IJST (2013) 37A4: 501-512
Iranian Journal of Science & Technology

http://ijsts.shirazu.ac.ir

A novel hybrid spectral-variational iteration method (H-S-VIM) for solving nonlinear equations arising in heat transfer

M. Heydari[1]*, G. B. Loghmani[1], S. M. Hosseini[2] and A. Yildirim[3]

[1]*Department of Mathematics, Yazd University P.O. Box: 89195-741 Yazd, Iran*
[2]*Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran*
[3]*Department of Mathematics, Science Faculty, Ege University, 35100 Bornova-Izmir, Turkey*
E-mail: m.heydari85@gmail.com

Abstract

The purpose of this study is to implement a new modification of the variational iteration method (H-S-VIM), which is a combination of spectral method and variational iteration method for heat transfer problems with high nonlinearity order. The merit of this method is that it does not require the solution of any linear or nonlinear system of equations unlike spectral method. Furthermore the proposed method is easy to implement and computationally very attractive. Here, H-S-VIM is used to solve an unsteady nonlinear convective-radiative equation containing two small parameters, ε_1 and ε_2. It is observed that H-S-VIM may be implemented on other strongly nonlinear models of physical nature.

Keywords: Heat transfer; spectral method; variational iteration method; gauss quadrature integration method

1. Introduction

Nonlinear phenomena in real world, that appear in many applications in scientific fields, such as fluid dynamics, solid state physics, electricity and magnetism, geophysics, plasma physics, kinetic theory of gases, quantum mechanics, mathematical economics, can be modeled by nonlinear differential equations [1-3]. Except in a limited number of these problems, we have difficulty in finding their exact analytical solutions.
Developing of new semi-analytical and numerical methods for approximation of the solutions of strongly nonlinear differential equations is an interesting research area of many engineers and mathematicians.

In recent years, such techniques like Adomian decomposition method (ADM) [4], the variational iteration method (VIM) [5, 6], the homotopy perturbation method (HPM) [7, 8], homotopy analysis methds (HAM) [9, 10] and the tanh method [11, 12] have drawn great attention from scientists and engineers.

The variational iteration method (VIM) is powerful in investigating approximate or analytical solutions of nonlinear ordinary and partial differential equations. This method is proposed by the Chinese mathematician He [5, 13] as a modification of a general Lagrange multiplier method proposed by Inokuti et al. in 1978 [14]. The main concepts in VIM, such as general Lagrange multiplier, restrictive variation, correction functional, are explained heuristically. Using the VIM we can find the exact solution of the given problem. The VIM has played an important role in recent researches for solving various kinds of problems. In this method the linear and nonlinear structures are handled in a similar manner without any need for restrictive assumptions.

The method is used successfully in several well-known problems such as delay differential equations [15], autonomous ordinary differential systems [16], Burger's and coupled Burger's equations [17], integro-differential equations [18], Helmholtz equations [19] and many other problems [20-27]. The convergence of the method is systematically discussed by Tatari and Dehghan [24]. Comparison of the method with the Adomian method was made by many authors via illustrative examples. Wazwaz in particular, gave a complete comparison between the two methods [26], revealing the variational iteration method has many merits over the Adomian method. It can completely overcome the difficulty arising in the calculation of the Adomian polynomial. Though the variational iteration method leads to fast convergent solutions, unnecessary calculation arises in the solution procedure.

In order to accelerate the convergent rate, various modifications have been suggested, for example,

*Corresponding author
Received: 3 October 2012 / Accepted: 18 December 2012

variational iteration-Padé method [28], variational iteration-Adomian method [29], variational iteration-differential transform method [30]. Abassy et al. suggested another effective modification [31]. Heydari et al. in [32], presented an efficient modification of the variational iteration method for solving boundary value problems using the Chebyshev polynomials.

In this paper a new modification of variational iteration method is proposed to find analytic approximate solution of heat transfer problems with high nonlinearity order [33]. This method is a combination of spectral method and variational iteration method, namely, hybrid spectral-variational iteration method (H-S-VIM).

Recently many analytical methods have been used to solve nonlinear equations arising in heat transfer. Ganji [34] used the homotopy perturbation method (HPM) to nonlinear equations arising in heat transfer. Tari et al. [35] obtained an approximate analytical solution of the nonlinear equations arising in heat transfer by usin VIM. Abbasbandy [36, 37] applied homotopy analysis method (HAM) to solve nonlinear equation arising in heat transfer and shown that the solutions obtained by HPM [34] and perturbation method are only special cases of the HAM solution. Marinca et al. [38] use optimal homotopy analysis method (OHAM) for solving nonlinear equations arising in heat transfer. Yaghoobi et al. in [39], applied differential transformation method (DTM) to nonlinear equations arising in heat transfer.

