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Solution of nonlinear PDE $\|\nabla u\|=c$ over a general space curve

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Abstract

A method has been presented for finding the solution surface of the NPDE: $\|\nabla u\| = c$, bounded by a general space curve. The method is based on the geometric characteristics of the surface, and is called the "*Cone-Slot Method*". It has been shown that such a surface can be obtained by movement of a cone inside the slot formed by the boundary space curve. An algorithm has been suggested on the basis of mathematics of these considerations. In previous methods the boundary curve had to be level. They obtain the surface as an assembly of its contour curves. In this method however, the solution surface is obtained as an assembly of its characteristics. The boundary curve can also be a general unlevel skew space curve. The method requires no mesh for calculation and allows the area of the integral surface and underneath volume to be readily determined.

Keywords: Characteristics; NPDE; eikonal; space curve

1. Introduction

Nonlinear PDEs appear in describing many physical phenomena of nature. Therefore, their solution is of interest to scientists. As in the case of linear ones, an attempt has been made to classify the nonlinear partial differential equations (NPDE) with regard to the order of partial derivatives they involve; but the degree of nonlinearity is also important in their case because it significantly affects the ease of finding their solution, which is usually made by linearization.

The equation that is the subject of this paper is a nonlinear first order PDE which appears in many areas of science and technology. These include wave propagation, electromagnetic, and geometric optics. It can be derived from the Maxwell's equations in electromagnetics [1], or equations of elastic waves under high frequency [2]. It is considered to link the physical (wave) optics and nonlinear geometrical (ray) optics [1]. From the classification point of view, it belongs to the Hamilton-Jacobi (H-J) generation.

The H-J equations appear in many areas of science. An important area is the propagation of fronts and interfaces in time. This has many applications in computer graphics, seismology, etc [3-5]. The more general time-dependent H-J equation can be solved by the Level Set Method (LSM) developed by Osher and Seithan [6-11].

It is a rather difficult task to track the motion and changes of the fronts and interfaces with time by studying themselves. In LSM, the front at any time is treated as the zero level set (contour) of a moving integral surface [8, 10, 11]. The LSM removes the difficulty of in-time tracking the fronts and interfaces by embedding them in a higher dimension level set function. The state of the front at any time is then determined by considering the zero level set of this function. In this way, the relatively unpredictable change and movement of the front would readily be under control [8, 10, 11].

The static H-J equations in which the velocity of propagation is only a function of location and not the time also appear in many areas of science. These include electromagnetic [1], wave propagation, seismology [4,5], non-linear optics [1], mesh generation [12], optimal control [13], geodesics and path planning [14], photolithography, and robotic navigation, etc. The wave propagation problem in this case is a single-pass one, meaning the front passes each point only once. If the medium is isotropic, the velocity of wave propagation does not depend on the direction and the NPDE in this case is called "*eikonal*", taken from "*eikon*", the Greek word for image or Figure. Two different single-pass methods have been developed for solution of this problem [15]. They both use the "one-pass" algorithm of Dijkstra for finding the shortest (optimal) path on a network [16]. Tsitsiklis's method for solution of *eikonal*

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equation can be interpreted as the isotropic min-time optimal trajectory approach [13]. This approach always gives a stair-step type of trajectory and does not converge to a continuous one as the size of cells in the computation mesh are refined. A more efficient method for solution of "eikonal equation" is the Fast Marching Method (FMM) developed by Sethian [17, 18]. The method is based on finite-difference upwinding discretization of the "eikonal" equation. In this method, the initial state of the front is taken as the boundary value of the stationary formulated problem and the computations are made at the mesh points in the order and direction in which the value of field variable is increasing. This direction would be that of the gradient which coincides with characteristics for eikonal equation. The characteristics can also be regarded as the optimal trajectories in the control-theoretic context. This property makes the Tsitsiklis and Sethian approaches efficient and fast because they only involve solving decoupled equations rather than coupled ones that require iteration. For the case of propagation in anisotropic medium in which the speed of propagation depends on the direction, the Ordered Upwind Method (OUM) has been developed by Sethian and Vladimirisky [15, 19]. The method utilizes partial information about the characteristics in order to decouple the nonlinear system of equations and keep the Dijkstra-like one-pass methodology.

