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Entire sequence spaces defined on locally convex Hausdorff topological space

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Abstract

In this paper we introduce entire sequence spaces defined by a sequence of modulus functions $F = (f_k)$. We study some topological properties of these spaces and prove some inclusion relations.

Keywords: Modulus function; solid; monotone; entire sequences; paranormed space

1. Introduction

A modulus function is a function $f : [0, \infty) \to [0, \infty)$ such that
(1) $f(x) = 0$ if and only if $x = 0$,
(2) $f(x + y) \le f(x) + f(y)$ for all $x \ge 0, y \ge 0$,
(3) f is increasing,
(4) f is continuous from right at 0.

It follows that f must be continuous everywhere on $[0, \infty)$. The modulus function may be bounded or unbounded. For example, if we take $f(x) = \frac{x}{x+1}$, then $f(x)$ is bounded. If $f(x) = x^p, 0 < p < 1$, then the modulus function $f(x)$ is unbounded. Subsequently, modulus function has been discussed in [Altin and Et, (2005); Altinok et. al (2008); Altin (2006); Altin et. al (2006); Malkowsky and Savas (2000); Raj and Sharma (2011); Raj and Sharma (2011)].

Let X be a linear metric space. A function $p: X \to \mathbb{R}$ is called paranorm, if
(1) $p(x) \ge 0$, for all $x \in X$,
(2) $p(-x) = p(x)$, for all $x \in X$,
(3) $p(x + y) \le p(x) + p(y)$, for all $x, y \in X$,
(4) if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see Wilansky (1984), Theorem 10.4.2, p-183).

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Mursaleen and Noman (2011) introduced the notion of λ-convergent and λ-bounded sequences as follows:

Let $\lambda = (\lambda_k)_{k=1}^{\infty}$ be a strictly increasing sequence of positive real numbers tending to infinity, i.e. $0 < \lambda_0 < \lambda_1 < \cdots$ and $\lambda_k \to \infty$ as $k \to \infty$.

We say that a sequence $x = (x_k) \in w$ is λ-convergent to the number L, called the λ-limit of x, if $\Lambda_m(x) \to L$ as $m \to \infty$, where

$$\Lambda_m(x) = \frac{1}{\lambda_m}\sum_{k=1}^{m}(\lambda_k - \lambda_{k-1})x_k.$$

The sequence $x = (x_k) \in w$ is λ-bounded if $sup_m|\Lambda_m(x)| < \infty$. It is well known that if $\lim_m x_m = a$ in the ordinary sense, then

$$\lim_m \left(\frac{1}{\lambda_m} \left(\sum_{k=1}^{m} (\lambda_k - \lambda_{k-1})|x_k - a| \right) \right) = 0.$$

This implies that

$$\lim_m |\Lambda_m(x) - a| = \lim_m \left| \frac{1}{\lambda_m} \sum_{k=1}^{m} (\lambda_k - \lambda_{k-1})(x_k - a) \right| = 0$$

which yields that $\lim_m \Lambda_m(x) = a$ and hence $x = (x_k) \in w$ is λ-convergent to a. For more detailed study of these sequence spaces, see Mursaleen, M., Noman, A.K. (2012), Mursaleen, M., Noman, A.K. (2012) and Mursaleen, M., Karakaya, V., Polat, H., Simsek, N. (2011).

The space consisting of all those sequences x in w such that $f_k\left(\frac{(k!|x_k|)^{\frac{1}{k}}}{\rho}\right) \to 0$ as $k \to \infty$ for some arbitrary fixed $\rho > 0$ is denoted by Γ_F and is known as a space of entire sequences defined by a sequence of modulus functions. The space Γ_F is a metric space with the metric

$$d(x,y) = sup_k f_k\left(\frac{(k!\,|x_k - y_k|)^{\frac{1}{k}}}{\rho}\right)$$

For all $x=(x_k)$ and $y=(y_k)$ in Γ_F (see [Subramanian and Misra (2010)]).

A sequence space E is said to be solid or normal if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ and for all sequences of scalars (α_k) with $|\alpha_k| \le 1$ (see [Kamthan and Gupta (1981)]).