The organization of this paper is as follows. The spectral method and variational iteration method (VIM) are introduced in Section 2 and 3, respectively. Some necessary definitions and mathematical preliminaries of the Chebysheve polynomials are introduced in Section 4. In Section 5, the hybrid spectral-variational iteration method (H-S-VIM) is presented. In Section 6, VIM and H-S-VIM are applied on nonlinear equations arising in heat transfer and comparisons between the VIM, HPM, HAM, DTM, H-S-VIM and the exact solution are presented. Section 7 ends this work with a brief conclusion.

2. spectral method

Spectral methods, in the context of numerical schemes for differential equations, belong to the family of weighted residual methods (WRMs), which are traditionally regarded as the foundation of many numerical methods such as finite element, spectral, finite volume and boundary element [40].

The base of spectral methods to solve differential equations is to expand the solution function as a finite series of very smooth basis function, as given

$$u^M(x) = \sum_{i=0}^{M} u_i \phi_i(x), \quad (1)$$

in which, the best choice of $\phi_i(x)$ are the eigenfunctions of a singular Sturm-Liouville problem, for example, Chebyshev or Legendre polynomials. If the function $u(x)$ belongs to $C^{\infty}[a, b]$, the produced error of approximation (1), when M tends to infinity, approaches to zero with exponential rate [41]. This phenomenon is usually referred to as spectral accuracy [42].

Remark 2.1. In the spectral methods for solving a differential equation, the problem of obtaining approximate solution by solving a system of algebraic equations is equivalent. Solving system of algebraic equations in general is not easy. This limitation is more apparent when M is a large number.

3. Variational iteration method (VIM)

To illustrate the procedure of this approach, we consider the following general differential equation:

$$Lu(x) + Nu(x) = g(x), \quad (2)$$

where L is a linear operator, N is a nonlinear operator and $g(x)$ is an inhomogeneous term. Then, we can construct a correction functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(Lu_n(t) + N\tilde{u}_n(t) - g(t))dt, \quad (3)$$

where λ is a general Lagrange multiplier which can be identified optimally via variational theory [17, 13]. Here $\tilde{u}_n$ is considered as a restricted variation which means $\delta\tilde{u}_n = 0$ [13]. Therefore, we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. The successive approximation $u_{n+1}(x)$, $n \geq 0$ of the solution $u(x)$ will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function $u_0(x)$. The zeroth approximation $u_0(x)$ may be selected from any function that just satisfies, at least, the initial and boundary conditions. With λ determined, several approximations $u_{n+1}(x)$, $n \geq 0$ follow immediately. Consequently, the exact solution may be obtained as,

$$u(x) = \lim_{n\to\infty} u_n(x). \quad (4)$$

4. Some preliminaries

4.1. Properties of Chebyshev polynomials

The well known Chebyshev polynomials of the first kind [43] of degree n are defined on the interval

$[-1,1]$ as

$$T_n(x) = \cos(n\arccos(x)). \quad (5)$$

Obviously $T_0(x) = 1, T_1(x) = x$ and they satisfy the recurrence relations:

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n = 1,2,\cdots. \quad (6)$$

Square integrable function $u(x)$ in $[-1,1]$, may be expressed in terms of Chebyshev polynomials as

$$u(x) = \sum_{j=0}^{\infty} u_j T_j(x), \quad (7)$$

where the coefficients u_j are given by

$$u_j = \frac{(u(x),T_j(x))_w}{(T_j(x),T_j(x))_w}, \quad j = 0,1,2,\cdots. \quad (8)$$

Here, $w(x) = \frac{1}{\sqrt{1-x^2}}$ and $(.,.)_w$ is the inner product of $L_w^2(-1,1)$.

Definition 4.1. [44] Let X be a normed linear space, let $u(x)$ in X be given, and let Y be a given subspace of X.

1. An approximation $u^*(x)$ in Y is said to be good (or acceptable) if

$$\left||u(x) - u^*(x)|\right| \le \varepsilon, \quad (9)$$

where ε is a prescribed level of absolute accuracy.

2. An approximation $u_B^*(x)$ in Y is a best approximation if, for any other approximation $u^*(x)$ in Y,

$$||u(x) - u_B^*(x)|| \le ||u(x) - u^*(x)||. \quad (10)$$

Theorem 4.2. [41] Let $u(x) \in H^k(-1,1)$ (Sobolev space), $u^M(x) = \sum_{j=0}^{M} u_j T_j(x)$ be the best approximation polynomial of $u(x)$ in L_w^2-norm, then

$$||u(x) - u^M(x)||_{L_w^2[-1,1]} \le C_0 M^{-k} ||u(x)||_{H_w^k(-1,1)}, \quad (11)$$

where C_0 is a positive constant, which depends on selected norm and is independent of $u(x)$ and M.