This paper discusses the solution of eikonal equation under an isotropic constant speed. In contrast to the previously mentioned wave propagation formulations in which the initial curve (front) is a level set (contour) of the solution surface; the initial curve considered here is a general skew space curve. As discussed before, the previous methods attempt to find the solution surface by resorting to its level sets (contour curves). In fact, in some special cases like constant speed eikonal equation under a level curve boundary, the graphical construction of contour curves is readily possible. The graphical construction of contours (level sets) has been considered as a method for solution of some PDE's [20, 21]. Examples of these types are the graphical construction of flow nets in potential problems and drawing contours of stress function in plastic torsion of bars [20-22]. A geometric-based numerical method has been presented for the case in which the perimeter curve is level [23]; but in many cases, the boundary curve is not level and therefore, these solution procedures do not apply. An example is the geometry of the rigid wedge formed beneath an inclinedly loaded area [24].

It is, therefore, the aim of this paper to find the solution surface of eikonal equation when the boundary is a general three-dimensional space curve. This is done by resorting to the characteristics of this nonlinear PDE. The proposed method is based on the geometric properties of the integral surface, which is ruled. The method is especially suitable for cases in which the knowledge of the surface area of the integral surface and its underneath volume are of particular interest.

2. Statement of the problem

Our main purpose in this paper is to develop a methodology for constructing the integral surface of the nonlinear partial differential equation $\|\nabla u\| = c$, over a general space curve. If u is a function of two independent variables x and y, the gradient of u is shown by the vector $\nabla u = (u_x, u_y)$. The Euclidean norm of ∇u would then be $\|\nabla u\| = \sqrt{\nabla u \bullet \nabla u} = \sqrt{u_x^2 + u_y^2} = c$. Therefore, in a more clear mathematical form, we seek the solution of the nonlinear partial differential equation:

$$u_x^2 + u_y^2 = c^2 \tag{1}$$

subject to the condition:

$$u = f(x, y) \text{ on } \Gamma, \tag{2}$$

where c is a constant, f in known function and Γ is a three dimensional space curve bounding the integral surface.

As mentioned before, this differential equation is the stationary form of Hamilton-Jacobi equations usually referred to as "*eikonal*" equation. Previous works on the subject have all been limited to the cases in which the boundary curve is level (contour). It has been for this reason that approaches like the level set method and fast marching method have come to the scene for solution of the problem. These methods are based on wave propagation approach to the problem, which treats the boundary level curve as the initial or boundary value of the problem. In contrast to these methods, our interest here is to find the solution surface for the case in which the boundary curve is not necessarily level. We do consider Dirichlet type boundary condition, but over an unlevel general space curve.

3. Solution by generalized method of characteristics

As for the other PDEs which have characteristics, the generalized method of characteristics can be applied for solution of such a problem. The information at the boundary perimeter curve Γ provides the data for Cauchy problem. If the partial differential equation is expressed in its implicit form as $F(x_i, u, p_i) = 0$, the Charpit-Lagrange equations for characteristics in its generalized form can be written as:

$$\frac{dx_i}{F_{p_i}} = \frac{du}{\Sigma p_i F_{p_i}} = \frac{-dp_i}{F_{x_i} + F_u p_i}, \tag{3}$$

where $p_i = \dfrac{\partial u}{\partial x_i}$.

In our case, Eq. 1 can be written as:

$$F(x_i, u, p_i) = p^2 + q^2 - c^2 = 0 \tag{4}$$

where $p = u_x$ and $q = u_y$, and we have $F_{x_i} = 0$, $F_u = 0$; and $F_p = 2p$ and $F_q = 2q$. With more simplification, Eq. (3) takes the form:

$$\frac{dx}{p} = \frac{dy}{q} = \frac{du}{c^2} = \frac{-dp}{0} = \frac{-dq}{0} = dt \tag{5}$$

where t is the running variable along the characteristic curve.

The first of these equations gives the direction of characteristics as:

$$\frac{dy}{dx} = \frac{q}{p}. \tag{6}$$

The last two equations indicate p and q are constant along the characteristics, so that it is enough to define them at the initial strip. Furthermore; this indicates the characteristics are straight lines so that the characteristic strips are also straight. With this information, Cauchy problem is solved by integration of $du = pdx + qdy$, which is obtained from the first two equations of (5). The gradient vector that is normal to the solution surface is (p, q,-1). The first two components give the same direction in x-y plane as that of characteristics given by Eq. 6. This is the direction of steepest ascent i.e., the direction along which the grade is maximum on the surface. Therefore, for *eikonal equation*, the direction of gradient and characteristics in x-y-plane coincide. The straight characteristic lines in x-y plane may cross each other as they move away from the initial boundary curve. This has been considered as a major concern of this method and has caused an approach to methods that describe the solution surface by its level curves. It is correct that the level curves (contours) do not cross each other; but they cross the boundary curve if it is not a level one, which is our case. It is especially in this case that the front propagation approaches like LSM and FMM are not applicable. Crossing of characteristics can be natural to an integral surface and therefore should be treated logically rather than being avoided. In the next section we propose a geometric approach for obtaining the solution surface that removes this difficulty. Furthermore, sometimes we are interested in knowing the area of the solution surface and the volume underneath; and these quantities are not readily determined when the solution surface is defined by a series of its contours, whereas this is vice versa when the solution surface is defined as the assembly of its characteristics, as we shall see.