Let $F = (f_k)$ be a sequence of modulus functions and X be locally convex Hausdorff topological linear space whose topology is determined by a set of continuous seminorms q. The symbol $\Gamma(X)$ denotes the space of all entire sequences defined over X. If $p=(p_k)$ be a bounded sequence of positive real numbers, then we define the following sequence spaces:

$$\Gamma_F(\Lambda,p,q) = \left\{x \,\epsilon\, \Gamma(X): \frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q\left(\frac{(k!|\Lambda_k(x)|)^{\frac{1}{k}}}{\rho}\right)\right)\right]^{p_k} \to 0 \; as\; n\to\infty, for\; some\; \rho>0\right\}.$$

For $p_k = 1$ for each k, we get

$$\Gamma_F(\Lambda,\ q) = \left\{x \,\epsilon\, \Gamma(X): \frac{1}{n}\sum_{k=1}^{n} f_k\left(q\left(\frac{(k!|\Lambda_k(x)|)^{\frac{1}{k}}}{\rho}\right)\right) \to 0 \; as\; n\to\infty, for\; some\; \rho>0\right\}.$$

The following inequality will be used throughout the paper. Let $p=(p_k)$ be a sequence of positive real numbers with $0 \le p_k \le \sup p_k = G$, $K = \max(1,\ 2^{G-1})$ then

$$|a_k + b_k|^{p_k} \le K\,\{|a_k|^{p_k} + |b_k|^{p_k}\} \qquad (1)$$

For all k and $a_k, b_k \in \mathbb{C}$. Also $|a_k|^{p_k} \le \max(1,|a|^G)$ for all $a \in \mathbb{C}$.

We examine some topological properties of our newly defined sequence spaces and prove inclusion relation between the spaces $\Gamma_F(\Lambda,p,q)$ and $\Gamma_F(\Lambda, q)$.

2. Some properties of the spaces $\Gamma_F(\Lambda,p,q)$ and $\Gamma_F(\Lambda,\ q)$

Theorem 2.1. Let $F=(f_k)$ be a sequence of modulus functions and $p=(p_k)$ be a bounded sequence of positive real numbers. Then the space $\Gamma_F(\Lambda,p,q)$ is a linear over the field of complex numbers $\mathbb{C}$.

Proof: Let $x = (x_k)$, $y=(y_k) \in \Gamma_F(\Lambda,p,q)$ and $\alpha,\beta \in \mathbb{C}$. In order to prove the result, we need to find some $\rho_3 > 0$ such that

$$\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q\left(\frac{(k!|\Lambda_k(\alpha x+\beta y)|)^{\frac{1}{k}}}{\rho_3}\right)\right)\right]^{p_k} \to 0\; as\; n\to\infty. \qquad (2)$$

Since $x=(x_k), y=(y_k) \in \Gamma_F(\Lambda,p,q)$, there exist some positive real numbers ρ_1 and ρ_2 such that

$$\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q\left(\frac{(k!|\Lambda_k(x)|)^{\frac{1}{k}}}{\rho_1}\right)\right)\right]^{p_k} \to 0\; as\; n\to\infty. \qquad (3)$$

and

$$\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q\left(\frac{(k!|\Lambda_k(y)|)^{\frac{1}{k}}}{\rho_2}\right)\right)\right]^{p_k} \to 0\; as\; n\to\infty. \qquad (4)$$

Since $F = (f_k)$ is a non-decreasing function, q is a seminorm, then

$$\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q\left(\frac{(k!|\Lambda_k(\alpha x+\beta y)|)^{\frac{1}{k}}}{\rho_3}\right)\right)\right]^{p_k} \le \frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q\left(\frac{|\alpha|^{\frac{1}{k}}(k!|\Lambda_k(x)|)^{\frac{1}{k}}}{\rho_3} + \frac{|\beta|^{\frac{1}{k}}(k!|\Lambda_k(y)|)^{\frac{1}{k}}}{\rho_3}\right)\right)\right]^{p_k}$$

so that

$$\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q\left(\frac{(k!|\Lambda_k(\alpha x+\beta y)|)^{\frac{1}{k}}}{\rho_3}\right)\right)\right]^{p_k} \le \frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q\left(\frac{|\alpha|(k!|\Lambda_k(x)|)^{\frac{1}{k}}}{\rho_3} + \frac{|\beta|(k!|\Lambda_k(y)|)^{\frac{1}{k}}}{\rho_3}\right)\right)\right]^{p_k}.$$

