $Cap_M(C) := \inf_u I(u,M)$, where the infimum is taken over the functions $u \in H_0(M)$ with $u = 1$ on C and $0 \leq u(x) \leq 1$ for all x. These functions are said to be admissible for C.

A relative continuum is a closed subset C of M such that $C \bigcup \{\infty\}$ is connected in Alexandrov's compactification $\overline{M} = M \bigcup \{\infty\}$. To avoid ambiguities, the connected closed subsets of M which are not reduced to one point will be called continua. Using these sets at every double point of M we define the function μ_M as follows.

Definition 4.2. Let (M,g) be a Finsler space. For all (x_1,x_2) in $M^2 := M \times M$ we set $\mu_M(x_1,x_2) = \inf_{C\in\alpha(x_1,x_2)} Cap_M(C)$, where $\alpha(x_1,x_2)$ is the set of all compact continua subsets of M, containing x_1 and x_2.

For any subset S of M and any $u \in \mathcal{C}(M)$, the oscillation $O(u,S)$ of u on S is defined by $O(u,S) := \sup_{x_1,x_2\in S} |u(x_1) - u(x_2)|$.

5. Fundamental properties of capacity

Lemma 5.1. Let (M, g) be an open Finsler sub-manifold of a Finsler space N then

(1) $Cap_M(C) \geq Cap_N(C)$ where C is a compact set in M.

(2) $\mu_M(x_1, x_2) \geq \mu_N(x_1, x_2)$

for all x_1, x_2 in M.

Proof: Every function $u \in H_0(M)$, admissible for C in M can be extended into an admissible function for C in N by setting $u = 0$ on $N \setminus M$. So we have the assertion (1). The assertion (2) follows from the definition of μ_M and the assertion (1) and we have the lemma.

Let (M, g) be an n-dimensional Finsler space, and $\sigma : [a, b] \to M$ a piecewise C^∞ curve with velocity $\frac{d\sigma}{dt} = \frac{d\sigma^i}{dt}\frac{\partial}{\partial x^i} \in T_{\sigma(t)}M$. Its integral length $L(\sigma)$ is defined by $\int_a^b F(\sigma, \frac{d\sigma}{dt})dt$. For $x_0, x_1 \in M$ we denote by $\Gamma(x_0, x_1)$ the collection of all piecewise C^∞ curves $\sigma : [a, b] \to M$ with $\sigma(a) = x_0$ and $\sigma(b) = x_1$. Define a map $d : M \times M \to [0, +\infty)$ by $d(x_0, x_1) := \inf_{\Gamma(x_0, x_1)} L(\sigma)$. It can be shown that (M, d) satisfies the first two axioms of a metric space except the symmetry property, but the topology defined by the distance d is equivalent to the original manifold topology of M. If the Finsler structure F is absolutely homogeneous, that is $F(x, -y) = F(x, y)$, then one also has the symmetry property.

Lemma 5.2. Let (M, g) be an n-dimensional Finsler space. For any $\varepsilon > 0, a \in M$ and any neighborhood U of a, there exists a compact connected neighborhood V of a, with $V \subseteq U$, and a function $u \in H(M)$ with compact support in U, satisfying $I(u, M) \leq \varepsilon$, $u = 1$ on V and $0 \leq u \leq 1$ everywhere.

Proof: There exists a local chart (U_a, φ) of M centered at a with $U_a \subseteq U$ such that φ is a bi-Lipschitzian map with ratio 2 of U_a onto a ball $B = B(0, 2R)$ of the n-dimensional Euclidean space E^n, namely for all $x_0, x_1 \in U_a$

$$\frac{1}{2}|\varphi(x_0) - \varphi(x_1)| \leq d(x_0, x_1) \leq 2|\varphi(x_0) - \varphi(x_1)|,$$

where d is the distance defined by Finsler structure ([11] P. 149). We can choose $r \geq 0$ such that

$$\ln(\frac{R}{r}) \geq (\frac{2^n \omega_{n-1}}{\varepsilon})^{\frac{1}{n-1}}, \tag{2}$$

where ω_{n-1} is the volume of S^{n-1}. We define a real function v on B by

$$V(X) = \begin{cases} 1 & \text{if} \quad |x| \leq r, \\ \dfrac{\ln R - \ln|x|}{\ln R - \ln r} & \text{if} \quad r \leq |x| \leq R, \\ 0 & \text{if} \quad |x| \geq R. \end{cases}$$