4. Geometric characteristics of the integral surface

Let us assume the problem has been solved and a typical solution surface has been constructed over the given space curve Γ. It is obvious that a solution surface can be defined by a series of its level sets (contour curves) as shown in Fig. 1. In fact, for some cases of partial differential equations, drawing the level sets of the integral function is considered as a graphical method of solution. Graphical construction of the flow net in potential problems for which the boundary of the problem is either a level contour or of Neumann type, may be mentioned as an example of this approach [20]. But this is difficult in our case in which the boundary curve is not level because the contours will cross the boundary curve. Instead, here, we try to construct the integral surface by resorting to its characteristics.

As mentioned, characteristics for eikonal equation in this case are straight lines and their directions coincide with that of the gradient. If t is the running variable along this direction, the slope of the solution surface which is nothing but the directional derivative of u along t can be written as:

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}. \tag{7}$$

This can be considered as the scalar product of (u_x, u_y) which is also shown by (p, q) and a unit vector $\left(\dfrac{dx}{dt}, \dfrac{dy}{dt}\right)$ that can also be shown as $(\cos\theta, \sin\theta)$, where θ is the angle between x and t directions. But this should be maximum because the direction of gradient is the direction of maximum ascent. For this, we should have the unit vector along the direction of gradient, i.e.:

$$(\cos\theta, \sin\theta) = \left(\frac{p}{\sqrt{p^2+q^2}}, \frac{q}{\sqrt{p^2+q^2}} \right). \quad (8)$$

Therefore, we get:

$$\frac{\partial u}{\partial t} = (p,q) \bullet \left(\frac{p}{\sqrt{p^2+q^2}}, \frac{q}{\sqrt{p^2+q^2}} \right). \quad (9)$$

This, together with the original equation $p^2+q^2=c^2$, reduces to the simple form:

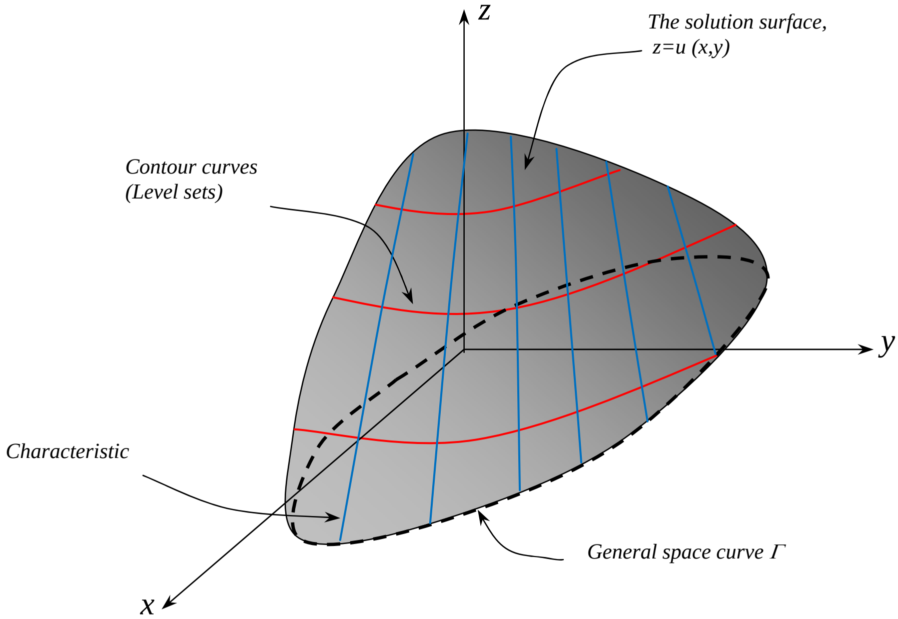

Fig. 1. The solution surface over 3D-boundary space curve

$$\frac{\partial u}{\partial t} = c \quad (10)$$

which indicates the solution surface of eikonal equation in which c is a constant is a surface whose maximum slope with the horizontal x-y plane at any point is constant and equal to c. If the slope angle is ϕ, we have:

$$\tan\phi = c \quad (11)$$

The integral surface in this case is therefore a *developable ruled* one made of the assembly of its straight characteristic lines, all making the same slope angle ϕ with horizon. Such a surface is obtained by movement of this characteristic line in a way that it always lies on the boundary curve on one side.

