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On almost statistical convergence of new type of generalized difference sequence of fuzzy numbers

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Abstract

In this paper, we introduce a new type of almost statistical convergence of generalized difference sequences of fuzzy numbers. We give the relations between the strongly almost Cesàro type convergence and almost statistical convergence in these spaces. Furthermore, we study some of their properties like completeness, solidity, symmetricity etc. We also give some inclusion relations related to these classes.

Keywords: Almost statistical convergence; difference sequence; fuzzy numbers; solidness; symmetricity; convergence free

1. Introduction

The notion of statistical convergence was introduced by Fast [1] and Schoenberg [2], independently. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Later, it was further investigated from the sequence space point of view and linked with summability theory by Connor [3], Fridy [4], Mursaleen et al. ([5], [6]), Šalát [7], Tripathy [8] and many others. In recent years, generalizations of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on locally compact spaces. Statistical convergence and its generalizations are also connected with subsets of the Stone-Cech compactification of the natural numbers. Moreover, statistical convergence is closely related to the concept of convergence in probability.

The existing literature on almost statistical convergence and strongly almost convergence appear to have been restricted to real or complex sequences, but Altınok et al. [9] extended the idea to apply to sequences of fuzzy numbers and also Altın et al. ([10], [11]), Et et al. ([12], [13]), Başarır and Mursaleen [14], Çolak et al. [15] Gökhan et al. [16], Nuray [17], Savaş [18], Tripathy et al. ([19], [20], [21]), Talo and Başar [22] studied the sequences of fuzzy numbers.

In the present paper, we introduce and examine the concepts of almost statistical convergence and strongly almost convergence of generalized difference sequences of fuzzy numbers. In section 2 we give a brief overview about statistical convergence, fuzzy numbers and using the generalized difference operator Δ_m^r and the sequence $\lambda=(\lambda_n)$. We define the concepts of almost Δ_m^r-statistical convergence and strongly almost Δ_m^r-convergence of sequences of fuzzy numbers. In section 3 we establish some inclusion relations between $w^{F}\left(\Delta_m^r,\lambda,p\right)$ and $S^{F}\left(\Delta_m^r,\lambda\right)$, between $S^{F}\left(\Delta_m^r,\lambda\right)$ and $S^{F}\left(\Delta_m^r\right)$.

2. Definitions and preliminaries

The definitions of statistical convergence and strongly $p-$Cesàro convergence of a sequence of real numbers were introduced in the literature independent of one another and have followed different lines of development since their first appearance. It turns out, however, that the two

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definitions can be simply related to one another in general and are equivalent for bounded sequences. The idea of statistical convergence depends on the density of subsets of the set $\mathbb{N}$ of natural numbers. The density of a subset E of $\mathbb{N}$ is defined by

$\delta(E) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}\chi_E(k)$ provided the limit exists,

where χ_E is the characteristic function of E. It is clear that any finite subset of $\mathbb{N}$ has zero natural density and $\delta(E^c) = 1-\delta(E)$.

A sequence (x_k) of complex numbers is said to be statistically convergent to ℓ if for every $\varepsilon > 0$, $\delta(\{k \in \mathbb{N} : |x_k - \ell| \geq \varepsilon\}) = 0$. In this case we write $stat - \lim x_k = \ell$ or $S - \lim x_k = \ell$.

Fuzzy sets are considered with respect to a nonempty base set X of elements of interest. The essential idea is that each element $x \in X$ is assigned a membership grade $u(x)$ a value in $[0,1]$, with $u(x) = 0$ corresponding to nonmembership, $0 < u(x) < 1$ to partial membership, and $u(x) = 1$ to full membership. According to Zadeh [23] a fuzzy subset of X is a nonempty subset $\{(x, u(x)) : x \in X\}$ of $X \times [0,1]$ for some function $u : X \to [0,1]$. The function u itself is often used for the fuzzy set.

Let $C(\mathbb{R}^n)$ denote the family of all nonempty, compact, convex subsets of $\mathbb{R}^n$. The space $C(\mathbb{R}^n)$ has linear structure induced by the operations $A + B = \{a + b : a \in A, b \in B\}$ and $\lambda A = \{\lambda a : a \in A\}$ for $A, B \in C(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}$. If $\alpha, \beta \in \mathbb{R}$ and $A, B \in C(\mathbb{R}^n)$, so

$$\alpha(A+B) = \alpha A + \alpha B, \quad (\alpha\beta)A = \alpha(\beta A), \quad 1A = A$$

and if $\alpha, \beta \geq 0$, then $(\alpha + \beta)A = \alpha A + \beta A$. The distance between A and B is defined by the Hausdorff metric

$$\delta_\infty(A,B) = \max\{\sup_{a\in A}\inf_{b\in B}\|a-b\|,\ \sup_{b\in B}\inf_{a\in A}\|a-b\|\},$$

where $\|.\|$ denotes the usual Euclidean norm in $\mathbb{R}^n$. It is well known that $(C(\mathbb{R}^n), \delta_\infty)$ is a complete metric space.

Denote

$L(\mathbb{R}^n) = \{u : \mathbb{R}^n \to [0,1]$: u satisfies $(i)-(iv)$ below$\}$,

where

$i)$ u is normal, that is, there exists an $x_0 \in \mathbb{R}^n$ such that $u(x_0) = 1$;

$ii)$ u is fuzzy convex, that is, for $x, y \in \mathbb{R}^n$ and $0 \leq \lambda \leq 1, u(\lambda x + (1-\lambda)y) \geq \min[u(x), u(y)]$;

$iii)$ u is upper semicontinuous;

$iv)$ the closure of $\{x \in \mathbb{R}^n : u(x) > 0\}$, denoted by $[u]^0$, is compact.