We choose the grid (interpolation) points to be the extrema

$$x_i = -\cos\left(\frac{i\pi}{M}\right) = \cos\left(\frac{(M-i)\pi}{M}\right), \quad i = 0,1,\cdots,M, \quad (12)$$

of the Mth order Chebyshev polynomial $T_M(x)$. These grids, $x_0 = -1 < x_1 < \cdots < x_{M-1} < x_M = 1$ are also viewed as the zeros of $(1-x^2)\dot{T}(x)$, where $\dot{T}_M(x) = \frac{dT_M(x)}{dx}$. Clenshaw and Curtis [45] introduced the following approximation of the function $u(x)$,

$$u^M(x) \simeq \sum_{j=0}^{M} \tilde{u}_j T_j(x), \quad (13)$$

where $\tilde{u}_j$ are the Chebyshev coefficients which are determined by the formulations

$$\tilde{u}_j = \frac{2}{M\tilde{c}_j}\sum_{i=0}^{M} \frac{1}{\tilde{c}_i} u(x_i)\cos\left(\frac{\pi(M-i)j}{M}\right) = \frac{2(-1)^j}{M\tilde{c}_j}\sum_{i=0}^{M} \frac{1}{\tilde{c}_i} u(x_i)\cos\left(\frac{\pi ij}{M}\right), \quad j = 0,1,\cdots,M, \quad (14)$$

and

$$\tilde{c}_j = \begin{cases} 2, & j = 0, M, \\ 1, & 1 \le j \le M-1. \end{cases} \quad (15)$$

Remark 4.3. This paper discusses using Chebyshev polynomials of the first kind to approximate functions on finite interval, that is, on the interval $[-1,1]$. Practically, other polynomials, which are orthogonal on finite interval, can also be applied for approximating functions. But the partial sums of a first-kind Chebyshev expansion of a continuous function in $[-1,1]$, converge faster than the partial sums of an expansion in any other orthogonal polynomials [44].

4.2. Legendre-Gauss nodes and weights

Let $L_{p+1}(x)$ be the Legendre polynomial of order $p+1$ on $[-1,1]$. Then the Legendre-Gauss nodes are

$$-1 < \xi_0 < \xi_1 < \cdots < \xi_p < 1, \quad (16)$$

where $\{\xi_i\}_{i=0}^{p}$ are the zeros of $L_{p+1}(x)$. No explicit formulas are known for the points ξ_i, and so they are computed numerically using subroutines [46]. Also, we approximate the integral of $f(x)$ on $[-1,1]$ as

$$\int_{-1}^{1} f(x)dx \simeq \sum_{i=0}^{p} w_i f(\xi_i) \quad (17)$$

where ξ_i are Legendre-Guass nodes in (16) and the weights w_i are given in [41] as follows:

$$w_i = \frac{2}{(1-\xi_i^2)[L'_{p+1}(\xi_i)]^2}, \quad i = 0,1,\cdots,p. \quad (18)$$

It is well known [46] that the integration in (17) is exact whenever $f(x)$ is a polynomial of degree $\le 2p+1$.

5. Analysis of the H-S-VIM

In this section, we present a new modified algorithm of the variational iteration method with the help of spectral method and Gauss quadrature integration method.

Consider the nonlinear differential equation,

$$Lu(x) + Nu(x) = g(x), \quad 0 < x \le T, \quad (19)$$

where L is a linear operator, N is a nonlinear

operator and $g(x)$ is a known analytic function, subject to the initial conditions,

$$u^{(k)}(0) = \gamma_k, \quad k = 0,1,\cdots,m-1, \tag{20}$$

where γ_k's are real numbers. According to variational iteration method, we obtain the following iteration formula for (19) as,

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(x,t)(Lu_n(t) + Nu_n(t) - g(t))dt, \tag{21}$$

where λ is a general Lagrange multiplier which can be identified optimally via variational theory. Here, according to initial condition (20), we can select the initial approximation $u_0(x)$.

Remark 5.1. For variational iteration method, the key is the identification of Lagrange multiplier. For linear problems, their exact solutions can be obtained by only one iteration step due to the fact that the Lagrange multiplier can be exactly identified. For nonlinear problems, the lagrange multiplier is difficult to identify exactly. To overcome the difficulty, we apply restricted variations to nonlinear term. Due to the approximate identification of the Lagrange multiplier, the approximate solutions converge to their exact solutions relatively slowly. It should be specially pointed out that the more accurate the identification of the multiplier, the faster the approximations converge to their exact solutions and the higher accuracy can be obtained.

Remark 5.2. It should be noted that in computation of integral in (21) two difficulties may arise:
i) The nonlinear operator N, Lagrange multiplier λ and nonhomogeneous part may be ill-conditioned such that the integration becomes very complicated.
ii) By increasing n the number of terms of approximate solution may increase so rapidly that the integration becomes both complicated and time consuming.