Take $\rho_3 > 0$ such that $\frac{1}{\rho_3} = \min\left\{\frac{1}{|\alpha|\rho_1}, \frac{1}{|\beta|\rho_2}\right\}$ and by using inequality (1), we get

$$\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q\left(\frac{(k!|\Lambda_k(\alpha x+\beta y)|)^{\frac{1}{k}}}{\rho_3}\right)\right)\right]^{p_k} \le \frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q\left(\frac{(k!|\Lambda_k(x)|)^{\frac{1}{k}}}{\rho_1} + \frac{(k!|\Lambda_k(y)|)^{\frac{1}{k}}}{\rho_2}\right)\right)\right]^{p_k}$$

$$\le \frac{1}{n}\sum_{k=1}^{n}\left[\left[f_k\left(q\left(\frac{(k!\,|\Lambda_k(x)|)^{\frac{1}{k}}}{\rho_1}\right)\right)\right]^{p_k} + \left[f_k\left(q\left(\frac{(k!\,|\Lambda_k(y)|)^{\frac{1}{k}}}{\rho_2}\right)\right)\right]^{p_k}\right]$$

$$\le K\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q\left(\frac{(k!\,|\Lambda_k(x)|)^{\frac{1}{k}}}{\rho_1}\right)\right)\right]^{p_k} + K\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q\left(\frac{(k!\,|\Lambda_k(y)|)^{\frac{1}{k}}}{\rho_2}\right)\right)\right]^{p_k}$$

$$\to 0\; as\; n \to \infty\;\; [\,by\; using\; (3) and\; (4)].$$

Hence

$$\sum_{k=1}^{n}\left[f_k\left(q\left(\frac{(k!\,|\alpha\Lambda_k(x)+\beta\Lambda_k(y)|)^{\frac{1}{k}}}{\rho_3}\right)\right)\right]^{p_k} \to 0 \ as\ n \to \infty.$$

This proves that $\Gamma_F(\Lambda,p,q)$ is a linear space. This completes the proof of the theorem.

Theorem 2.2. Let $F=(f_k)$ be a sequence of modulus functions and $p=(p_k)$ be a bounded sequence of positive real numbers. $\Gamma_F(\Lambda,p,q)$ is a paranormed space with paranorm defined by

$$g(x)=\inf\left\{\rho^{\frac{p_m}{H}} : sup_{k\geq 1}\left[f_k\left(q\left(\frac{(k!|\Lambda_k(x)|)^{\frac{1}{k}}}{\rho}\right)\right)\right]^{p_k}\leq 1,\ \rho>0,\ m\in\mathbb{N}\right\}, where\ H=\max(1, sup_k\, p_k).$$

Proof: Clearly $g(x)\geq 0$, $g(x)=g(-x)$ and $g(\theta)=0$, where θ is the zero sequence of X. Let $(x_k),(y_k)\in\Gamma_F(\Lambda,p,q)$. Let $\rho_1,\rho_2>0$ be such that

$$sup_{k\geq 1}\left[f_k\left(q\left(\frac{(k!\,|\Lambda_k(x)|)^{\frac{1}{k}}}{\rho_1}\right)\right)\right]^{p_k}\leq 1$$

and

$$sup_{k\geq 1}\left[f_k\left(q\left(\frac{(k!\,|\Lambda_k(y)|)^{\frac{1}{k}}}{\rho_2}\right)\right)\right]^{p_k}\leq 1.$$

