IJST (2014) 38A2: 111-115
Iranian Journal of Science & Technology

http://ijsts.shirazu.ac.ir

On the range of a derivation

A. Hosseini[1]*, M. Hassani[2] and A. Niknam[3]

[1]*Kashmar Higher Education Institute, Kashmar, Iran*
[2]*Department of Mathematics, Mashhad Branch, Islamic Azad University, Mashhad, Iran*
[2]*Department of Mathematics, Mashhad Branch, Islamic Azad University, Mashhad, Iran*
E-mail: hosseini.amin82@gmail.com

Abstract

Let $\mathcal{A}$ be a Banach algebra and $D: \mathcal{A} \to \mathcal{A}$ a derivation. In this paper, it is proved, under certain conditions, that $D(\mathcal{A}) \subseteq rad(\mathcal{A})$, where $rad(\mathcal{A})$ is the Jacobson radical of $\mathcal{A}$. Moreover, we prove that if $\mathcal{A}$ is unital and $D: \mathcal{A} \to \mathcal{A}$ is a continuous derivation, then $D(\mathcal{A}) \subseteq \bigcap_{\mathcal{P} \in P_c(\mathcal{A})} \mathcal{P} \subseteq \bigcap_{\mathcal{M} \in M_c(\mathcal{A})} \mathcal{M} \subseteq \bigcap_{\varphi \in \Phi_{\mathcal{A}}} M_\varphi$, where $P_c(\mathcal{A})$ denotes the set of all primitive ideals such that $\frac{\mathcal{A}}{\mathcal{P}}$ is commutative, $M_c(\mathcal{A})$ denotes the set of all maximal (modular) ideals such that $\frac{A}{\mathcal{M}}$ is commutative, and $\Phi_{\mathcal{A}}$ is the set of all non-zero multiplicative linear functionals from $\mathcal{A}$ into $\mathbb{C}$. In addition, we present several results about the range of a derivation on algebras having the property $(\mathbb{B})$.

***Keywords**:* Derivation; derivation generator; pre-prime ideal; zero prime; property $(\mathbb{B})$

1. Introduction

Bounded derivations initiated in 1953 by Kaplansky and had its peak development around 1966-68 with the work by Kadison and Sakai, and was essentially finished in 1977-78 with work by Elliott, Akemann, and Pedersen, but the subject of bounded derivations from one algebra into another is still under development.

General theory of unbounded derivations was started by Sakai, Powers, Helemkii, Sinai, and Robinson around 1974. Derivations appear in various branches of mathematics and physics. The study of derivation theory in operator algebras is motivated by questions in quantum physics and statistical mechanics (Bratteli, 1987; Bratteli, 1997). Now, we are going to recall the definition of a derivation. Suppose that $\mathcal{A}$ is an algebra and $\mathcal{N}$ is an $\mathcal{A}$-bimodule. A linear mapping $D: \mathcal{A} \to \mathcal{N}$ is called a derivation if $D(ab) = D(a)b + aD(b)$ for all $a, b \in \mathcal{A}$. Let us introduce the background of our investigation. In 1955, Singer and Wermer obtained a fundamental result which started investigation into the ranges of derivations on Banach algebras. The result states that every bounded derivation on a commutative Banach algebra maps into the Jacobson radical. In the same paper they conjectured that the assumption of boundedness is not necessary. In 1988, Thomas proved this conjecture. Indeed, he proved that every derivation on a commutative Banach algebra maps into its Jacobson radical. Recall that the set $Z(\mathcal{A}) = \{a \in \mathcal{A} \ | \ ab = ba \quad \forall b \in \mathcal{A}\}$ is named the center of $\mathcal{A}$. By putting $\frac{Z_2(\mathcal{A})}{Z(\mathcal{A})} = Z(\frac{\mathcal{A}}{Z(\mathcal{A})})$, we have $Z_2(\mathcal{A}) = \{a \in \mathcal{A} \ | \ [[a,b],c] = 0 \quad \forall\ b, c \in \mathcal{A}\}$, where $[a,b] = ab - ba$ for $a, b \in \mathcal{A}$. In this study, it is proved that if $D: \mathcal{A} \to \mathcal{A}$ is a derivation and $Z_2(\mathcal{A}) = \mathcal{A}$, then $D(\mathcal{A}) \subseteq rad(\mathcal{A})$.

Suppose that $\mathcal{J}$ is a closed bi-ideal of $\mathcal{A}$ and $D: \mathcal{A} \to \mathcal{A}$ is a map. A pair $(D, \mathcal{J})$ is called a derivation generator if $\{D(ab) - D(a)b - aD(b) \ | \ a, b \in \mathcal{A}\} \subseteq \mathcal{J}$. In this article the following result is proved:

Let D be a linear mapping and $(D, \mathcal{J})$ be a derivation generator such that $D(\mathcal{J}) \subseteq \mathcal{J}$. If $\frac{\mathcal{A}}{\mathcal{J}}$ is a commutative, semi-simple Banach algebra, then $D(\mathcal{A}) \subseteq \mathcal{J}$. By this result, we prove that $D(\mathcal{A}) \subseteq \bigcap_{\mathcal{P} \in P_c(\mathcal{A})} \mathcal{P} \subseteq \bigcap_{\mathcal{M} \in M_c(\mathcal{A})} \mathcal{M} \subseteq \bigcap_{\varphi \in \Phi_{\mathcal{A}}} M_\varphi$ if $\mathcal{A}$ is unital and $D: \mathcal{A} \to \mathcal{A}$ is a continuous derivation, where $P_c(\mathcal{A})$ denotes the set of all primitive ideals such that $\frac{\mathcal{A}}{\mathcal{P}}$ is commutative, $M_c(\mathcal{A})$ denotes the set of all maximal (modular) ideals such that $\frac{A}{\mathcal{M}}$ is commutative and $\Phi_{\mathcal{A}}$ is the set of all non-zero multiplicative linear functionals on $\mathcal{A}$. Moreover, in this paper an ideal which is called zero prime is defined as follows:

A bi-ideal $\mathcal{J}$ of $\mathcal{A}$ is called a zero prime ideal if ab=0 implies that a $\in \mathcal{J}$ or b $\in \mathcal{J}$.