5. Geometric cone-slot method for construction of the integral surface

For the purpose of constructing this surface, we consider a right circular cone with the base angle ϕ. If we bring the cone into contact with the boundary space curve at a point; the generator of the cone drawn from that point would be a characteristic line of the solution surface. As such cone moves while having contact with the boundary curve, the required surface is traced by the generator of the cone. If the boundary curve is a loop, the surface develops a ridge. This is where the characteristic lines coming from two points N_1 and N_2 of the boundary curve cross, indicating discontinuity in u_x and u_y. In order to get the complete solution surface and its ridge, we let the cone in contact with the boundary curve at N_1 slide along its generator and penetrate further into the slot made by the boundary curve to the extent that it contacts the other side at N_2 (see Fig. 2a). It is obvious that the generators SN_1 and SN2 are two conjugate characteristics of the solution surface, and S is a point of its ridge (Fig. 2b). All we have to do in order to complete the solution surface and its ridge is to move the cone inside the slot made by the boundary curve in such a way that it is always in contact with this curve in at least two points (Fig. 2c). Figure 3 shows the side and top views of some stages of movement of such a cone inside the slot of a typical closed skew space curve and development of the solution surface.

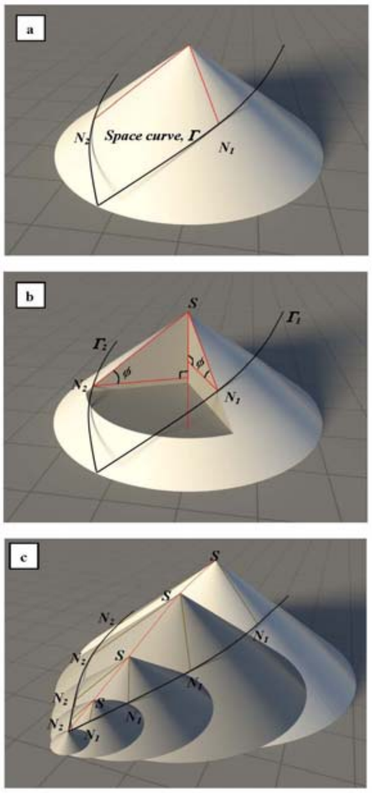

Fig. 2. Geometric properties of the solution surface

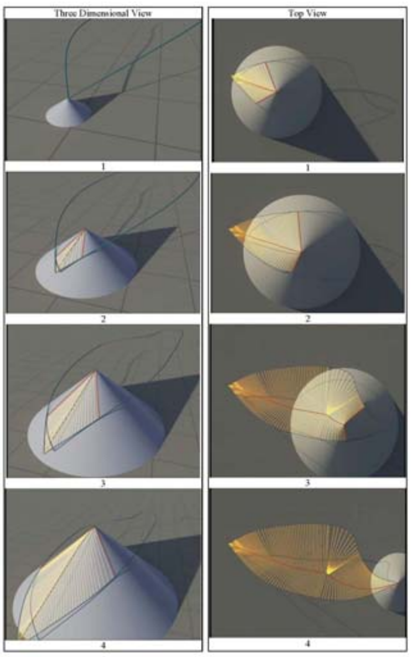

Fig. 3. Development of the solution surface by the "*Cone-Slot Method*"

6. Mathematical basis of the cone-slot method

In this way, the solution surface is obtained as an integral of its characteristics by the so called "*cone-slot method*". This physical interpretation of the construction of the solution surface that was based on its geometric properties can also be stated mathematically. It is possible to find p and q at any point of the boundary curve and, because in this case the characteristic curves of the solution surface are straight lines (generator of the cone), determination of p and q at the boundary curve would be enough for defining them, since these quantities do not change along the characteristics. The perimeter curve Γ in its general space form can be expressed by:

$$\frac{dx}{l(x,y,z)} = \frac{dy}{m(x,y,z)} = \frac{dz}{n(x,y,z)} \tag{12}$$

where l, m, and n are direction parameters of a line segment along the perimeter curve at the point (x,y,z). Therefore, the vector (l, m, n) represents a tangent vector to the perimeter space curve Γ, which itself is part of the solution surface. The denominators of the first three terms in Charpit-Lagrange equations indicate the vector along the characteristics of the solution surface can be expressed by (p, q, c^2). The gradient vector $(p, q, -1)$ is normal to the solution surface. It then must be parallel to $(l, m, n)\times(p, q, c^2)$. This requirement yields to two simultaneous equations in two unknowns p and q as the following:

$$\begin{cases} lp + mq = n \\ p^2 + q^2 = c^2 \end{cases} \tag{13}$$

Solving for q in terms of l, m, n and c yields:

$$q = \frac{mn \pm \sqrt{(mn)^2 - (m^2 + l^2)(n^2 - l^2 c^2)}}{(m^2 + l^2)} \tag{14}$$