If $u \in L(\mathbb{R}^n)$, then u is called a fuzzy number, and $L(\mathbb{R}^n)$ is said to be a fuzzy number space.

For $0 < \alpha \leq 1$, the α-level set $[u]^\alpha$ of $u \in L(\mathbb{R}^n)$ is defined by

$$[u]^\alpha = \{x \in \mathbb{R}^n : u(x) \geq \alpha\}.$$

Then from $(i)-(iv)$, it follows that the α-level sets $[u]^\alpha$ are in the space $C(\mathbb{R}^n)$.

For the addition and scalar multiplication in $L(\mathbb{R}^n)$, we have

$$[u+v]^\alpha = [u]^\alpha + [v]^\alpha, \quad [ku]^\alpha = k[u]^\alpha$$

where $u, v \in L(\mathbb{R}^n)$, $k \in \mathbb{R}$.

The aritmetic operations for α-level sets are defined as follows:

Let $u, v \in L(\mathbb{R}^n)$ with the $\alpha -$level sets be $[u]^\alpha = [a_1^\alpha, b_1^\alpha]$, $[v]^\alpha = [a_2^\alpha, b_2^\alpha]$, $\alpha \in [0,1]$. Then we have

$$[u+v]^\alpha = [a_1^\alpha + a_2^\alpha, b_1^\alpha + b_2^\alpha]$$

$$[u-v]^\alpha = [a_1^\alpha - b_2^\alpha, b_1^\alpha - a_2^\alpha]$$

$$[u.v]^\alpha = \left[\min_{i,j\in[1,2]} a_i^\alpha b_j^\alpha, \max_{i,j\in[1,2]} a_i^\alpha b_j^\alpha\right].$$

Define, for each $1 \leq q < \infty$,

$$d_q(u,v) = \left(\int_0^1 \left[\delta_\infty([u]^\alpha, [v]^\alpha)\right]^q d\alpha\right)^{1/q}$$

and $d_\infty(u,v)=\sup_{0\le\alpha\le1}\delta_\infty([u]^\alpha,[v]^\alpha)$, where δ_∞ is the Hausdorff metric. Clearly $d_\infty(u,v)=\lim_{q\to\infty}d_q(u,v)$ with $d_q\le d_s$ if $q\le s$ ([24], [25]).

For simplicity in notation, throughout the paper d will denote the notation d_q with $1\le q\le\infty$.

The generalized de la Vallée-Poussion mean is defined by

$$t_n(x)=\frac{1}{\lambda_n}\sum_{k\in I_n}x_k,$$

where $\lambda=(\lambda_n)$ is a non-decreasing sequence of positive numbers such that $\lambda_{n+1}\le\lambda_n+1$, $\lambda_1=1$, $\lambda_n\to\infty$ as $n\to\infty$ and $I_n=[n-\lambda_n+1,n]$.

A sequence $x=(x_k)$ is said to be $(V,\lambda)-$summable to a number ℓ, if $t_n(x)\to\ell$ as $n\to\infty$. $(V,\lambda)-$summability reduces to $(C,1)$ summability when $\lambda_n=n$ for all $n\in\mathbb{N}$.

A sequence $X=(X_k)$ of fuzzy numbers is a function X from the set $\mathbb{N}$ of all natural numbers into $L(\mathbb{R}^n)$. Thus, a sequence (X_k) of fuzzy numbers is a correspondence from the set of natural numbers to a set of fuzzy numbers, i.e., to each natural number k there corresponds a fuzzy number $X(k)$. It is more common to write X_k rather than $X(k)$ and to denote the sequence by (X_k) rather than X. The fuzzy number X_k is called the k th term of the sequence.

Let $X=(X_k)$ be a sequence of fuzzy numbers. The sequence $X=(X_k)$ of fuzzy numbers is said to be bounded if the set $\{X_k:k\in\mathbb{N}\}$ of fuzzy numbers is bounded and convergent to the fuzzy number X_0, written as $\lim_k X_k=X_0$, if for every $\varepsilon>0$ there exists a positive integer k_0 such that $d(X_k,X_0)<\varepsilon$ for $k>k_0$. Let ℓ_∞^{F} and c^{F} denote the set of all bounded sequences and all convergent sequences of fuzzy numbers, respectively [26].

The famous space $\hat{c}$ of all almost convergent sequences was introduced by Lorentz [27] and a sequence $x=(x_k)$ is said to be strongly almost convergent to a number ℓ (see Maddox [28]) if

$$\lim_n\frac{1}{n}\sum_{k=1}^{n}|x_{k+i}-\ell|=0,\quad \text{uniformly in } i.$$

The difference sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$, consisting of all real valued sequences $x=(x_k)$ such that $\Delta x=(x_k-x_{k+1})$ in the sequence spaces ℓ_∞, c and c_0, were defined by Kızmaz [29]. The idea of difference sequences is generalized by Et and Çolak [30], Başar and Altay [31], Mursaleen [32], Tripathy et al. ([19], [33]) and many others.