Let $\mathcal{A}$ be a unital Banach algebra with the property $(\mathbb{B})$, which introduced in (Alaminos,

*Corresponding author
Received: 12 August 2012 / Accepted: 14 October 2013

2009), and $\mathcal{J}$ is a closed zero prime ideal of $\mathcal{A}$ which is invariant under a linear mapping $D: \mathcal{A} \to \mathcal{A}$, and furthermore there exists a positive number N such that $\| D(a) + \mathcal{J} \| \leq N \| a + \mathcal{J} \|$ for all $a \in \mathcal{A}$. If $D(1) = 0$ then $D(\mathcal{A}) \subseteq \mathcal{J}$. Brašar and Mathieu (Brešar, 1995) proved that every spectrally bounded derivation on a unital Banach algebra maps into the Jacobson radical. We recall that a linear mapping $T: \mathcal{A} \to \mathcal{B}$ is called spectrally bounded if there is a constant $M \geq 0$ such that $r(T(a)) \leq Mr(a)$ for all $a \in \mathcal{A}$, where $r(a) = \lim_{n\to\infty} \| a^n \|^{\frac{1}{n}}$ denotes the spectral radius of an element a and $\mathcal{B}$ is also a complex Banach algebra.

2. Main results

Throughout this paper $\mathcal{A}$ denotes a complex Banach algebra and $\Phi_{\mathcal{A}}$ denotes the set of all non-zero multiplicative linear functionals on $\mathcal{A}$. Moreover, $Q(\mathcal{A})$ is the set of all quasi-nilpotent elements of $\mathcal{A}$, i.e. the set of elements a in $\mathcal{A}$ such that $\lim_{n\to\infty} \| a^n \|^{\frac{1}{n}} = 0$.

It is well-known that $Z(\mathcal{A})$, the center of $\mathcal{A}$, is a closed subalgebra of $\mathcal{A}$. We define $\frac{Z_{n+1}(\mathcal{A})}{Z_n(\mathcal{A})} = Z(\frac{\mathcal{A}}{Z_n(\mathcal{A})})$ for all positive integer n. It is clear that $Z_0(\mathcal{A}) = \{0\}$ if and only if $Z_1(\mathcal{A}) = Z(\mathcal{A})$. If n = 2, for example, we have

$$\begin{aligned}\frac{Z_2(\mathcal{A})}{Z(\mathcal{A})} &= Z\left(\frac{\mathcal{A}}{Z(\mathcal{A})}\right)\\ &= \{a + Z(\mathcal{A}) \mid ab - ba \in Z(\mathcal{A}) \ \forall\ b \in \mathcal{A}\}\\ &= \{a + Z(\mathcal{A}) \mid (ab - ba)c = c(ab - ba) \ \forall\ b, c \in \mathcal{A}\}\\ &= \{a + Z(\mathcal{A}) \mid [[a, b], c] = 0 \ \forall\ b, c \in \mathcal{A}\}.\end{aligned}$$

Hence, $Z_2(\mathcal{A}) = \{a \in \mathcal{A} \mid [[a, b], c] = 0 \ \forall\ b, c \in \mathcal{A}\}$. Moreover, if n = 3 then we have $Z_3(\mathcal{A}) = \{a \in \mathcal{A} \mid [[[a, b], c], d] = 0 \ \forall\ b, c, d \in \mathcal{A}\}$. Thus, by induction on n, we obtain that

$$Z_n(\mathcal{A}) = \{a \in \mathcal{A} \mid [\underbrace{[[\ldots[}_{n-\text{times}} a, a_1], a_2], a_3], \ldots, a_n] = 0 \ \forall\ a_1, a_2, a_3, \ldots, a_n \in \mathcal{A}\}.$$

Clearly, $\{0\} = Z_0(\mathcal{A}) \subseteq Z_1(\mathcal{A}) \subseteq Z_2(\mathcal{A}) \subseteq \ldots \subseteq Z_n(\mathcal{A}) \subseteq \ldots \subseteq \mathcal{A}$ and $Z_n(\frac{\mathcal{A}}{Z_m(\mathcal{A})}) = \frac{Z_{n+m}(\mathcal{A})}{Z_m(\mathcal{A})}$, where $m, n \in \mathbb{N}$ and $m \leq n$. Moreover, $Z_2(\mathcal{A}) = \mathcal{A}$ if and only if $Z(\frac{\mathcal{A}}{Z(\mathcal{A})}) = \frac{\mathcal{A}}{Z(\mathcal{A})}$. It means that $\frac{\mathcal{A}}{Z(\mathcal{A})}$ is commutative if and only if $Z_2(\mathcal{A}) = \mathcal{A}$.