Let $\rho=\rho_1+\rho_2$. Then by using Minkowski's inequality, we have

$$sup_{k\geq 1}\left[f_k\left(q\left(\frac{(k!\,|\Lambda_k(x+y)|)^{\frac{1}{k}}}{\rho}\right)\right)\right]^{p_k}$$
$$\leq\left(\frac{\rho_1}{\rho_1+\rho_2}\right) sup_{k\geq 1}\left[f_k\left(q\left(\frac{(k!\,|\Lambda_k(x)|)^{\frac{1}{k}}}{\rho_1}\right)\right)\right]^{p_k}$$
$$+\left(\frac{\rho_1}{\rho_1+\rho_2}\right) sup_{k\geq 1}\left[f_k\left(q\left(\frac{(k!\,|\Lambda_k(y)|)^{\frac{1}{k}}}{\rho_2}\right)\right)\right]^{p_k}\leq 1.$$

Hence

$$g(x+y)\leq \inf\left\{(\rho_1+\rho_2)^{\frac{p_m}{H}} : sup_{k\geq 1}\left[f_k\left(q\left(\frac{(k!\,|\Lambda_k(x)|)^{\frac{1}{k}}}{\rho_1+\rho_2}\right)\right)\right]^{p_k}\leq 1,\quad \rho_1,\rho_2>0; m\in\mathbb{N}\right\}$$

$$\leq \inf\left\{(\rho_1)^{\frac{p_m}{H}} : sup_{k\geq 1}\left[f_k\left(q\left(\frac{(k!\,|\Lambda_k(x)|)^{\frac{1}{k}}}{\rho_1}\right)\right)\right]^{p_k}\leq 1,\ \rho_1>0; m\in\mathbb{N}\right\}\lambda$$
$$+\inf\left\{(\rho_2)^{\frac{p_m}{H}} : sup_{k\geq 1}\left[f_k\left(q\left(\frac{(k!\,|\Lambda_k(y)|)^{\frac{1}{k}}}{\rho_2}\right)\right)\right]^{p_k}\leq 1,\ \rho_2>0; m\in\mathbb{N}\right\}.$$

Thus we have $g(x+y)\leq g(x)+g(y)$. Hence g satisfies the triangle inequality

$$g(\mu x)=\inf\left\{(\rho)^{\frac{p_m}{H}} : sup_{k\geq 1}\left[f_k\left(q\left(\frac{(k!\,|\mu\Lambda_k(x)|)^{\frac{1}{k}}}{\rho}\right)\right)\right]^{p_k}\leq 1,\quad \rho>0; m\in\mathbb{N}\right\}$$
$$=\inf\left\{(r|\mu|)^{\frac{p_m}{H}} : sup_{k\geq 1}\left[f_k\left(q\left(\frac{(k!\,|\Lambda_k(x)|)^{\frac{1}{k}}}{r}\right)\right)\right]^{p_k}\leq 1,\quad r>0; m\in\mathbb{N}\right\},$$

where $r=\frac{\rho}{|\lambda|}$. Hence $\Gamma_F(\Lambda,p,q)$ is a paranormed space. This completes the proof of the theorem.

Theorem 2.3. Let $F'=(f'_k)$ and $F''=(f''_k)$ be two sequences of modulus functions. Then

$$\Gamma_{F'}(\Lambda,p,q)\bigcap\Gamma_{F''}(\Lambda,p,q)\subseteq\Gamma_{F'+F''}(\Lambda,p,q).$$

Proof: Let $x=(x_k)\in\Gamma_{F'}(\Lambda,p,q)\cap\Gamma_{F''}(\Lambda,p,q)$. Then there exist ρ_1 and ρ_2 such that

$$\frac{1}{n}\sum_{k=1}^{n}\left[f'_k\left(q\left(\frac{(k!|\Lambda_k(x)|)^{\frac{1}{k}}}{\rho_1}\right)\right)\right]^{p_k}\to 0\ as\ n\to\infty \qquad (5)$$

and

$$\frac{1}{n}\sum_{k=1}^{n}\left[f''_k\left(q\left(\frac{(k!|\Lambda_k(y)|)^{\frac{1}{k}}}{\rho_2}\right)\right)\right]^{p_k}\to 0\ as\ n\to\infty. \qquad (6)$$