Let w^{F} be the set of all sequences of fuzzy numbers. The operator $\Delta_m^r:w^{\mathrm{F}}\to w^{\mathrm{F}}$ is defined by $(\Delta^0X)_k=X_k$, $(\Delta_m^1X)_k=(\Delta_mX)_k=X_k-X_{k+m}$, and $(\Delta_m^rX)_k=\sum_{\upsilon=0}^{r}(-1)^\upsilon\binom{r}{\upsilon}x_{k+m\upsilon}$ for all $k\in\mathbb{N}$. Throughout the paper m,r will denote any positive integers and for convenience we will write $\Delta_m^rX_k$ instead of $(\Delta_m^rX)_k$.

Definition 2.1. Let $X=(X_k)$ be a sequence of fuzzy numbers. Then the sequence $X=(X_k)$ is said to be Δ_m^r-bounded if the set $\{\Delta_m^rX_k:k\in\mathbb{N}\}$ of fuzzy numbers is bounded, and Δ_m^r-convergent to the fuzzy number X_0, written as $\lim_k\Delta_m^rX_k=X_0$, if for every $\varepsilon>0$ there exists a positive integer k_0 such that $d(\Delta_m^rX_k,X_0)<\varepsilon$ for all $k>k_0$. By $\ell_\infty^{\mathrm{F}}(\Delta_m^r)$ and $c^{\mathrm{F}}(\Delta_m^r)$ we denote the sets of all Δ_m^r-bounded sequences and all Δ_m^r-convergent sequences of fuzzy numbers, respectively.

Definition 2.2. Let $\lambda=(\lambda_n)$ be a non-decreasing sequence of positive numbers such that $\lambda_{n+1}\le\lambda_n+1$, $\lambda_1=1$, $\lambda_n\to\infty$ as $n\to\infty$ and $X=(X_k)$ be a sequence of fuzzy numbers. Then the sequence $X=(X_k)$ of fuzzy numbers is said to be almost Δ_m^r-statistically convergent to the fuzzy number X_0, if for every $\varepsilon>0$,

$$\lim_{n\to\infty}\lambda_n^{-1}\left|\left\{k\in I_n:d\left(\Delta_m^rX_{k+i},X_0\right)\ge\varepsilon\right\}\right|=0,\ \text{uniformly in } i\in\mathbb{N}.$$

The set of all almost Δ_m^r-statistically convergent sequences of fuzzy numbers is denoted by $S^{\mathrm{F}}(\Delta_m^r,\lambda)$. In this case we write $X_k\to X_0(S^{\mathrm{F}}(\Delta_m^r,\lambda))$. In the special case

$\lambda_n = n$ for all $n \in \mathbb{N}$, we shall write $S^{\mathsf{F}}(\Delta_m^r)$ instead of $S^{\mathsf{F}}(\Delta_m^r, \lambda)$.

Definition 2.3. Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers such that $\lambda_{n+1} \le \lambda_n + 1$, $\lambda_1 = 1$, $\lambda_n \to \infty$ as $n \to \infty$, $X = (X_k)$ be a sequence of fuzzy numbers and $p = (p_k)$ be any sequence of strictly positive real numbers. We define the following sets

$$w^{\mathsf{F}}(\Delta_m^r, \lambda, p) = \left\{ \begin{array}{r} X = (X_k) : \lim\limits_{n \to \infty} \frac{1}{\lambda_n} \sum\limits_{k \in I_n} \left[d(\Delta_m^r X_{k+i}, X_0) \right]^{p_k} = 0 \\ \text{uniformly in } i \in \mathbb{N} \end{array} \right\},$$

$$w_0^{\mathsf{F}}(\Delta_m^r, \lambda, p) = \left\{ \begin{array}{r} X = (X_k) : \lim\limits_{n \to \infty} \frac{1}{\lambda_n} \sum\limits_{k \in I_n} \left[d(\Delta_m^r X_{k+i}, \overline{0}) \right]^{p_k} = 0 \\ \text{uniformly in } i \in \mathbb{N} \end{array} \right\},$$

$$w_\infty^{\mathsf{F}}(\Delta_m^r, \lambda, p) = \left\{ \begin{array}{r} X = (X_k) : \sup\limits_{n,i} \frac{1}{\lambda_n} \sum\limits_{k \in I_n} \left[d(\Delta_m^r X_{k+i}, \overline{0}) \right]^{p_k} < \infty \\ \text{uniformly in } i \in \mathbb{N} \end{array} \right\},$$

where

$$\overline{0}(t) = \begin{cases} 1, & t = (0,0,0,...,0) \\ 0, & \text{otherwise.} \end{cases}$$

If $X \in w^{\mathsf{F}}(\Delta_m^r, \lambda, p)$, we say that X is strongly almost $\Delta_m^r -$Cesaro convergent to the fuzzy number X_0 and is written as $X_k \to X_0 (w^{\mathsf{F}}(\Delta_m^r, \lambda, p))$.

We get the following sequence spaces from the above sequence spaces giving particular values to m, λ and p.