Clearly by a classical result ([4] P. 80), $I(v, B) = \omega_{n-1}(\ln\frac{R}{r})^{1-n}$. Let us put $u = vo\varphi$, since φ is 2 quasi-conform ([8]), from relation (2) we obtain $I(u, M) \leq \varepsilon$, where we have put $V = \varphi^{-1}(\overline{B(0, r)})$. Hence the proof is complete.

Lemma 5.3. For any given $\varepsilon > 0$ and any $a \in M$ there exists a neighborhood V of a such that for any $u \in H^*(M), O^n(u, V) \leq \varepsilon I(u, M)$.

Proof: Let (U_a, φ) be a chart of M centered at a such that φ is a bi-Lipschitzian map with ratio 2 of u_a onto a ball $B = B(0, R)$ of E^n. If $u \in H^*(M)$ then $v = uo\varphi^{-1}$ is monotone on the ball B, hence its oscillation $\Omega(t)$ on the sphere $\partial B(0, t)$ $(0 \leq t \leq R)$ is the same as that on the $B(0, t)$ and is an increasing function of t. Hence for all $0 \leq r \leq R$, $\int_r^R \frac{\Omega^n(t)}{t}dt \geq \Omega^n(r)\ln\frac{R}{r}$. On the other hand, from a classical result in [4] we have $\int_0^R \frac{\Omega^n(t)}{t}dt \leq A_n I(v, B)$, where A_n is an absolute constant. By choosing $r \geq 0$ such that $\ln \frac{R}{r} \geq \frac{2^n A_n}{\varepsilon}$ we find that

$$O^n(u, \varphi^{-1}(B(0,r))) = \Omega^n(r) \leq \frac{\int_r^R \frac{\Omega^n(t)}{t} dt}{\ln \frac{R}{r}} \leq$$

$$\frac{A_n I(v,B)}{\ln \frac{R}{r}} \leq \frac{\varepsilon}{2^n} I(v,B) \leq \varepsilon I(u,M),$$

where the last inequality follows from the fact that φ is 2-quasi-conform. Thus the set $V = \varphi^{-1}(B(0,r))$ satisfies the claim and we have the lemma.

Lemma 5.4. For any compact continuum set C in the n-dimensional Finsler space (M,g) there exists a constant $K(C)$, only depending on C, such that for all $u \in H^*(M)$, $O^n(u,C) \leq K(C) I(u,M)$.

Proof: We apply Lemma 5.3 with $\varepsilon = 1$. For all $a \in C$ there exists an open neighborhood $V(a)$ of a such that $O^n(u, V(a)) \leq I(u,M)$. We can choose a finite covering of C by neighborhoods $V(a_k)$ $(k = 1,...,p)$ such that, for all $u \in H^*(M)$, $O(u,C) \leq \sum_{k=1}^{p} O(u, V(a_k))$.

Thus we have $O(u,C) \leq p(I(u,M))^{\frac{1}{n}}$. Hence the announced maximization, with $K(C) = p^n$. This completes the proof of the Lemma.

Remark: The capacity of a compact subset of a non compact Finsler space is finite.

This fact can be easily verified as follows. Let C be a compact subset of a non compact Finsler space M. We apply Lemma 5.2 with $\varepsilon = 1$. Denote by V_x and u_x the compact connected neighborhood and the function associated with each point x of C, respectively. There exists a covering of C by a finite number of open sets $\mathrm{Int}V_x$, denoted by $\mathrm{Int}V_{x_i}$ for $i = 1,...,m$, where $x_i \in C$. The function $u = \sup_{1\leq i\leq m} u_{x_i}$ is admissible for C and we have $I(u,M) \leq m$.

Lemma 5.5. If $C = \{a\}$ then $\mathrm{Cap}_M(C) = 0$.

Proof: This follows from the definition of capacity and Lemma 5.2.