Which, when found, can be put in the first equation to give p as:

$$p = \frac{1}{l}(n - mq)\,. \tag{15}$$

These equations give the direction of characteristics for any point on the perimeter space curve. It has to be mentioned that only one of the roots of Eq. 14 applies in a practical problem.

In general, characteristic lines drawn from two different points of the boundary curve are skew with respect to each other (see Fig. 4). In order to construct the solution surface we need to have a means of finding the conjugate point N_2 on the branch Γ_2 of the boundary curve, for an assumed N_1 on the branch Γ_1, so that the characteristic lines drawn from these two points cross each other. Once proper conjugate points N_2s are found for some assumed N_1s, the solution surface can be developed over the boundary curve. The conjugate of an assumed point N_1, i.e., N_2, is found when the distance between straight characteristic lines drawn from these two points is zero. The distance H_1H_2, between two skew characteristic lines can be found if the scalar triple product of $\overrightarrow{N_1N_2}$ and the vectors along the characteristics are divided by the norm of the cross product of vectors along the characteristics. If q and p are evaluated from Eqs. 14 and 15 for points N_1 and N_2, the vectors along the characteristics coming from these two points would be (p_1, q_1, c^2) and (p_2, q_2, c^2) respectively. We can write:

$$H_1H_2 = \frac{\overrightarrow{N_2N_1} \bullet (p_1, q_1, c^2) \times (p_2, q_2, c^2)}{\left|(p_1, q_1, c^2) \times (p_2, q_2, c^2)\right|} \tag{16}$$

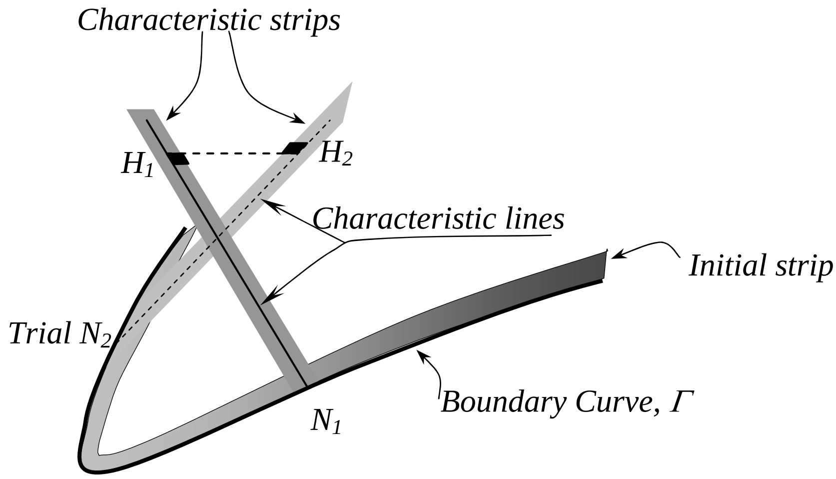

Fig. 4. Characteristic lines drawn from two points of the boundary space curve

For characteristics to cross, it is sufficient to have the numerator of the right hand side of Eq. 16 equal to zero. A zero value for the scalar triple product of $\overrightarrow{N_1N_2}$ and vectors along the characteristics indicate they lie in a common plane. Therefore, in order to locate the correct position for N_2, we can define the numerator as function F as:

$$F = \overrightarrow{N_2N_1} \bullet (p_1, q_1, c^2) \times (p_2, q_2, c^2) \tag{17}$$

and study when it becomes zero. Being equal to the volume of the parallel pipe formed on vectors $\overrightarrow{N_1N_2}$, (p_1, q_1, c^2) and (p_2, q_2, c^2); this function approaches zero and changes in sign as the trial N_2 approaches the correct position and passes by it.