i) $w^{\mathsf{F}}(\Delta_m^r, \lambda, p) = w^{\mathsf{F}}(\Delta_m^r, p)$, $w_0^{\mathsf{F}}(\Delta_m^r, \lambda, p) = w_0^{\mathsf{F}}(\Delta_m^r, p)$ and $w_\infty^{\mathsf{F}}(\Delta_m^r, \lambda, p) = w_\infty^{\mathsf{F}}(\Delta_m^r, p)$ when $\lambda_n = n$ for all $n \in \mathbb{N}$,

ii) If $p_k = 1$ for all $k \in \mathbb{N}$ then $w^{\mathsf{F}}(\Delta_m^r, \lambda, p) = w^{\mathsf{F}}(\Delta_m^r, \lambda)$, $w_0^{\mathsf{F}}(\Delta_m^r, \lambda, p) = w_0^{\mathsf{F}}(\Delta_m^r, \lambda)$, and $w_\infty^{\mathsf{F}}(\Delta_m^r, \lambda, p) = w_\infty^{\mathsf{F}}(\Delta_m^r, \lambda)$,

iii) If $m = 1$ then $w^{\mathsf{F}}(\Delta_m^r, \lambda, p) = w^{\mathsf{F}}(\Delta^r, \lambda, p)$, $w_0^{\mathsf{F}}(\Delta_m^r, \lambda, p) = w_0^{\mathsf{F}}(\Delta^r, \lambda, p)$ and $w_\infty^{\mathsf{F}}(\Delta_m^r, \lambda, p) = w_\infty^{\mathsf{F}}(\Delta^r, \lambda, p)$.

A sequence space E^{F} is said to be normal (or solid) if $(X_k) \in E^{\mathsf{F}}$ and (Y_k) is such that $d(Y_k, \overline{0}) \le d(X_k, \overline{0})$ implies $(Y_k) \in E^{\mathsf{F}}$.

A sequence space E^{F} is said to be monotone if E^{F} contains the canonical pre-image of all its step spaces. Let $K = \{k_n : k_n < k_{n+1}, n \in \mathbb{N}\} \subseteq \mathbb{N}$ and E^{F} be a sequence space. A $K-$step space of E^{F} is a sequence space $\mu_K^{E^{\mathsf{F}}} = \{(X_{k_n}) \in w^{\mathsf{F}} : (X_n) \in E^{\mathsf{F}}\}$.

A canonical pre-image of a sequence $(X_{k_n}) \in \mu_K^{E^{\mathsf{F}}}$ is a sequence $(Y_n) \in w^{\mathsf{F}}$ defined as

$$Y_n = \begin{cases} X_n, & \text{if } n \in K, \\ \overline{0}, & \text{otherwise} \end{cases}$$

A canonical pre-image of a step space $\mu_K^{E^{\mathsf{F}}}$ is a set of canonical pre-images of all elements in $\mu_K^{E^{\mathsf{F}}}$, i.e., Y is in canonical pre-image $\mu_K^{E^{\mathsf{F}}}$ if and only if Y is canonical pre-image of some $X \in \mu_K^{E^{\mathsf{F}}}$.

Remark: If a sequence space E^{F} is solid, then E^{F} is monotone.

A sequence space E^{F} is said to be symmetric if $(X_{\pi(n)}) \in E^{\mathsf{F}}$, whenever $(X_k) \in E^{\mathsf{F}}$ where π is a permutation of $\mathbb{N}$.

A sequence space E^{F} is said to be convergence free if $(Y_k) \in E^F$ whenever $(X_k) \in E^{\mathsf{F}}$ and $X_k = \overline{0}$ implies $Y_k = \overline{0}$.

The sequence spaces $w_0^{\mathsf{F}}(\Delta_m^r, \lambda, p)$, $w^{\mathsf{F}}(\Delta_m^r, \lambda, p)$, $w_\infty^{\mathsf{F}}(\Delta_m^r, \lambda, p)$ and $S^{\mathsf{F}}(\Delta_m^r, \lambda)$ contain some unbounded sequences of fuzzy numbers which are divergent too. To show that let $m = 1$, $p_n = 1$ and $\lambda_n = n$ for all $n \in \mathbb{N}$. Then the sequence $X = (X_k) = (\overline{k}^r)$ belongs to $w^{\mathsf{F}}(\Delta_m^r, \lambda, p)$, but the sequence X is divergent and is not bounded.

For the classical sets, (x_k) converges to ℓ which implies $(\Delta_m^r x_k)$ converges to 0, but this case does not hold for the sequences of fuzzy numbers.

Example 1. Let $p_n = 1$, $\lambda_n = n$ for all $n \in \mathbb{N}$ and $r = m = 1$. Consider the sequence (X_k) as follows:

$$X_k(t)=\begin{cases}\frac{k}{k+1}t+\frac{1-k}{1+k}, & \text{if } t\in\left[\frac{k-1}{k},2\right]\\ -\frac{k}{k+1}t+\frac{3k+1}{1+k}, & \text{if } t\in\left[2,\frac{3k+1}{k}\right]\\ 0, & \text{otherwise,}\end{cases}$$

then the sequence $X=(X_k)$ is convergent to the fuzzy number L, where

$$L(t)=\begin{cases}t-1, & \text{if } t\in[1,2]\\ -t+3, & \text{if } t\in[2,3].\\ 0, & \text{otherwise}\end{cases}$$

We find the $\alpha-$level sets of X_k and ΔX_k as follows respectively:

$$[X_k]^\alpha=\left[\frac{k-1}{k}+\frac{k+1}{k}\alpha,\frac{3k+1}{k}-\frac{k+1}{k}\alpha\right]$$

and

$$[\Delta X_k]^\alpha=\left[\frac{-2k^2-4k-1}{k^2+k}+\left(\frac{k+1}{k}+\frac{k+2}{k+1}\right)\alpha,\frac{2k^2+4k+1}{k^2+k}-\left(\frac{k+1}{k}+\frac{k+2}{k+1}\right)\alpha\right].$$

Then we have $\Delta X_k\to L_1$, where $[L_1]^\alpha=[-2+2\alpha,2-2\alpha]\neq\bar{0}$.