Since $\rho>0$ such that $\frac{1}{\rho}=\min\left(\frac{1}{\rho_1},\frac{1}{\rho_2}\right)$. Then we have

$$\frac{1}{n}\sum_{k=1}^{n}\left[f'_k+f''_k\left(q\left(\frac{(k!|\Lambda_k(x)|)^{\frac{1}{k}}}{\rho}\right)\right)\right]^{p_k}\leq$$
$$K\left[\frac{1}{n}\sum_{k=1}^{n}\left[f'_k\left(q\left(\frac{(k!|\Lambda_k(x)|)^{\frac{1}{k}}}{\rho_1}\right)\right)\right]^{p_k}\right]+$$
$$K\left[\frac{1}{n}\sum_{k=1}^{n}\left[f''_k\left(q\left(\frac{(k!|\Lambda_k(x)|)^{\frac{1}{k}}}{\rho_2}\right)\right)\right]^{p_k}\right]$$
$$\to 0\ as\ n\to\infty\ [\,by\ using\ (5) and\ (6)].$$

Then

$$\frac{1}{n}\sum_{k=1}^{n}\left[(f'_k+f''_k)\left(q\left(\frac{(k!\,|\Lambda_k(x)|)^{\frac{1}{k}}}{\rho}\right)\right)\right]^{p_k}\to 0\ as\ n\to\infty.$$

Therefore $x = (x_k) \in \Gamma_{F'+F''}(\Lambda, p, q)$. This completes the proof of the theorem.

Theorem 2.4. Let $0 \le p_k \le r_k$ and let $\left\{\frac{r_k}{p_k}\right\}$ be bounded. Then $\Gamma_F(\Lambda, r, q) \subset \Gamma_F(\Lambda, p, q)$.

Proof: Let $x = (x_k) \in \Gamma_F(\Lambda, r, q)$. Then

$$\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q\left(\frac{(k!|\Lambda_k(x)|)^{\frac{1}{k}}}{\rho}\right)\right)\right]^{r_k} \to 0 \; as \; n \to \infty. \quad (7)$$

Let $t_k = \frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q\left(\frac{(k!|\Lambda_k(x)|)^{\frac{1}{k}}}{\rho}\right)\right)\right]^{q_k}$ and $\lambda_k = \frac{p_k}{r_k}$. Since $p_k \le r_k$, we have $0 \le \lambda_k \le 1$. Take $0 < \lambda < \lambda_k$. Define

$$u_k = \begin{cases} t_k, if \; t_k \ge 1 \\ 0, if \; t_k < 1 \end{cases}$$

and

$$v_k = \begin{cases} 0, if \; t_k \ge 1 \\ t_k, if \; t_k < 1 \end{cases}$$

$t_k = u_k + v_k$, $t_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}$. It follows that $u_k^{\lambda_k} \le u_k \le t_k$, $v_k^{\lambda_k} \le v_k^{\lambda}$. Since $t_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}$, then $t_k^{\lambda_k} = t_k + v_k^{\lambda_k}$

$$\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q\left(\frac{(k!\,|\Lambda_k(x)|)^{\frac{1}{k}}}{\rho}\right)\right)^{r_k}\right]^{\lambda_k}$$

$$\le \frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q\left(\frac{(k!\,|\Lambda_k(x)|)^{\frac{1}{k}}}{\rho}\right)\right)\right]^{r_k}$$

$$\Rightarrow \frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q\left(\frac{(k!\,|\Lambda_k(x)|)^{\frac{1}{k}}}{\rho}\right)\right)^{r_k}\right]^{\frac{p_k}{r_k}}$$

$$\le \frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q\left(\frac{(k!\,|\Lambda_k(x)|)^{\frac{1}{k}}}{\rho}\right)\right)\right]^{r_k}$$

$$\Rightarrow \frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q\left(\frac{(k!\,|\Lambda_k(x)|)^{\frac{1}{k}}}{\rho}\right)\right)\right]^{p_k}$$

$$\le \frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q\left(\frac{(k!\,|\Lambda_k(x)|)^{\frac{1}{k}}}{\rho}\right)\right)\right]^{r_k}.$$