Lemma 5.6. For any compact continuum sets C_1 and C_2 on M we have

$$\mathrm{Cap}_M(C_1 \bigcup C_2) \leq \mathrm{Cap}_M(C_1) + \mathrm{Cap}_M(C_2),$$

and for any compact set C of a finite number of points we have $\mathrm{Cap}_M(C) = 0$.

Proof: Let $u_i, (i = 1,2)$ be an admissible function on C_i, then $u = \sup(u_1,u_2)$ is an admissible function on $C_1 \bigcup C_2$. This proves the inequality.

Now we are in position to show that the function μ_M is a continuous distance function.

Lemma 5.7. Let (M,g) be a Finsler space, then we have

(1) $\mu_M(x_1,x_2) = \mu_M(x_2,x_1)$,

(2) $\mu_M(x,x) = 0$,

(3) $\mu_M(x_1,x_3) \leq \mu_M(x_1,x_2) + \mu_M(x_2,x_3)$,

(4) μ_M is a continuous function on $M \times M$.

Proof: The first assertion follows from definition of the function μ_M. The second assertion follows easily from Lemma 5.2. To prove 3, we notice that

$$\mu_M(x_1,x_3) = \inf_{x_1,x_3 \in C} \mathrm{Cap}_M(C) \leq$$

$$\inf_{x_1,x_2,x_3 \in C} \mathrm{Cap}_M(C) \leq \inf_{x_1,x_2 \in C_1} \mathrm{Cap}_M(C_1) +$$

$$\inf_{x_2,x_3 \in C_2} \mathrm{Cap}_M(C_2) = \mu_M(x_1,x_2) + \mu_M(x_2,x_3),$$

where C_1 and C_2 are compact continuum sets in M. Finally, to prove 4, continuity of μ_M on the $\{(x,x) \in M^2 \mid x \in M\}$ follows easily from Lemma 5.2, otherwise on the $\{(x_1,x_2) \in M^2 \mid x_1,x_2 \in M, x_1 \neq x_2\}$, we use 3 and the assertion follows. Hence the proof is complete.

6. An invariance property of capacity

Here we obtain another invariance property of capacity. Let D be an open proper subset of a locally connected topological space E. For any real number c, and any continuous function f on

D, let cDf denote the union of all connected components of $D - f^{-1}(c)$ whose closures lie in D. For any real number a, let $f.a$ denote the function on $\overline{D}$ defined by

$$(f.a)(x) = \begin{cases} a, & \text{if} \quad x \in aDf, \\ f(x), & \text{if} \quad x \notin aDf. \end{cases} \qquad (3)$$

Lemma 6.1. (Lebesgue's straightening Lemma [4]). Let D be an open proper subset of a locally connected topological space and let f be a continuous real valued function on $\overline{D}$ with values in the bounded interval $[r,s]$. Let $a_1, a_2, ..., a_n, ...$ be an enumeration of the rational numbers in the interval $[r,s]$. Set $f_n = (...((f.a_1).a_2)...).a_n$ then f_n uniformly converge on $\overline{D}$ to a monotone function.

Lemma 6.2. Let (M,g) be a Finsler space and u_p a sequence in $H(M)$ which is uniformly convergent on every compact subset of M, then its limit v belongs to $H(M)$ and we have $I(v,M) \le \liminf_{p\to\infty} I(u_p, M)$.

Proof: This Lemma is an easy extension of the classical one in [8].

•Extension of the Lebesgue's straightening Lemma to the Finsler spaces.

Lemma 6.3. Let U be a proper open subset of a Finsler space (M,g). Then for any bounded function $u \in \mathcal{C}(M)$ there exists a function $v \in \mathcal{C}(M)$ with the same bounds as u, which is monotone on U and equal to u on $M \setminus U$. Moreover, if $u \in H(M)$, then $v \in H(M)$ and $I(v,M) \le I(u,M)$.

Proof: Let U be an open proper subset of the Finsler space M. For any real number a, and any function $u \in H(M)$, let aUu denote the union of all connected components of $U - u^{-1}(a)$ whose closures lie in U, then from relation (1) we have $I(u.a, M) = \int_{S(aUu)} |\nabla u^V|^n \, \eta(g) \le I(u,M)$, where $u.a$ is defined by relation (3), $S(aUu)$ is the sphere bundle of aUu and $|\nabla u^V|$ is defined with respect to the Sasaki metric. The proof of this lemma follows from Lemmas 6.1, 6.2 and last inequality.