3. Main results

In this section we give some inclusion relations between $w^{F}(\Delta_m^r,\lambda,p)$ and $S^{F}(\Delta_m^r,\lambda)$, between $S^{F}(\Delta_m^r)$ and $S^{F}(\Delta_m^r,\lambda)$.

Theorem 3.1. Let the sequence (p_k) be bounded. Then $w_0^{F}(\Delta_m^r,\lambda,p)\subset w^{F}(\Delta_m^r,\lambda,p)\subset w_\infty^{F}(\Delta_m^r,\lambda,p)$ and the inclusions are strict.

Proof: The inclusion $w_0^{F}(\Delta_m^r,\lambda,p)\subset w^{F}(\Delta_m^r,\lambda,p)$ is obvious. So, we will only show that $w^{F}(\Delta_m^r,\lambda,p)\subset w_\infty^{F}(\Delta_m^r,\lambda,p)$. Let $X\in w^{F}(\Delta_m^r,\lambda,p)$. Then we have

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\left[d\left(\Delta_m^r X_{k+i},\bar{0}\right)\right]^{p_k}\le\frac{D}{\lambda_n}\sum_{k\in I_n}\left[d\left(\Delta_m^r X_{k+i},X_0\right)\right]^{p_k}+\frac{D}{\lambda_n}\sum_{k\in I_n}\left[d\left(X_0,\bar{0}\right)\right]^{p_k}$$

$$\le\frac{D}{\lambda_n}\sum_{k\in I_n}\left[d\left(\Delta_m^r X_{k+i},X_0\right)\right]^{p_k}+D\max\left\{1,\sup\left[d\left(X_0,\bar{0}\right)\right]^{H}\right\},$$

where $\sup_k p_k=H$, and $D=\max(1,2^{H-1})$. Thus we get $X\in w_\infty^{F}(\Delta_m^r,\lambda,p)$.

To show that the inclusion is strict, consider the following example:

Let $p_n=1$, $\lambda_n=n$ for all $n\in\mathbb{N}$ and $r=m=1$. Consider the sequence (X_k) of fuzzy numbers as follows:

$$X_k(t)=\begin{cases}\begin{Bmatrix}\frac{2}{k}t+1, & -\frac{k}{2}\le t\le 0\\ -\frac{2}{k}t+1, & 0\le t\le\frac{k}{2}\\ 0, & \text{otherwise}\end{Bmatrix}, & \text{if } k=10^j,(j=1,2,3,...)\\ \bar{0}, & \text{otherwise.}\end{cases}$$

Then, for $\alpha\in(0,1]$, we have $\alpha-$level sets of X_k and ΔX_k as follows:

$$[X_k]^\alpha=\begin{cases}\left[\frac{k}{2}(\alpha-1),\frac{k}{2}(1-\alpha)\right], & \text{if } k=10^j,(j=1,2,3,...)\\ [0,0], & \text{otherwise}\end{cases}$$

and

$$[\Delta X_k]^\alpha=\begin{cases}\left[\frac{k}{2}(\alpha-1),\frac{k}{2}(1-\alpha)\right], & \text{if } k=10^j,(j=1,2,3,...)\\ \left[\frac{(k+1)}{2}(\alpha-1),\frac{(k+1)}{2}(1-\alpha)\right], & \text{if } k+1=10^j,(j=1,2,3,...)\\ [0,0], & \text{otherwise.}\end{cases}$$

Now it is easy to see that $-\bar{2}<[\sigma_n]^\alpha<\bar{2}$ for $\alpha\in(0,1]$ and all $n\in\mathbb{N}$, where $[\sigma_n]^\alpha=\frac{1}{n}\sum_{k=1}^{n}[\Delta X_k]^\alpha$. Thus, the sequence (σ_n) of fuzzy numbers is bounded but is not convergent.

We give the following three theorems without proof.

Theorem 3.2. The spaces $w^{F}(\Delta_m^r,\lambda,p)$, $w_0^{F}(\Delta_m^r,\lambda,p)$, $w_\infty^{F}(\Delta_m^r,\lambda,p)$ and $S^{F}(\Delta_m^r,\lambda)$ are closed under the operations of addition and scalar multiplication.

Theorem 3.3. Let $0<p_k\le q_k$ and $\left(\frac{q_k}{p_k}\right)$ be bounded. Then $w^{F}(\Delta_m^r,\lambda,q)\subseteq w^{F}(\Delta_m^r,\lambda,p)$.

Theorem 3.4. If $\liminf_n(\lambda_n/n)>0$, then $S^{F}(\Delta_m^r)\subset S^{F}(\Delta_m^r,\lambda)$.

Theorem 3.5. The spaces $w^{F}(\Delta_m^r,\lambda,p)$, $w_0^{F}(\Delta_m^r,\lambda,p)$ and $w_\infty^{F}(\Delta_m^r,\lambda,p)$ are complete metric space with the metric

$$\delta_\Delta(X,Y)=\sum_{k=1}^{r} d(X_k,Y_k)+\sup_{n,i}\left(\lambda_n^{-1}\sum_{k\in I_n}\left[d\left(\Delta_m^r X_{k+i},\Delta_m^r Y_{k+i}\right)\right]^{p_k}\right)^{\frac{1}{K}}$$

where $K=\max(1,\sup_k p_k)$.