But

$$\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q\left(\frac{(k!|\Lambda_k(x)|)^{\frac{1}{k}}}{\rho}\right)\right)\right]^{r_k} \to 0 \; as \; n \to \infty \;\; (by\,(7)).$$

Therefore

$$\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q\left(\frac{(k!\,|\Lambda_k(x)|)^{\frac{1}{k}}}{\rho}\right)\right)\right]^{p_k} \to 0 \; as \; n \to \infty.$$

Hence $x = (x_k) \in \Gamma_F(\Lambda, p, q)$. From (7), we get $\Gamma_F(\Lambda, r, q) \subset \Gamma_F(\Lambda, p, q)$. This completes the proof of the theorem.

Theorem 2.5. (i) Let $0 < \inf p_k \le p_k \le 1$. Then $\Gamma_F(\Lambda, p, q) \subset \Gamma_F(\Lambda, q)$,
(ii) Let $1 \le p_k \le \sup p_k < \infty$. Then $\Gamma_F(\Lambda, q) \subset \Gamma_F(\Lambda, p, q)$.

Proof: (i) Let $x = (x_k) \in \Gamma_F(\Lambda, p, q)$. Then

$$\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q\left(\frac{(k!|\Lambda_k(x)|)^{\frac{1}{k}}}{\rho}\right)\right)\right]^{p_k} \to 0 \; as \; n \to \infty. \quad (8)$$

Since $0 < \inf p_k \le p_k \le 1$,

$$\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q\left(\frac{(k!|\Lambda_k(x)|)^{\frac{1}{k}}}{\rho}\right)\right)\right] \le$$

$$\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q\left(\frac{(k!|\Lambda_k(x)|)^{\frac{1}{k}}}{\rho}\right)\right)\right]^{p_k} \to 0 \; as \; n \to \infty. \quad (9)$$

From (8) and (9), it follows that $x = (x_k) \in \Gamma_F(\Lambda, q)$. Thus $\Gamma_F(\Lambda, p, q) \subset \Gamma_F(\Lambda, q)$.
(ii) Let $p_k \ge 1$ for each k and $\sup p_k < \infty$ and let $x = (x_k) \in \Gamma_F(\Lambda, q)$. Then

$$\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q\left(\frac{(k!|\Lambda_k(x)|)^{\frac{1}{k}}}{\rho}\right)\right)\right] \to 0 \; as \; n \to \infty. \quad (10)$$

Since $1 \le p_k \le \sup p_k < \infty$, we have

$$\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q\left(\frac{(k!\,|\Lambda_k(x)|)^{\frac{1}{k}}}{\rho}\right)\right)\right]^{p_k} \le \frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q\left(\frac{(k!\,|\Lambda_k(x)|)^{\frac{1}{k}}}{\rho}\right)\right)\right].$$

Thus

$$\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q\left(\frac{(k!\,|\Lambda_k(x)|)^{\frac{1}{k}}}{\rho}\right)\right)\right]^{p_k} \to 0 \; as \; n \to \infty.$$

This implies that $x = (x_k) \in \Gamma_F(\Lambda, p, q)$. Therefore $\Gamma_F(\Lambda, q) \subset \Gamma_F(\Lambda, p, q)$. This completes the proof of the theorem.

Theorem 2.6. Suppose $\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q\left(\frac{(k!|\Lambda_k(x)|)^{\frac{1}{k}}}{\rho}\right)\right)\right]^{p_k} \le |x_k|^{\frac{1}{k}}$, then $\Gamma \subset \Gamma_F(\Lambda, p, q)$.

Proof: Let $x = (x_k) \in \Gamma$. Then we have,

$$|x_k|^{\frac{1}{k}} \to 0 \; as \; k \to \infty. \quad (11)$$

But $\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q\left(\frac{(k!|\Lambda_k(x)|)^{\frac{1}{k}}}{\rho}\right)\right)\right]^{p_k} \leq |x_k|^{\frac{1}{k}}$, by our assumption it implies that

$$\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q\left(\frac{(k!\,|\Lambda_k(x)|)^{\frac{1}{k}}}{\rho}\right)\right)\right]^{p_k} \to 0\ as\ n \to \infty. \qquad [by\ (11)]$$

Then $x = (x_k) \in \Gamma_F(\Lambda, p, q)$ and $\Gamma \subset \Gamma_F(\Lambda, p, q)$.