Corollary: The value of $Cap_M(C)$ remains invariant if we replace the admissible functions on M by the monotone functions on $M \setminus C$.

Proof: If we replace $U = M \setminus C$ in Lebesgue's Lemma 6.3, then we have the corollary.

7. A class of Finsler spaces

The following lemma is an extension of the corresponding Euclidean one.

Lemma 7.1. Let (M,g) be a Finsler space. If there exists a pair of points (x_1, x_2) of M with $x_1 \ne x_2$ and $\mu_M(x_1,x_2) = 0$, then $Cap_M(C) = 0$ for all compact continuum C of M and the function μ_M is identically zero.

Proof: First assume that the compact continuum C does not contain both points x_1 and x_2, and let $x_1 \notin C$ for precision. Then we can choose a compact continuum γ containing C and x_2 in $M \setminus \{x_1\}$. Applying Lemma 5.4 to $M \setminus \{x_1\}$ we obtain a constant k such that for all $u \in H^*(M \setminus \{x_1\})$ we have $O(u, C \bigcup \gamma) \le k I^{\frac{1}{n}}(u,M)$. Now $\in > 0$ be given, it is possible to assume that $\varepsilon < \frac{1}{k}$. As $\mu_M(x_1,x_2) = 0$ there exists a function $u \in H_o(M)$ satisfying $0 \le u \le 1$ everywhere, $u = 1$ on a compact continuum C_ε containing x_1 and x_2, and $I(u,M) \le (\frac{\varepsilon}{2})^n$. In accordance to Corollary 6 we can assume that u is monotone on $M \setminus C_\varepsilon$, hence also on $M \setminus \{x_1\}$. Then we have $O(u, C \bigcup \gamma) \le \frac{k\varepsilon}{2}$. As $u(x_2) = 1$, it follows that $u(x) \ge 1 - \frac{k\varepsilon}{2} \ge \frac{1}{2}$ for all $x \in C$. Then the function $v = \inf(2u, 1)$ is admissible for C, hence $Cap_M(C) \le I(v,M) \le 2^n I(u,M) \le \varepsilon^n$. As ε is arbitrarily small we have at last $Cap_M(C) = 0$.

Next, for any $x \in M \setminus \{x_1\}$, there exists a compact continuum C containing x and x_2 in $M \setminus \{x_1\}$ and the relation $Cap_M(C) = 0$ involves $\mu_M(x, x_2) = 0$. If we observe that a compact set of M is a proper subset of M, then we can use the above argument with a point x of $M \setminus \{C\}$ in the place of x_1. Therefore the relation $Cap_M(C) = 0$ still holds if the compact continuum C contains x_1 and x_2. The first assertion is now completely proved, and the second one follows immediately. Hence, the proof of the Lemma is complete.

Applying the last part of the preceding lemma we see that the two following classes of non-compact Finsler spaces are complementary.

- The class *I* of manifolds M for which the function μ_M is identically zero.
- The class *II* of manifolds M for which the function μ_M is not identically zero. From the above lemma it is equivalent to say that $\mu_M(x, x_2) = 0$ gives rise to $x_1 = x_2$.

Example 1. Let N be a compact Finsler space. Then for any compact Continuum C of N the open submanifold $N \setminus C$ is of class *II* and for any finite set $S = \{a_1, ..., a_k\}$ of points in N, the punctured manifold $N \setminus S$ is of class *I*. Moreover, if two manifolds are conformally equivalent, then they belong to the same class. In particular, E^n is of class *I* and every bounded domain of E^n is of class *II*.

Using Lemma 5.7 we have the following theorem.

Theorem 7.2. Let (M, g) be a Finsler space of class *II*, then for every point x_1, x_2 and $x_3 \in M$ we have

(1) $\mu_M(x_1, x_2) = \mu_M(x_2, x_1)$,

(2) $\mu_M(x_1, x_2) = 0 \Leftrightarrow x_1 = x_2$,

(3) $\mu_M(x_1, x_3) \le \mu_M(x_1, x_2) + \mu_M(x_2, x_3)$.