Proof: We shall prove only for $w_\infty^{F}(\Delta_m^r,\lambda,p)$. The others can be treated similarly. Let (X^s) be a Cauchy sequence in $w_\infty^{F}(\Delta_m^r,\lambda,p)$, where $X^s=(X_j^s)_j=(X_1^s,X_2^s,\ldots)\in w_\infty^{F}(\Delta_m^r,\lambda,p)$ for each $s\in\mathbb{N}$. Then

$$\delta_\Delta(X^s,X^t)=\sum_{k=1}^{r} d(X_k^s,X_k^t)+\sup_{n,i}\left(\lambda_n^{-1}\sum_{k\in I_n}\left[d\left(\Delta_m^r X_{k+i}^s,\Delta_m^r X_{k+i}^t\right)\right]^{p_k}\right)^{\frac{1}{K}}\to 0,\text{ as } s,t\to\infty.$$

Therefore $\sum_{k=1}^{r} d(X_k^s,X_k^t)\to 0$ and $\lambda_n^{-1}\sum_{k\in I_n}\left[d\left(\Delta_m^r X_{k+i}^s,\Delta_m^r X_{k+i}^t\right)\right]^{p_k}\to 0$ as $s,t\to\infty$, for each fixed $i\in\mathbb{N}$. Now from

$$d(X_{j+r}^s,X_{j+r}^t)\le d(\Delta^r X_j^s,\Delta^r X_j^t)+\binom{r}{0}d(X_j^s,X_j^t)+\ldots+\binom{r}{r-1}d(X_{j+r-1}^s,X_{j+r-1}^t)$$

we have $d(X_j^s,X_j^t)\to 0$, as $s,t\to\infty$, for each $j\in\mathbb{N}$. Therefore $(X_j^s)_s=(X_j^1,X_j^2,\ldots)$ is a Cauchy sequence in $L(\mathbb{R}^n)$. Since $L(\mathbb{R}^n)$ is complete, it is convergent

$$\lim_s X_j^s = X_j$$

say, for each $j\in\mathbb{N}$. Since (X^s) is a Cauchy sequence, for each $\varepsilon>0$, there exists $n_0=n_0(\varepsilon)$ such that

$$\delta_\Delta(X^s,X^t)<\varepsilon \text{ for all } s,t\ge n_0.$$

Hence for each $i\in\mathbb{N}$ we get

$$\sum_{k=1}^{r} d(X_k^s,X_k^t)<\varepsilon \text{ and } \lambda_n^{-1}\sum_{k\in I_n}\left[d\left(\Delta_m^r X_{k+i}^s,\Delta_m^r X_{k+i}^t\right)\right]^{p_k}<\varepsilon^K,\qquad s,t\ge n_0.$$

So we have

$$\lim_t\sum_{k=1}^{r} d(X_k^s,X_k^t)=\sum_{k=1}^{r} d(X_k^s,X_k)<\varepsilon$$

and

$$\lim_t \lambda_n^{-1}\sum_{k\in I_n}\left[d\left(\Delta_m^r X_{k+i}^s,\Delta_m^r X_{k+i}^t\right)\right]^{p_k}=\lambda_n^{-1}\sum_{k\in I_n}\left[d\left(\Delta_m^r X_{k+i}^s,\Delta_m^r X_{k+i}\right)\right]^{p_k}<\varepsilon^K$$

for all $n,i\in\mathbb{N}$ and $s\ge n_0$. This implies that $\delta_\Delta(X^s,X)<2\varepsilon$, for all $s\ge n_0$, that is $X^s\to X$ as $s\to\infty$, where $X=(X_j)$. Since

$$\lambda_n^{-1}\sum_{k\in I_n}\left[d\left(\Delta_m^r X_{k+i},\bar{0}\right)\right]^{p_k}\le\lambda_n^{-1}\left\{\sum_{k\in I_n}\left[d\left(\Delta_m^r X_{k+i}^{n_0},\bar{0}\right)+d\left(\Delta_m^r X_{k+i}^{n_0},\Delta_m^r X_{k+i}\right)\right]^{p_k}\right\}$$

$$\le\frac{D}{\lambda_n}\sum_{k\in I_n}\left[d\left(\Delta_m^r X_{k+i}^{n_0},\bar{0}\right)\right]^{p_k}+\frac{D}{\lambda_n}\sum_{k\in I_n}\left[d\left(\Delta_m^r X_{k+i}^{n_0},\Delta_m^r X_{k+i}\right)\right]^{p_k}$$

we obtain $X\in w_\infty^{F}(\Delta_m^r,\lambda,p)$. Therefore $w_\infty^{F}(\Delta_m^r,\lambda,p)$ is a complete metric space.

Theorem 3.6. If $\liminf_k p_k>0$ and X is strongly almost Δ_m^r-Cesaro convergent to the fuzzy number X_0, then $X_k\to X_0\left(w^{F}(\Delta_m^r,\lambda,p)\right)$ uniquely.

Proof: Suppose that $\liminf p_k=s>0$. Let $X_k\to X_0\left(w^{F}(\Delta_m^r,\lambda,p)\right)$, $X_k\to X_1\left(w^{F}(\Delta_m^r,\lambda,p)\right)$ and $X_0\ne X_1$. Then

$$\lim_{n\to\infty}\lambda_n^{-1}\sum_{k\in I_n}\left[d\left(\Delta_m^r X_{k+i},X_0\right)\right]^{p_k}=0,$$

and

$$\lim_{n\to\infty}\lambda_n^{-1}\sum_{k\in I_n}\left[d\left(\Delta_m^r X_{k+i},X_1\right)\right]^{p_k}=0,\text{ uniformly in i.}$$

Then we have

$$\lambda_n^{-1}\sum_{k\in I_n}\left[d(X_0,X_1)\right]^{p_k}\le\frac{D}{\lambda_n}\sum_{k\in I_n}\left[d\left(\Delta_m^r X_{k+i},X_0\right)\right]^{p_k}$$

$$+\frac{D}{\lambda_n}\sum_{k\in I_n}\left[d\left(\Delta_m^r X_{k+i},X_1\right)\right]^{p_k}\to 0,\text{ uniformly in } i\ (n\to\infty),$$

Hence

$$\lim_n\lambda_n^{-1}\sum_{k\in I_n}\left[d(X_0,X_1)\right]^{p_k}=0$$

and so $X_0=X_1$, a contradiction. Thus the limit is unique.