Recall that an algebra $\mathcal{A}$ is called semi-prime if $a\mathcal{A}a = \{0\}$ implies that a = 0 for each $a \in \mathcal{A}$.

Proposition 2.1. (i)Suppose that $Q(\mathcal{A}) = \{0\}$. Then $\mathcal{A}$ is commutative if and only if $[a, [a, b]] = 0$ for all $a, b \in \mathcal{A}$.
(ii) Suppose that $\mathcal{A}$ is a semi-prime Banach algebra. Then $Z_n(\mathcal{A}) = Z_{n+1}(\mathcal{A})$ for all $n \in \mathbb{N}$. Hence, $\mathcal{A}$ is commutative if and only if $Z_n(\mathcal{A}) = \mathcal{A}$ for some $n \in \mathbb{N}$.

Proof: (i) Suppose that $\mathcal{A}$ is commutative. Clearly $[a, [a, b]] = 0$ for all $a, b \in \mathcal{A}$. Conversely, assume that $[[a, b], a] = 0$ for all $a, b \in \mathcal{A}$. It follows from Proposition 18.13 of (Bonsall, 1973), that $[a, b] \in Q(\mathcal{A})$ for all $a, b \in \mathcal{A}$. Since $Q(\mathcal{A}) = \{0\}$, $\mathcal{A}$ is commutative.
(ii) We shall prove this part of the statement by induction on n. Suppose that $a \in Z_2(\mathcal{A})$, i.e. $[a, b] \in Z(\mathcal{A})$ for all $b \in \mathcal{A}$. It follows from Lemma 3.1 of (Hejazian, 2005) that $a \in Z(\mathcal{A})$. So, $Z_2(\mathcal{A}) \subseteq Z(\mathcal{A})$ and since $Z(\mathcal{A}) \subseteq Z_2(\mathcal{A})$, $Z(\mathcal{A}) = Z_2(\mathcal{A})$. Assume that $a \in Z_n(\mathcal{A})$ implies that $a \in Z(\mathcal{A})$. Let $a \in Z_{n+1}(\mathcal{A})$. Therefore, $\underbrace{[[\ldots[}_{n-\text{times}} a, a_1], a_2], \ldots, a_{n-1}], a_n] \in Z(\mathcal{A})$ for all $a_1, a_2, a_3, \ldots, a_n \in \mathcal{A}$.
Set $b = \underbrace{[[\ldots[}_{(n-1)-\text{times}} a, a_1], a_2], \ldots], a_{n-1}]$. Hence, $[b, a_n] \in Z(\mathcal{A})$ for all $a_n \in \mathcal{A}$. By the case n = 2, $b \in Z(\mathcal{A})$.
It means that $\underbrace{[[\ldots[}_{(n-1)-\text{times}} a, a_1], a_2], \ldots], a_{n-1}] \in Z(\mathcal{A})$. Thus, $a \in Z_n(\mathcal{A})$. Induction hypothesis results in $a \in Z(\mathcal{A})$. Hence, $Z_{n+1}(\mathcal{A}) \subseteq Z(\mathcal{A}) \subseteq Z_{n+1}(\mathcal{A})$, so $Z_n(\mathcal{A}) = Z_{n+1}(\mathcal{A})$ for all $n \in \mathcal{N}$.
Suppose that $\mathcal{A}$ is commutative. Clearly, $Z_n(\mathcal{A}) = \mathcal{A}$ for all $n \in \mathbb{N}$. Conversely, assume that $Z_n(\mathcal{A}) = \mathcal{A}$ for some positive integer n, so $\mathcal{A} = Z_n(\mathcal{A}) = Z(\mathcal{A})$. It is well-known that if $Z(\mathcal{A}) = \mathcal{A}$ then $\mathcal{A}$ is commutative.

Mathieu (2005) proved that the following result holds:

Theorem 2.2. Let d be a derivation on a Banach algebra $\mathcal{A}$. Then the following three conditions are equivalent:
(i) $[a, d(a)] \in \mathrm{rad}(\mathcal{A})$ for all $a \in \mathcal{A}$;
(ii) d is spectrally bounded;
(iii) $d(\mathcal{A}) \subseteq \mathrm{rad}(\mathcal{A})$,
where $[a, b] = ab - ba$ and $\mathrm{rad}(\mathcal{A})$ is denoted the Jacobson radical of $\mathcal{A}$.

Theorem 2.3. Suppose that $D: \mathcal{A} \to \mathcal{A}$ is a derivation. If $Z_2(\mathcal{A}) = \mathcal{A}$ then $D(\mathcal{A}) \subseteq \mathrm{rad}(\mathcal{A})$.

Proof: First of all, we define another product on $\mathcal{A}$ by the following form: $a \bullet b = \frac{ab+ba}{2}$ $(a, b \in \mathcal{A})$. Clearly, $a \bullet b = b \bullet a$, i.e. $\bullet$ is commutative. In addition, $a \bullet (b + c) = a \bullet b + a \bullet c$ and $(a + b) \bullet$