7. Algorithm for numerical solution

In order to find the solution surface of the NPDE for the Cauchy problem defined on the three-dimensional space curve Γ, we first consider a number of N_1 points on Γ. For each of them we try to locate its correct conjugate N_2. For this we try a trial point for N_2 and evaluate the function F from Eq. 17. If this trial point is the correct N_2 i.e., the conjugate of the assumed N_1, the value of F would be zero. The value of F would be either negative or positive when the trial N_2 is not the correct one. As we move our trial point toward the correct location along Γ, the absolute value of F decreases. The value of F approaches zero and changes its sign and increases in absolute value as the trial N_2 passes the correct location. We then dictate the trial point to move back and proceed in the same way until we get a value of F that is less than an acceptable error. In this way an answer is found for the triple (N_1, S, N_2). We repeat the same procedure for the next N_1 and get another triple. Proceeding in the same way will result in the solution surface, which can be constructed by connecting N_1 to S, and S to N_2 for each of the (N_1, S, N_2)-triples found. A simple computer code has been written for this purpose. The flowchart of the program has been shown in Fig. 5.

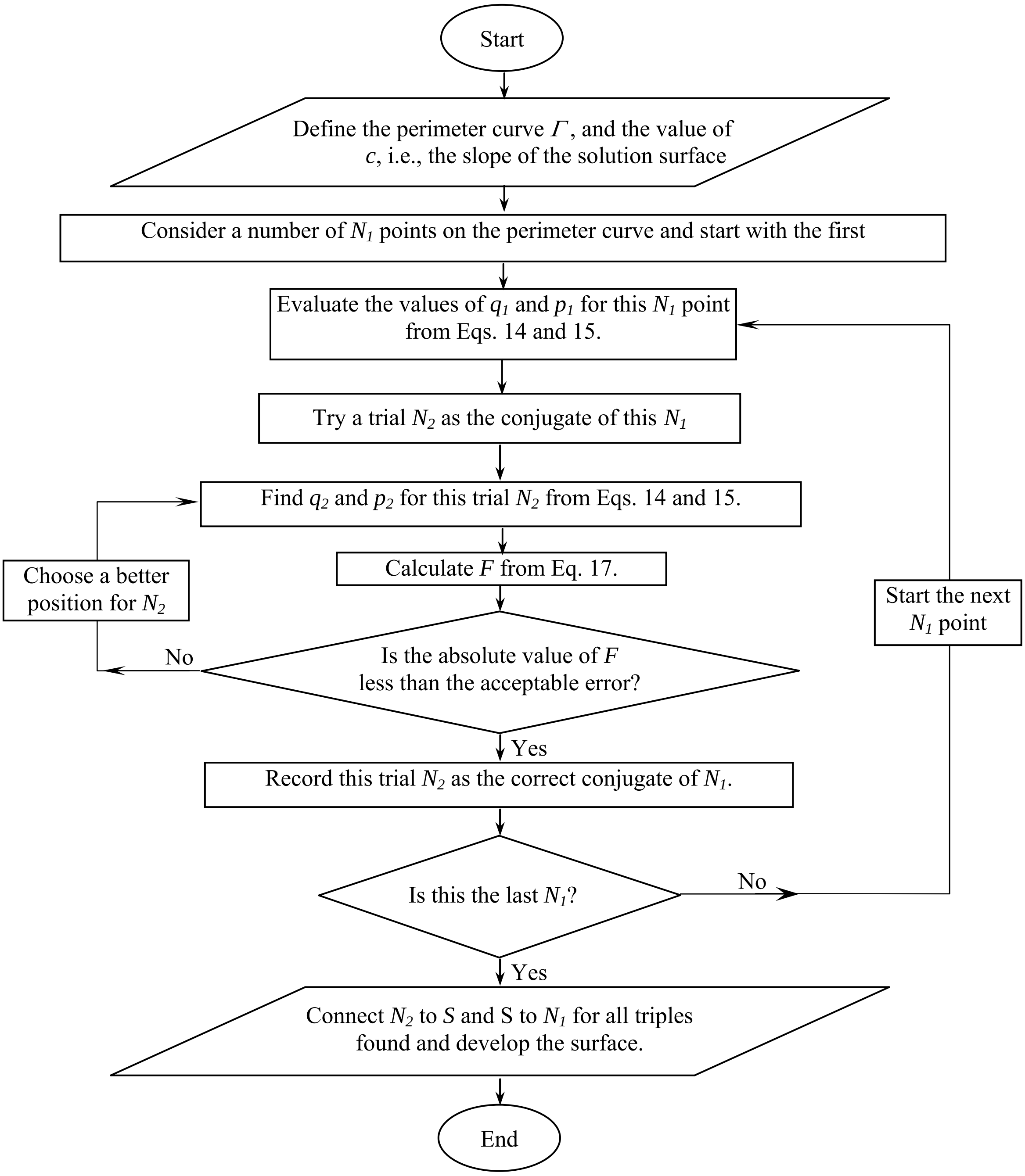

Fig. 5. Flow chart of the algorithm for finding the solution surface

8. Examples

The described geometric "*cone-slot method*" is practically applicable and does not need verification. Nevertheless, we have provided two examples here to show the functionality of the suggested algorithm.