Theorem 3.7. Let $X=(X_k)$ be a sequence of fuzzy numbers, then we have $w_\infty^F(\Delta_m^r,\lambda)=\ell_\infty^F(\Delta_m^r)$.

Proof: Let $X\in w_\infty^F(\Delta_m^r,\lambda)$. Then there exists a constant $K_1>0$ such that

$$\frac{1}{\lambda_1}d\left(\Delta_m^r X_{1+i},\overline{0}\right)\le \sup\frac{1}{\lambda_n}\sum_{k\in I_n}d\left(\Delta_m^r X_{k+i},\overline{0}\right)\le K_1 \text{ for all } i$$

and so we have $X\in\ell_\infty^F(\Delta_m^r)$.

Conversely, let $X\in\ell_\infty^F(\Delta_m^r)$. Then there exists a constant $K_2>0$ such that $d(\Delta_m^r X_j,\overline{0})\le K_2$ for all j, and so

$$\frac{1}{\lambda_n}\sum_{k\in I_n}d\left(\Delta_m^r X_{k+i},\overline{0}\right)\le\frac{K_2}{\lambda_n}\sum_{k\in I_n}1\le K_2 \text{ for all k and } i.$$

Thus $X\in w_\infty^F(\Delta_m^r,\lambda)$.

Theorem 3.8. Let $X=(X_k)$ be a sequence of fuzzy numbers and $0<h=\inf_k p_k\le p_k\le\sup_k p_k=H$, then $w^F(\Delta_m^r,\lambda,p)\subset S^F(\Delta_m^r,\lambda)$.

Proof: The proof follows from the following inequality

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\left[d\left(\Delta_m^r X_{k+i},X_0\right)\right]^{p_k}\ge\frac{1}{\lambda_n}\sum_{\substack{k\in I_n\\ d(\Delta_m^r X_{k+i},X_0)\ge\varepsilon}}\left[d\left(\Delta_m^r X_{k+i},X_0\right)\right]^{p_k}$$

$$\ge\frac{1}{\lambda_n}\left|\left\{k\in I_n:d\left(\Delta_m^r X_{k+i},X_0\right)\ge\varepsilon\right\}\right|.\min\left(\varepsilon^h,\varepsilon^H\right)$$

Theorem 3.9. Let $X=(X_k)$ be a Δ_m^r-bounded sequence of fuzzy numbers, then $S^F(\Delta_m^r,\lambda)\subset w^F(\Delta_m^r,\lambda,p)$.

Proof: Suppose that $X_k\to X_0\left(S^F(\Delta_m^r,\lambda)\right)$. Since $X\in\ell_\infty^F(\Delta_m^r)$, there is a constant $T>0$ such that $d(\Delta_m^r X_k,X_0)\le T$. Given $\varepsilon>0$ we have

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\left[d\left(\Delta_m^r X_{k+i},X_0\right)\right]^{p_k}\le\frac{1}{\lambda_n}\sum_{\substack{k\in I_n\\ d(\Delta_m^r X_{k+i},X_0)\ge\varepsilon}}\max\left(T^h,T^H\right)+\frac{1}{\lambda_n}\sum_{\substack{k\in I_n\\ d(\Delta_m^r X_{k+i},X_0)<\varepsilon}}\varepsilon^{p_k}$$

$$\le\max\left(T^h,T^H\right)\frac{1}{\lambda_n}\left|\left\{k\in I_n:d\left(\Delta_m^r X_{k+i},X_0\right)\ge\varepsilon\right\}\right|$$
$$+\max\left(\varepsilon^h,\varepsilon^H\right).$$

Hence $X\in w^F(\Delta_m^r,\lambda,p)$.

Theorem 3.10. The sequence spaces $w_0^F(\lambda)$ and $w_\infty^F(\lambda)$ are solid and hence monotone, but the sequence spaces $S^F(\Delta_m^r,\lambda)$, $w^F(\Delta_m^r,\lambda,p)$, $w_0^F(\Delta_m^r,\lambda,p)$ and $w_\infty^F(\Delta_m^r,\lambda,p)$ are not solid.

Proof: Let $(X_k)\in w_\infty^F(\lambda)$ and (Y_k) be such that $d(Y_k,\overline{0})\le d(X_k,\overline{0})$ for each $k\in\mathbb{N}$. Then we get

$$\frac{1}{\lambda_n}\sum_{k\in I_n}d\left(Y_k,\overline{0}\right)\le\frac{1}{\lambda_n}\sum_{k\in I_n}d\left(X_k,\overline{0}\right).$$

Hence $w_\infty^F(\lambda)$ is solid and hence monotone. The space $w_\infty^F(\Delta_m^r,\lambda,p)$ is not solid. This follows from the following example:

Let $p_n=1$, $\lambda_n=n$ for all $n\in\mathbb{N}$ and $m=1$. Let us consider the sequences $X=(X_n)=(\overline{n})=(\overline{1},\overline{2},\overline{3},...)\in w_\infty^F(\Delta_m^r,\lambda,p)$ and

$$\alpha_n=\begin{cases}0, & if\ n \text{ is odd}\\ 1, & if\ n \text{ is even}\end{cases}$$

then $(\alpha_n X_n)=(\overline{0},\overline{2},\overline{0},\overline{4},\overline{0},\overline{6},...)\notin w_\infty^F(\Delta_m^r,\lambda,p)$. Hence $w_\infty^F(\Delta_m^r,\lambda,p)$ is not solid.