$c = a \bullet c + b \bullet c$ for all $a, b, c \in \mathcal{A}$. Since $Z_2(\mathcal{A}) = \mathcal{A}$, we have

$$a \bullet (b \bullet c) = a \bullet (\frac{bc+cb}{2}) = \frac{abc+acb+bca+cba}{4} = \frac{abc+cba+bac+cab}{4} = (a \bullet b) \bullet c$$

for all $a, b, c \in \mathcal{A}$. Let a and b be two arbitrary elements of $\mathcal{A}$. Then $\| a \bullet b \| = \| \frac{ab+ba}{2} \| \leq \| a \| \| b \|$. Therefore, $\mathcal{A}$ is a Banach algebra with the original norm and this new product. We denote this algebra by $\tilde{\mathcal{A}}$. D is a derivation on $\tilde{\mathcal{A}}$, because

$$D(a \bullet b) = D(\frac{ab+ba}{2}) = \frac{D(a)b+aD(b)+D(b)a+bD(a)}{2} = \frac{D(a)b+bD(a)}{2} + \frac{aD(b)+D(b)a}{2} = D(a) \bullet b + a \bullet D(b).$$

Since $\tilde{\mathcal{A}}$ is commutative Banach algebra, it follows from Theorem 4.4 of (Thomas, 1988) that $D(\tilde{\mathcal{A}}) \subseteq \mathrm{rad}(\tilde{\mathcal{A}}) = Q(\tilde{\mathcal{A}})$. Since a^n in $\tilde{\mathcal{A}}$ is equivalent to a^n in $\mathcal{A}$, $Q(\tilde{\mathcal{A}}) = Q(\mathcal{A})$ and thus, $D(\mathcal{A}) \subseteq Q(\mathcal{A})$. It means that D is spectrally bounded and by using the Theorem 2.2 $D(\mathcal{A}) \subseteq \mathrm{rad}(\mathcal{A})$ is obtained.

Definition 2.4. Let $\sigma: \mathcal{A} \to \mathcal{A}$ be a linear mapping. A linear mapping $d: \mathcal{A} \to \mathcal{A}$ is called a $\sigma-$derivation if $d(ab) = d(a)\sigma(b) + \sigma(a)d(b)$ for all $a, b \in \mathcal{A}$ (for more details see (Hosseini et al., 2011; Hosseini et al., 2013; Hosseini et al., 2011; Mirzavaziri and Moslehian, 2007; Mirzavaziri and Moslehian, 2006; Mirzavaziri, 2008).

Definition 2.5. (Derivation Generator) Suppose that $D: \mathcal{A} \to \mathcal{A}$ is a map and $\mathcal{I}$ is a closed bi-ideal of $\mathcal{A}$. A pair $(D, \mathcal{I})$ is called a derivation generator if $\{D(ab) - D(a)b - aD(b) \mid a, b \in \mathcal{A}\} \subseteq \mathcal{I}$.

Example 2.6. Suppose that $D: \mathcal{A} \to \mathcal{A}$ is a map and $\sigma: \mathcal{A} \to \mathcal{A}$ is an endomorphism. It is well-known that $\overline{\ker(\sigma)}$ is a closed bi-ideal of $\mathcal{A}$. $(D, \overline{\ker(\sigma)})$ is a derivation generator, whenever $\sigma D: \mathcal{A} \to \mathcal{A}$ is a σ-derivation. Since $\sigma D(ab) = \sigma D(a)\sigma(b) + \sigma(a)\sigma D(b)$, $\{D(ab) - D(a)b - aD(b) \mid, b \in \mathcal{A}\} \subseteq \ker(\sigma) \subseteq \overline{\ker(\sigma)}$. Thus, $(D, \overline{\ker(\sigma)})$ is a derivation generator.

Example 2.7. Set $\mathfrak{B} = \mathcal{A} \times \mathcal{A}$. Then $\mathfrak{B}$ is a Banach algebra by the following action and norm: $(a, b) \cdot (c, d) = (ac, bd)$ and $\| (a, b) \| = \| a \| + \| b \|$ for all $a, b, c \in \mathcal{A}$. We define $D: \mathfrak{B} \to \mathfrak{B}$ by $D((a, b)) = (0, b + c_0)$, where c_0 is a fixed element of $\mathcal{A}$. Obviously, D is a non-linear mapping. Assume that $\mathcal{I} = \{(0, a) \mid a \in \mathcal{A}\}$. One can easily show that $\mathcal{I}$ is a closed bi-ideal of $\mathfrak{B}$. We have

$$D((a, b) \cdot (c, d)) - (a, b) \cdot D((c, d)) - D((a, b))(c, d) = (0, bd + c_0) - (0, bd + bc_0) - (0, bd + c_0 d) = (0, c_0 - bc_0 - bd - c_0 d) \in \mathcal{I}.$$

Therefore, $(D, \mathcal{I})$ is a derivation generator.

Remark 2.8. If D is a linear mapping, then $(D, \mathcal{I})$ is a derivation generator if and only if $d: \mathcal{A} \to \frac{\mathcal{A}}{\mathcal{I}}$ defined by $d(a) = D(a) + \mathcal{I}$ is a derivation. Let $\mathcal{A}$ be a unital algebra with unit 1. Since d is a derivation, $d(1) = \mathcal{I}$, thus $D(1) \in \mathcal{I}$.

Proposition 2.9. Suppose that $(D, \mathcal{I})$ is a derivation generator such that D is a linear mapping. If $\mathcal{I}$ is a bounded set in $\mathcal{A}$ then D is a derivation.

Proof: Since $\mathcal{I}$ is bounded, there exists a positive number m such that $\| x \| \leq m$ for all $x \in \mathcal{I}$. Hence, $\| D(ab) - D(a)b - aD(b) \| \leq m$ for all $a, b \in \mathcal{A}$. Let ε be an arbitrary positive number. We have $\| D(\frac{2m}{\varepsilon} ab) - D(\frac{2m}{\varepsilon} a)b - \frac{2m}{\varepsilon} aD(b) \| \leq m$. It implies that $\| D(ab) - aD(b) - D(a)b \| \leq \frac{\varepsilon}{2} < \varepsilon$. Since ε was arbitrary, $D(ab) - D(a)b - aD(b) = 0$ for all $a, b \in \mathcal{A}$ and consequently D is a derivation.