Example 1.

For this example we have assumed: $c = \tan\phi = \tan 30 = \sqrt{3}/3$. The 3D space curve assumed is a closed skew one, described by the following parametric equations:

$$x = (3 + 2\cos t)\cos t \cos 30$$

$$y = (3 + 2\cos t)\sin t \cos 30 \qquad (18)$$

$$z = 7 - (3 + 2\cos t)\sin 30$$

Fig. 6 shows this space curve. The numerical solution surface obtained based on the suggested algorithm is shown in Fig. 7. This Figure indicates that the obtained integral surface is part of a cone, i.e., the characteristics drawn from different points of the curve all cross each other at a common point which is nothing but the vertex of the cone. This is not surprising because we have intentionally chosen the space curve on a cone of equation:

$$x^2 + y^2 = (7 - z)^2 \cot^2 30 \qquad (19)$$

in order to see whether we get such a result or not. In this example, for any point N_1 taken on the curve, all other points of the curve are the answer as N_2 because characteristics drawn from them all cross the one drawn from N_1. Figure 8 shows the values of F calculated when different points of the curve are taken as N_2. As shown, the maximum absolute value of F calculated is less than 10^{-12}, which is acceptable. This indicates high accuracy of the suggested procedure.

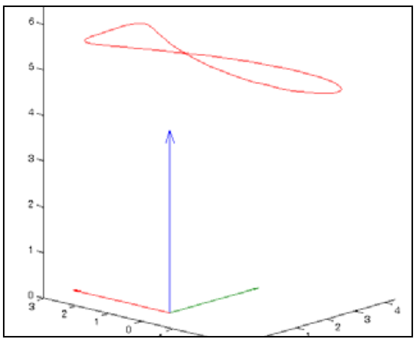

Fig. 6. Space curve for Example 1

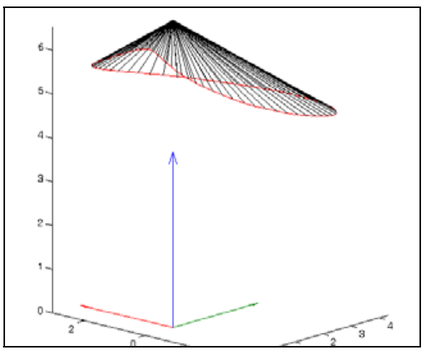

Fig. 7. The integral surface for Example 1

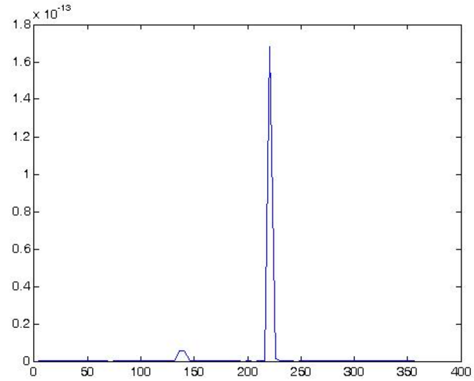

Fig. 8. The value of F for different locations of N_2 in Example 1

Example 2.

The value of the constant c in Eq. 1 for this example has been taken equal to 0.8391. This is equivalent to ϕ=40°. The curve, bounding the solution surface in this case is a circle of unit radius and we want to get the surface for cases in which the plane of circle makes different angles with the horizontal *x-y* plane. Obviously when the plane of circle is horizontal, the answer is the surface of a right circular cone of base angle 40°. The computer code gives the same answer (see Fig. 9a). The surface in this case has a tip which is the vertex of the cone. As the plane of the circle is inclined, a ridge is developed from the vertex of the cone. Figures 9-b, c, and d show the solution surface when the plane of circle is inclined at 15, 20 and 25 degrees with respect to the horizon. As shown, the ridge extends more to the sides and approaches the top side of the circle as the grade of the plane of circle is increased. When the slope angle i becomes equal to ϕ, the surface would be the same as the circle itself. One of the advantages of the described method is the ease of calculation of the area of the solution surface and the volume underneath, numerically. These quantities are important and have applications in many problems. The surface area is the area swept by the generator of the cone in contact with the boundary curve. The volume underneath is developed by the movement of the triangular area between this generator and axis of the cone that lies above the base circle. Both surface area and volume quantities are calculated numerically on the basis of the theorems of Pappus and Guldinus. When the slope of the plane of circle is zero, the area of the surface and the underneath volume would be the surface area of the cone and its volume that are:

$$S = \frac{\pi r^2}{\cos\phi} = \frac{\pi}{\cos 40} = 4.101 \qquad (20)$$

and:

$$V = \pi r^2 . \frac{1}{3} r \tan\phi = \frac{\pi}{3} \tan 40^\circ = 0.8787 \qquad (21)$$