Theorem 3.11. Let μ denote any of the sequence spaces $S^F(\Delta_m^r,\lambda)$, $w^F(\Delta_m^r,\lambda,p)$, $w_0^F(\Delta_m^r,\lambda,p)$ and $w_\infty^F(\Delta_m^r,\lambda,p)$. Then, the following statements hold:

$a)$ μ is not symmetric,

$b)$ μ is not convergence free.

Proof: Since the proof can be obtained for the spaces $S^F(\Delta_m^r,\lambda)$, $w^F(\Delta_m^r,\lambda,p)$, $w_0^F(\Delta_m^r,\lambda,p)$ in a similar way, we consider the only the space $w_\infty^F(\Delta_m^r,\lambda,p)$.

$a)$ Let $p_k=1$ for all $k\in\mathbb{N}$ and $m=2$. Consider the sequence $(X_k)=(\overline{1},\overline{2},\overline{3},...)\in$

$w_\infty^F(\Delta_m^r,\lambda,p)$. Define $(Y_k)=(\bar{1},\bar{3},\bar{2},\bar{5},\bar{4},\bar{6},...)$ by a rearrangement of (X_k). Then $(Y_k)\notin w_\infty^F(\Delta_m^r,\lambda)$.

b) Let $p_k=1$ for all $k\in\mathbb{N}$ and $m=3$. Suppose that $\lambda=n$, then $I_n=[1,n]$. Define the sequence $\{X_n(t)\}$ by

$$X_n(t)=\begin{cases}\dfrac{nt+n+1}{n+1}, & -1-\dfrac{1}{n}\le t\le 0,\\[2mm] \dfrac{n+1-nt}{n+1}, & 0\le t\le 1+\dfrac{1}{n},\\[2mm] 0, & \text{otherwise.}\end{cases}$$

Then, we have

$$\Delta_3X_n(t)=\begin{cases}\dfrac{(t+2)n^2+3nt+8n+3}{2n^2+8n+3}, & -2-\dfrac{1}{n}-\dfrac{1}{n+3}\le t\le 0,\\[2mm] \dfrac{(-t+2)n^2-3tn+8n+3}{2n^2+8n+3}, & 0\le t\le 2+\dfrac{1}{n}+\dfrac{1}{n+3},\\[2mm] 0, & \text{otherwise.}\end{cases}$$

Therefore, $\lim_{n\to\infty}\Delta_3X_n(t)=X(t)$, where $X(t)$ is defined by

$$X(t)=\begin{cases}\dfrac{(t+2)}{2}, & -2\le t\le 0,\\[2mm] \dfrac{(2-t)}{2}, & 0\le t\le 2,\\[2mm] 0, & \text{otherwise.}\end{cases}$$

Hence, $\{X_n(t)\}\in w_\infty^F(\Delta_m^r,\lambda)$. Now, consider $\{Y_n(t)\}$ defined by

$$Y_n(t)=\begin{cases}\dfrac{(t+n)}{n}, & -n\le t\le 0,\\[2mm] \dfrac{(n-t)}{n}, & 0\le t\le n,\\[2mm] 0, & \text{otherwise.}\end{cases}$$

At this stage,

$$\Delta_3Y_n(t)=\begin{cases}\dfrac{t+2n+3}{2n+3}, & -2n-3\le t\le 0,\\[2mm] \dfrac{2n+3-t}{2n+3}, & 0\le t\le 2n+3,\\[2mm] 0, & \text{otherwise.}\end{cases}$$

It is clear that $\{Y_n(t)\}\notin w_\infty^F(\Delta_m^r,\lambda,p)$. This shows that the space $w_\infty^F(\Delta_m^r,\lambda,p)$ is not convergence free.

4. Conclusion

The sequences of fuzzy numbers were introduced and studied by Matloka [26] and the first difference sequences of fuzzy numbers were studied by Savaş [18], Talo and Başar [34]. Now in this paper we study the m^{th} difference sequences of fuzzy numbers for some sequence classes. The results obtained in this study are much more general than those obtained by others. To do this some fairly wide classes of sequences of fuzzy numbers using the generalized difference operator Δ_m^r and a nondecreasing sequence $\lambda=(\lambda_n)$ of positive real numbers such that $\lambda_{n+1}\le\lambda_n+1$, $\lambda_1=1$, $\lambda_n\to\infty$ as $n\to\infty$ have been introduced. Furthermore, using these concepts we establish some inclusion relations between $w^F(\Delta_m^r,\lambda,p)$ and $S^F(\Delta_m^r,\lambda)$, between $S^F(\Delta_m^r,\lambda)$ and $S^F(\Delta_m^r)$ and show that the sequence spaces $w^F(\Delta_m^r,\lambda,p)$, $w_0^F(\Delta_m^r,\lambda,p)$ and $w_\infty^F(\Delta_m^r,\lambda,p)$ are complete metric spaces with suitable metric.

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