Theorem 2.10. Suppose that $(D, \mathcal{I})$ is a derivation generator such that D is a linear mapping and $\mathcal{I}$ is invariant under D. Furthermore, assume that $\frac{\mathcal{A}}{\mathcal{I}}$ is a commutative, semi-simple Banach algebra. Then $D(\mathcal{A}) \subseteq \mathcal{I}$.

Proof: We define $d: \frac{\mathcal{A}}{\mathcal{I}} \to \frac{\mathcal{A}}{\mathcal{I}}$ by $d(a + \mathcal{I}) = D(a) + \mathcal{I}$. If $a + \mathcal{I} = b + \mathcal{I}$ then $a - b \in \mathcal{I}$. Since $D(\mathcal{I}) \subseteq \mathcal{I}$, $D(a - b) \in \mathcal{I}$. Hence, $D(a) + \mathcal{I} = D(b) + \mathcal{I}$ and it implies that d is well-defined. Clearly, d is a linear mapping. For convenience, we denote $a + \mathcal{I}$ by $\tilde{a}$. Since $(D, \mathcal{I})$ is a derivation generator, $\{D(ab) - aD(b) - D(a)b \mid a, b \in \mathcal{A}\} \subseteq \mathcal{I}$. Thus $d(\tilde{a}\tilde{b}) - \tilde{a}d(\tilde{b}) - d(\tilde{a})\tilde{b} = D(ab) - aD(b) - D(a)b + \mathcal{I} = \mathcal{I}$ for all $\tilde{a}, \tilde{b} \in \frac{\mathcal{A}}{\mathcal{I}}$. So, d is a derivation. By Theorem 4.4. (Thomas, 1988), $d(\frac{\mathcal{A}}{\mathcal{I}}) \subseteq \mathrm{rad}(\frac{\mathcal{A}}{\mathcal{I}})$. Since $\frac{\mathcal{A}}{\mathcal{I}}$ is semi-simple, $d(\frac{\mathcal{A}}{\mathcal{I}}) = \{\mathcal{I}\}$ and so, $D(a) + \mathcal{I} = \mathcal{I}$ for all $a \in \mathcal{A}$. It implies that $D(\mathcal{A}) \subseteq \mathcal{I}$.

We will denote the kernel of $\varphi \in \Phi_{\mathcal{A}}$ by M_φ.

Corollary 2.11. Let $\mathcal{A}$ be a unital Banach algebra and $D: \mathcal{A} \to \mathcal{A}$ be a continuous derivation. Then $D(\mathcal{A}) \subseteq \bigcap_{\mathcal{P} \in P_c(\mathcal{A})} \mathcal{P} \subseteq \bigcap_{\mathcal{M} \in M_c(\mathcal{A})} \mathcal{M} \subseteq \bigcap_{\varphi \in \Phi_{\mathcal{A}}} M_\varphi$, where $M_c(\mathcal{A})$, $P_c(\mathcal{A})$ and $\Phi_{\mathcal{A}}$ are the symbols which introduced in the introduction. In particular, If $\bigcap_{\mathcal{P}\in \mathrm{P_c}(\mathcal{A})} \mathcal{P} = \{0\}$, then $\mathrm{D} \equiv 0$.

Proof: Assume that $\mathcal{P} \in \mathrm{P_c}(\mathcal{A})$. According to Theorem 6.2.3 (Dales, 2003), $\mathrm{D}(\mathcal{P}) \subseteq \mathcal{P}$. Since $(\mathrm{D}, \mathcal{P})$ is a derivation generator, Theorem 2.10 implies that $\mathrm{D}(\mathcal{A}) \subseteq \mathcal{P}$ and so, $\mathrm{D}(\mathcal{A}) \subseteq \bigcap_{\mathcal{P}\in \mathrm{P_c}(\mathcal{A})} \mathcal{P}$, since $\mathcal{P}$ was an arbitrary element of $\mathrm{P_c}(\mathcal{A})$. We know that if $\mathcal{A}$ is unital then every (maximal) ideal of $\mathcal{A}$ is a (maximal) modular ideal (Murphy, 1990). Moreover, it follows from Proposition 1.4.34 (iii) (Dales, 2000) that each maximal modular ideal in $\mathcal{A}$ is primitive. Thus, $\mathrm{M_c}(\mathcal{A}) \subseteq \mathrm{P_c}(\mathcal{A})$ and it results that $\bigcap_{\mathcal{P}\in \mathrm{P_c}(\mathcal{A})} \mathcal{P} \subseteq \bigcap_{\mathcal{M}\in \mathrm{M_c}(\mathcal{A})} \mathcal{M}$. According to Proposition 3.1.2 (Dales, 2003) M_φ is a maximal ideal of $\mathcal{A}$ for every $\varphi \in \Phi_{\mathcal{A}}$. Since $\varphi(\mathrm{ab}) = \varphi(\mathrm{a})\varphi(\mathrm{b}) = \varphi(\mathrm{b})\varphi(\mathrm{a}) = \varphi(\mathrm{ba})$ for all $\mathrm{a}, \mathrm{b} \in \mathcal{A}$, it means that $\mathrm{ab} - \mathrm{ba} \in \mathrm{M}_\varphi$. Thus, $(\mathrm{a} + \mathrm{M}_\varphi)(\mathrm{b} + \mathrm{M}_\varphi) = (\mathrm{b} + \mathrm{M}_\varphi)(\mathrm{a} + \mathrm{M}_\varphi)$, i.e. $\frac{\mathcal{A}}{\mathrm{M}_\varphi}$ is a commutative Banach algebra. Thus, $\{\mathrm{M}_\varphi \mid \varphi \in \Phi_{\mathcal{A}}\} \subseteq \mathrm{M_c}(\mathcal{A})$. We have