Lemma 7.3. Let (M, g) be a Finsler space of class *II*, then μ_M is a distance on M. Moreover, for any relatively compact domain D of M, we have $\mu_M(x_1, x_2) \ge \inf_{x_3 \in \partial D} \mu_M(x_1, x_3) > 0$ for all $x_1 \in D$ and all $x_2 \in M \setminus D$.

Proof: The first assertion follows from Lemma 5.7 and Lemma 7.1. For any $x_1 \in D$, $x_2 \in M \setminus D$ and $\varepsilon > 0$ given, there exists a compact continuum C containing x_1 and x_2 with $Cap_M(C) \le \mu_M(x_1, x_2) + \varepsilon$. Then C meets ∂D and for all $x_3 \in C \bigcap \partial D$, we have $\mu_M(x_1, x_3) \le Cap_M(C) \le \mu_M(x_1, x_2) + \varepsilon$.

Since the continuous function $\mu_M(x_1, x_3)$ does not vanish on ∂D, we obtain the lemma by letting ε tend to zero. This completes the proof of the Lemma.

Theorem7.4. For any *n*-dimensional Finsler space (M, g) of class *II*, the topology defined by distance μ_M coincides with the intrinsic manifold topology.

Proof: Every neighborhood of a point x_1 of M contains an open connected and relatively compact neighborhood D of x_1, and according to Lemma 7.3, D contains the μ-ball with center x_1 and radius $\inf_{x_2 \in \partial D} \mu_M(x_1, x_2)$. In the opposite direction it follows from the continuity of μ_M that every μ-ball $B = \{x_3 \in M \mid \mu_M(x_1, x_3) < r\}$ $(r > 0)$ is a neighborhood of x_1. This completes the proof of the theorem.

References

[1] Asanjarani, A. & Bidabad, B. (2008). A classification of complete Finsler manifolds through a second order differential equation. *Diff. Geom. and its Appl. 26*, 434-444.

[2] Bidabad, B. & Hedayatian, S. (2006). Invariant conformal geometry on Finsler manifolds. *The Fifth Conf. of Balkan Society of Geometers Romania-Mangalia, Balkan Press*, 34-43.

[3] Bidabad, B. & Hedayatian, S. (2008). Capacity on Finsler spaces., *IJST, Trans. A, 32*(A1), 17-24.

[4] Mostow, G. D. (1998). Quasiconformal mappings in *n*-space and the rigidity of hyperbolic space forms. *Inst. Hautes Etudes Sci. Publ. Math, 34*, 53-104.

[5] Vuorinen, M. (1988). Conformal geometry and quasiregular mappings. *Lecture Notes in Mathematics*, 1319. *Springer-Verlag, Berlin*.

[6] Ferrand, J. (1973). Invariants conformes globaux sur les vari´et´es Riemanniennes. *J. Differential Geometry*, 8, 487-510.
[7] Ferrand, J. (1996). Conformal capacities and conformally invariant functions on Riemannian manifolds. *Geometriae Dedicata*, 61, 103-120.
[8] Ferrand, J. (1996). The action of conformal transformations on a Riemannian manifold. *Mathematische Annalen* 304, 277-291.
[9] Akbar-Zadeh, H. (2006). *Initiation to Global Finslerian Geometry*. North-Holland, Mathematical Library.
[10] Cartan, E. (1934). Les escpaces de Finsler, Hermann et C^{ie} , *´Editeurs, Paris*.
[11] Bao, D., Chern, S. S. & Shen, Z. (2000). An Introduction to Riemann-Finsler Geometry. Springer-Verlag.
[12] Lichnerowicz, A. (1949). Quelques the´ore`mes de geome`trie diffe´rentie`lle globale, *Comm.Math. Helv*., *22*, 271-301.
[13] Nakahara, M. (1990). Geometry Topology and Physics. Graduate student series in Physics, *University of Sussex*.
[14] Najafi, B., Tayebi, A. & Rezaii, M. M. (2005). On general relatively isotropic L-curvature Finsler metric. *IJST, Trans. A, 29*(A3),