Theorem 2.7. $\Gamma_F(\Lambda, p, q)$ is solid.

Proof: Let $|x_k| \leq |y_k|$ and let $y = (y_k) \in \Gamma_F(\Lambda, p, q)$, because $F = (f_k)$ is non-decreasing

$$\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q\left(\frac{(k!\,|\Lambda_k(x)|)^{\frac{1}{k}}}{\rho}\right)\right)\right]^{p_k} \leq \frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q\left(\frac{(k!\,|\Lambda_k(y)|)^{\frac{1}{k}}}{\rho}\right)\right)\right]^{p_k}.$$

Since $y = (y_k) \in \Gamma_F(\Lambda, p, q)$,

$$\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q\left(\frac{(k!\,|\Lambda_k(y)|)^{\frac{1}{k}}}{\rho}\right)\right)\right]^{p_k} \to 0\ as\ n \to \infty$$

and so that

$$\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q\left(\frac{(k!\,|\Lambda_k(x)|)^{\frac{1}{k}}}{\rho}\right)\right)\right]^{p_k} \to 0\ as\ n \to \infty.$$

Therefore $x = (x_k) \in \Gamma_F(\Lambda, p, q)$.

Theorem 2.8. $\Gamma_F(\Lambda, p, q)$ is monotone.

Proof: It is trivial so we omit it.

References

Altin, Y., & Et, M. (2005). Generalized difference sequence spaces defined by a modulus function in a locally convex space. *Soochow J. Math*., 31, 233–243.

Altinok, H., Et, M., & Altin, Y. (2008). The sequence space $BV_\sigma(M, P, Q, S)$ on seminormed spaces. *Indian J. Pure Appl. Math*., 39, 49–58.

Altin, Y. (2006). Properties of some sets of sequences defined by a modulus function. *Acta Math. Sci. Ser. B Engl. Ed*., 29, 427–434.

Altin, Y., Altinok, & H., Colak, R. (2006). On some seminormed sequence spaces defined by a modulus function. *Kragujevac J. Math*., 29, 121–132.

Kamthan, P. K., & Gupta, M. (1981). *Sequence spaces and series, Lecture Notes in Pure and Applied Mathematics*. 65, Marcel Dekker, Inc., New York.

Malkowsky, E., & Savas, E. (2000). Some λ- sequence spaces defined by a modulus. *Arch. Math*., 36, 219–228.

Mursaleen, M., Karakaya, V., Polat, H., & Simsek, N. (2011). Measure of noncompactness of matrix operators on some difference sequence spaces of weighted means. *Comput. Math. Appl*., 62, 814–820.

Mursaleen, M., & Noman, A. K. (2011). On some new sequence spaces of non absolute type related to the spaces l_p and l_∞ I. *Filomat*, 25, 33–51.

Mursaleen, M., & Noman, A. K. (2011). On some new sequence spaces of non absolute type related to the spaces l_p and l_∞ II. *Math. Commun*., 16, 383–398.

Mursaleen, M., & Noman, A. K. (2012). Compactness of matrix operators on some new difference sequence spaces. *Linear Algebra Appl*., 436(1), 41--52.

Raj, K., & Sharma, S. K. (2011). Difference sequence spaces defined by sequence of modulus function. *Proyecciones J. Math*., 30, 180–199.

Raj, K., & Sharma, S. K. (2011). Some difference sequence spaces defined by sequence of modulus function. *Int. J. Math. Archive*, 2, 236–240.

Subramanian, N., & Misra, U. K. (2010). The semi normed space defined by a double gai sequence of modulus function. *Fasciculi Math*., 46, 111–120.

Wilansky, A. (1984). *Summability through Functional Analysis*. North-Holland Math. Stud.