$$\bigcap_{\mathcal{P}\in \mathrm{P_c}(\mathcal{A})} \mathcal{P} \subseteq \bigcap_{\mathcal{M}\in \mathrm{M_c}(\mathcal{A})} \mathcal{M} \subseteq \bigcap_{\varphi\in\Phi_{\mathcal{A}}} \mathrm{M}_\varphi$$

and it causes that $\mathrm{D}(\mathcal{A}) \subseteq \bigcap_{\mathcal{P}\in \mathrm{P_c}(\mathcal{A})} \mathcal{P} \subseteq \bigcap_{\mathcal{M}\in \mathrm{M_c}(\mathcal{A})} \mathcal{M} \subseteq \bigcap_{\varphi\in\Phi_{\mathcal{A}}} \mathrm{M}_\varphi$.

Definition 2.12. Let $\phi: \mathcal{A} \times \mathcal{A} \to \mathcal{X}$, where $\mathcal{X}$ is a Banach space, be a bilinear map. We say that ϕ preserves zero product if $\mathrm{ab} = 0$ then $\phi(\mathrm{a}, \mathrm{b}) = 0$ for all $\mathrm{a}, \mathrm{b} \in \mathcal{A}$ (B_1).

Definition 2.13. A Banach algebra $\mathcal{A}$ has the property $(\mathbb{B})$ if for every continuous bilinear map $\phi: \mathcal{A} \times \mathcal{A} \to \mathcal{X}$, where $\mathcal{X}$ is an arbitrary Banach space, (B_1) implies that $\phi(\mathrm{ab}, \mathrm{c}) = \phi(\mathrm{a}, \mathrm{bc})$ for all $\mathrm{a}, \mathrm{b}, \mathrm{c} \in \mathcal{A}$ (for more details see (Alaminos, 2009).
We say that a proper bi-ideal $\mathcal{J}$ is pre-prime if $\mathrm{ab} \in \mathcal{J}$ implies that $\mathrm{a} \in \mathcal{J}$ or $\mathrm{b} \in \mathcal{J}$. It is clear that if $\mathcal{J}$ is a pre-prime ideal of $\mathcal{A}$ then $\frac{\mathrm{A}}{\mathcal{J}}$ is a domain. Obviously, every pre-prime ideal is zero prime. According to Proposition 1.3.46 (i) (Dales, 2000), every prime ideal of a commutative algebra is zero prime. Moreover, if $\mathcal{A}$ is a domain then every ideal in $\mathcal{A}$ is a zero prime ideal. In fact, $\mathcal{A}$ is domain if and only if $\mathcal{J} = \{0\}$ is a zero prime ideal. Set $\mathrm{P_P}(\mathcal{A}) = \{\mathcal{J} \subseteq \mathcal{A} \ : \ \mathcal{J} \text{ is a pre}-\text{prime ideal}\}$ and
$\mathfrak{Z}_\mathrm{P}(\mathcal{A}) = \{\mathcal{J} \subseteq \mathcal{A} \ : \ \mathcal{J} \text{ is a zero prime ideal}\}$. Clearly, $\mathrm{P_P}(\mathcal{A}) \subseteq \mathfrak{Z}_\mathrm{P}(\mathcal{A})$. So,

$$\bigcap_{\mathcal{J}\in \mathfrak{Z}_\mathrm{P}(\mathcal{A})} \mathcal{J} \subseteq \bigcap_{\mathcal{J}\in \mathrm{P_P}(\mathcal{A})} \mathcal{J}.$$

Definition 2.14. A bilinear map $\phi: \mathcal{A} \times \mathcal{A} \to \mathcal{N}$ is called a left two variables derivation if $\phi(\mathrm{ab}, \mathrm{c}) = \phi(\mathrm{a}, \mathrm{c})\mathrm{b} + \mathrm{a}\phi(\mathrm{b}, \mathrm{c})$ for all $\mathrm{a}, \mathrm{b}, \mathrm{c} \in \mathcal{A}$. Similarly, ϕ is called a right two variables derivation whenever $\phi(\mathrm{a}, \mathrm{bc}) = \phi(\mathrm{a}, \mathrm{b})\mathrm{c} + \mathrm{b}\phi(\mathrm{a}, \mathrm{c})$ for all $\mathrm{a}, \mathrm{b}, \mathrm{c} \in \mathcal{A}$. A bilinear map $\phi: \mathcal{A} \times \mathcal{A} \to \mathcal{N}$ is said to be a two variables derivation if it is a left as well as a right two variables derivation.

Example 2.15. The bilinear map $\phi: \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ which is defined by $\phi(\mathrm{a}, \mathrm{b}) = [\mathrm{a}, \mathrm{b}] = \mathrm{ab} - \mathrm{ba}$ is a two variables derivation.