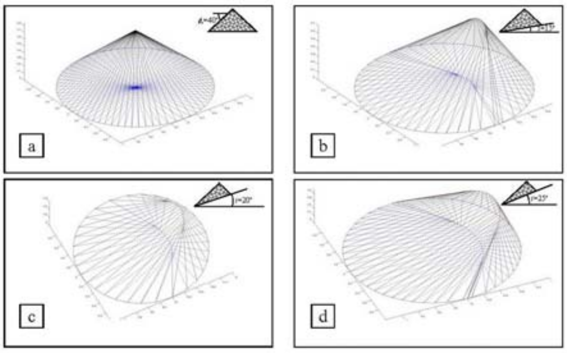

Fig. 9. The integral surfaces for Example 2

When the inclination of the plane of the circle i is the same as ϕ, i.e., 40°, the area of the surface should be equal to the area of the circle and the volume should become zero. The values of the surface area and volume have been numerically calculated for different inclination angles. These quantities have been plotted vs. the inclination angle in Fig. 10. As shown, starting from that of a cone, the area of the surface approaches that of a circle as the slope angle is increased. The volume underneath approaches zero from that of a cone.

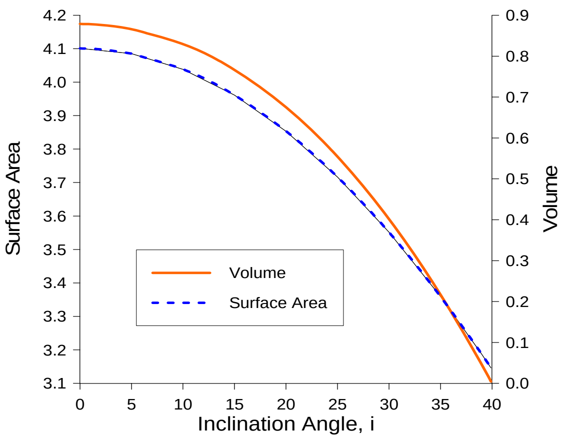

Fig. 10. The area of the surface and the volume underneath for Example 2

Obviously the solution surface would be determined more precisely as the number of divisions and points considered on the boundary curve is increased. The surface area and volume underneath would then be determined more precisely. The convergence in the numerical solution and answer of the problem with the increase in number of divisions on boundary curve has been investigated for the case of i=25°. Fig. 11 shows variation of the surface area and volume with the number of divisions considered on Γ. The Figure indicates relatively rapid convergence in the solution so that not much improvement is obtained when the number of divisions considered exceeds 30.

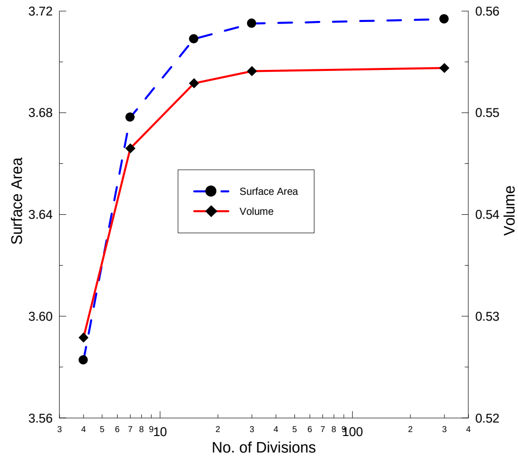

Fig. 11. Convergence of solution in Example 2

9. Conclusion

In this paper, the solution surface of the NPDE $\|\nabla u\| = c$, under Dirichlet boundary conditions over a 3D space curve as a Cauchy problem was investigated. The geometric properties indicated that the solution surface is a developable ruled surface that can be constructed by moving a cone of base angle $\tan^{-1} c$ inside the slot formed by the space curve. The algorithm suggested on the basis of this cone-slot method is functional. It obtains the solution surface as the integral of its characteristics and does not require any mesh for calculations. It was concluded that the method works well, especially where the boundary curve is not level, and where the interest is on knowing the area of the solution surface and the volume underneath.

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