Remark 2.16. Suppose that $\mathcal{A}, \mathcal{N}$ are unital and $\phi: \mathcal{A} \times \mathcal{A} \to \mathcal{N}$ is a left two variables derivation. We have $\phi(1, \mathrm{a}) = \phi(1.1, \mathrm{a}) = 1\phi(1, \mathrm{a}) + \phi(1, \mathrm{a})1$, so $\phi(1, \mathrm{a}) = 0$. Similarly, if ϕ is a right two variables derivation, then $\phi(\mathrm{a}, 1) = 0$. Clearly, if $\phi(\mathrm{ab}, \mathrm{c}) = \phi(\mathrm{a}, \mathrm{bc})$ for all $\mathrm{a}, \mathrm{b}, \mathrm{c} \in \mathcal{A}$ then $\phi(\mathrm{a}, \mathrm{b}) = \phi(1\mathrm{a}, \mathrm{b}) = \phi(1, \mathrm{ab}) = 0$ and it means that ϕ is identically zero.

Proposition 2.17. Suppose that $\mathcal{A}$ has an approximate identity and $\mathcal{J}$ is a closed zero prime ideal of $\mathcal{A}$. If $\mathcal{A}$ has the property $(\mathbb{B})$ then $[\mathrm{a}, \mathrm{b}] \in \mathcal{J}$ for all $\mathrm{a}, \mathrm{b} \in \mathcal{A}$, i.e. $\frac{\mathcal{A}}{\mathcal{J}}$ is commutative. In particular, if $\bigcap_{\mathcal{J}\in \mathfrak{Z}_\mathrm{P}(\mathcal{A})} \mathcal{J} = \{0\}$ then $\mathcal{A}$ is commutative.
Proof: We define a bilinear map $\phi: \mathcal{A} \times \mathcal{A} \to \frac{\mathcal{A}}{\mathcal{J}}$ by $\phi(\mathrm{a}, \mathrm{b}) = [\mathrm{a}, \mathrm{b}] + \mathcal{J}$. Clearly, ϕ preserves zero product, furthermore, it is a continuous two variables derivation. According to Remark 2.16, ϕ is identically zero. Hence, $[\mathrm{a}, \mathrm{b}] \in \mathcal{J}$ for all $\mathrm{a}, \mathrm{b} \in \mathcal{A}$.

Proposition 2.18. Suppose that $\mathcal{A}$ is a unital domain with the property $(\mathbb{B})$. Then every continuous linear mapping $\mathrm{T}: \mathcal{A} \to \mathcal{A}$ is a centralizer, i.e. $\mathrm{T}(\mathrm{ab}) = \mathrm{T}(\mathrm{a})\mathrm{b} = \mathrm{a}\mathrm{T}(\mathrm{b})$ for all $\mathrm{a}, \mathrm{b} \in \mathcal{A}$. In particular, if $\mathrm{T}(1) = 0$ then $\mathrm{T} \equiv 0$.

Proof: We define a bilinear map $\phi: \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ by $\phi(\mathrm{a}, \mathrm{b}) = \mathrm{T}(\mathrm{a})\mathrm{b}$ $(\mathrm{a}, \mathrm{b} \in \mathcal{A})$ which is clearly continuous. If $\mathrm{ab} = 0$ then $\mathrm{a} = 0$ or $\mathrm{b} = 0$, since $\mathcal{A}$ is a domain. It implies that $\phi(\mathrm{a}, \mathrm{b}) = 0$ and hence ϕ preserves zero product. By the hypothesis, $\mathcal{A}$ has the property $(\mathbb{B})$ and so, $\phi(\mathrm{ab}, \mathrm{c}) = \phi(\mathrm{a}, \mathrm{bc})$ for all $\mathrm{a}, \mathrm{b}, \mathrm{c} \in \mathcal{A}$. If we put $\mathrm{c} = 1$, then $\mathrm{T}(\mathrm{ab})1 = \mathrm{T}(\mathrm{a})\mathrm{b}$. Using the same strategy, we can observe that $\mathrm{T}(\mathrm{ab}) = \mathrm{a}\mathrm{T}(\mathrm{b})$ which means that T is a centralizer. Obviously, if $\mathrm{T}(1) = 0$ then $\mathrm{T} \equiv 0$.

Theorem 2.19. Suppose that $\mathcal{A}$ is a unital Banach algebra with the property $(\mathbb{B})$. Let $\mathrm{D}: \mathcal{A} \to \mathcal{A}$ be a linear mapping and $\mathcal{J}$ be a closed zero prime ideal of $\mathcal{A}$ which is invariant under D. If $\mathrm{D}(1) \in \mathcal{J}$ and there exists a positive number N such that $\| D(a) + \mathcal{J} \| \leq N \| a + \mathcal{J} \|$ for all $a \in \mathcal{A}$ then $D(\mathcal{A}) \subseteq \mathcal{J}$.

Proof: We define $\phi: \mathcal{A} \times \mathcal{A} \to \frac{\mathcal{A}}{\mathcal{J}}$ by $\phi(a, b) = D(a)b + \mathcal{J}$ $(a, b \in \mathcal{A})$. Obviously, ϕ is a continuous bilinear map. Assume that ab = 0 then $a \in \mathcal{J}$ or $b \in \mathcal{J}$. If $a \in \mathcal{J}$, then $D(a) \in \mathcal{J}$, since $\mathcal{J}$ is invariant under D. Hence, $\phi(a, b) = D(a)b + \mathcal{J} = \mathcal{J}$ and if $b \in \mathcal{J}$ then $\phi(a, b) = D(a)b + \mathcal{J} = \mathcal{J}$. So, ϕ preserves zero product. Since $\mathcal{A}$ has the property $(\mathbb{B})$, $\phi(ab, c) = \phi(a, bc)$ for all $a, b, c \in \mathcal{A}$. Hence, $\phi(a, 1) = \phi(1a, 1) = \phi(1, a)$. It implies that $D(a) + \mathcal{J} = \mathcal{J}$ for all $a \in \mathcal{A}$. It means that $D(a) \in \mathcal{J}$ and so, $D(\mathcal{A}) \subseteq \mathcal{J}$.

Corollary 2.20. Suppose that $\mathcal{A}$ is a unital domain with the property $(\mathbb{B})$. If $D: \mathcal{A} \to \mathcal{A}$ is a continuous derivation then D is identically zero.

Proof: Set $\mathcal{J} = \{0\}$. Then $\mathcal{J}$ is a closed zero prime ideal. Previous theorem gives the result. Moreover, we can obtain this result by using Proposition 2.18.

Theorem 2.21. Let $\mathcal{N}$ be a unital Banach $\mathcal{A}$-bimodule, where $\mathcal{A}$ has the property $(\mathbb{B})$ and let $D: \mathcal{A} \to \mathcal{N}$ be a linear mapping for which $\| D(ab) - D(a)b - aD(b) \| \leq m \| a \| \| b \|$ for some positive number m. If ab = 0 implies that $D(a)b + aD(b) = 0$ then D is a derivation if and only if $D(1) = 0$.

Proof: Let D be a derivation. It is routine to show that $D(1) = 0$. Conversely, assume that $D(1) = 0$. We define $\phi: \mathcal{A} \times \mathcal{A} \to \mathcal{N}$ by $\phi(a, b) = D(ab) - D(a)b - aD(b)$ $(a, b \in \mathcal{A})$. Obviously, ϕ is a continuous bilinear map. If $ab = 0$, then it follows from the hypothesis that $\phi(a, b) = D(ab) - aD(b) - D(a)b = 0$, i.e. ϕ preserves zero product. Since $\mathcal{A}$ has the property $(\mathbb{B})$, $\phi(ab, c) = \phi(a, bc)$ for all $a, b, c \in \mathcal{A}$. Hence, $\phi(a, b) = \phi(1a, b) = \phi(1, ab) = D(ab) - D(1)ab - 1D(ab) = 0$ for all $a, b \in \mathcal{A}$ as a consequence $D(ab) = D(a)b + aD(b)$. So, D is a derivation.

We present an example about an algebra which is a domain with the property $(\mathbb{B})$. Suppose that $\mathcal{A}$ has the property $(\mathbb{B})$ and $\mathcal{J} \in P_P(\mathcal{A})$. By [(Alaminos, 2009), Proposition 2.4] $\frac{\mathcal{A}}{\mathcal{J}}$ has the property $(\mathbb{B})$. It is easy to see that $\frac{\mathcal{A}}{\mathcal{J}}$ is also a domain.

References

Alaminos, J., Brešar, M., Extremera, J., & Villena, A. R. (2009). Maps preserving zero products. *Studia Math*, 193, 131–159.

Bonsall, F. F., & Duncan, J. (1973). *Complete Normed Algebras*. New York: Springer.

Brešar, M., & Mathieu, M. (1995). Derivations mapping into the radical III. *J Func Anal*, 133, 21–29.

Bratteli, O., & Robinson, D. W. (1987). *Operator algebras and quantum statistical mechanics 1*. New York: Springer.

Bratteli, O., & Robinson, D. W. (1997). Operator algebras and quantum statistical mechanics 2. Berlin: Springer.

Dales, H. G. (2000). *Banach Algebras and Automatic Continuity*. New York: Oxford University Press.

Dales, H. G., Aiena, P., Eschmeier, J., Laursen, K., & Willis, G. A. (2003). *Introduction to Banach Algebras*, Operators and Harmonic Analysis. Cambridge University Press.

Hosseini, A., Hassani, M., & Niknam, A. (2011). Generalized σ-derivation on Banach algebras. *Bull Iranian Math Soc*, 37, 87–100.

Hosseini, A., Hassani, M., & Niknam, A. (2013). Some resultes on generalized σ-derivations. *Journal of advanced reaserch in pure mathematics*, 5, 61–72.

Hosseini, A., Hassani, M., Niknam, A., & Hejazian, S. (2011). Some results on σ-derivations. *Ann Funct Anal*, 2, 75–84.

Hejazian, S., Mirzavaziri, M., & Moslehian, M. S. (2005). n-Homomorphismss. *Bull Iranian Math Soc*, 31(1), 13–23.

Mathieu, M. (1994). Where to find the image of aderivation. *Functional analysis and operator theory*, 38, 237–249.

Mirzavaziri, M., & Moslehian, M. S. (2007). σ-derivations in Banach algebras. *Bull Iranian Math Soc*, 32, 65–78.

Mirzavaziri, M., & Moslehian, M. S. (2006). Automatic continuity of σ-derivations in C*-algebras. *Proc Amer Math Soc*, 134, 3312–3327.

Mirzavaziri, M. (2008). σ-derivations on $\mathbb{C}[x]$. Int J Algebra, 2, 571–580.

Murphy, G. J. (1990). *C^*-Algebras and Operator Theory. Boston*: Academic Press.

Thomas, M. P. (1988). The image of a derivation is contained in the radical. *Ann of math*, 128, 435–460.