So, to overcome these problems a new technique is presented here.

At first, based on initial condition, the initial approximation $u_0(x)$ is selected. By using iteration formula (21), we have

$$u_1(x) = u_0(x) + \int_0^x \lambda(x,t)\big(Lu_0(t) + Nu_0(t) - g(t)\big)dt. \tag{22}$$

From (13) and (14), the function $u_1(x)$ on $[0,T]$ can be approximated as

$$u_1(x) \simeq u_1^M(x) = \sum_{j=0}^M \tilde{u}_{1j} T_j\left(\frac{2}{T}x - 1\right), \tag{23}$$

where

$$\tilde{u}_{1j} = \frac{2(-1)^j}{M\tilde{c}_j}\sum_{i=0}^M \frac{1}{\tilde{c}_i} u_1(\tilde{x}_i)\cos\left(\frac{\pi ij}{M}\right), \quad j = 0,1,\cdots,M, \tag{24}$$

and $\tilde{x}_i = \frac{T}{2}(x_i + 1), i = 0,1,\cdots,M$. For finding the unknown coefficients $u_1(\tilde{x}_i), i = 0,1,\cdots,M$, by substituting the grid points $\tilde{x}_i, i = 0,1,\cdots,M$ in (22), we have

$$u_1(\tilde{x}_i) = u_0(\tilde{x}_i) + \int_0^{x_i} \lambda(\tilde{x}_i,t)\big(Lu_0(t) + Nu_0(t) - g(t)\big)dt, \quad i = 0,1,\cdots,M. \tag{25}$$

By change of variable $t = \frac{\tilde{x}_i}{2}(\xi + 1)$ (25) can be written as:

$$u_1(\tilde{x}_i) = u_0(\tilde{x}_i) + \frac{\tilde{x}_i}{2}\int_{-1}^1 \lambda\left(\tilde{x}_i, \frac{\tilde{x}_i}{2}(\xi+1)\right)F_{0,i}(\xi)d\xi, \quad i = 0,1,\cdots,M, \tag{26}$$

where

$$F_{0,i}(\xi) = \big(Lu_0(t) + Nu_0(t) - g(t)\big)\big|_{t=\frac{\tilde{x}_i}{2}(\xi+1)}, \quad i = 0,1,\cdots,M. \tag{27}$$

By applying numerical integration method given in (17), we can approximate the integral in the right hand of (26) and get:

$$u_1(\tilde{x}_i) = u_0(\tilde{x}_i) + \frac{\tilde{x}_i}{2}\sum_{l=0}^p w_l\lambda\left(\tilde{x}_i, \frac{\tilde{x}_i}{2}(\xi_l+1)\right)F_{0,i}(\xi_l), \quad i = 0,1,\cdots,M. \tag{28}$$

So, from (28), (24) and (23), we obtain the approximation of $u_1(x)$. For finding the approximation of $u_2(x)$, by substituting (23) in (21), we can obtain

$$u_2(x) \simeq u_1^M(x) + \int_0^x \lambda(x,t)(Lu_1^M(t) + Nu_1^M(t) - g(t))dt. \tag{29}$$

In a similar way, the function $u_2(x)$ on $[0,T]$ can be approximated as

$$u_2(x) \simeq u_2^M(x) = \sum_{j=0}^M \tilde{u}_{2j} T_j\left(\frac{2}{T}x - 1\right), \tag{30}$$

where

$$\tilde{u}_{2j} = \frac{2(-1)^j}{M\tilde{c}_j}\sum_{i=0}^M \frac{1}{\tilde{c}_i} u_2(\tilde{x}_i)\cos\left(\frac{\pi ij}{M}\right), \quad j = 0,1,\cdots,M. \tag{31}$$

Similarly, for finding the unknown coefficients $u_2(\tilde{x}_i), i = 0,1,\cdots,M$, by substituting the grid points $\tilde{x}_i,\ i = 0,1,\cdots,M$ in (29), we have

$$u_2(\tilde{x}_i) \simeq u_1^M(\tilde{x}_i) + \int_0^{\tilde{x}_i} \lambda(\tilde{x}_i,t)\left(Lu_1^M(t) + Nu_1^M(t) - g(t)\right)dt, \quad i = 0,1,\cdots,M. \tag{32}$$

By change of variable $t = \frac{\tilde{x}_i}{2}(\xi + 1)$, (32) can be written as:

$$u_2(\tilde{x}_i) \simeq u_1^M(\tilde{x}_i) + \frac{\tilde{x}_i}{2}\int_{-1}^{1} \lambda\left(\tilde{x}_i, \frac{\tilde{x}_i}{2}(\xi+1)\right) F_{1,i}(\xi)d\xi, \quad i = 0,1,\cdots,M, \tag{33}$$

where

$$F_{1,i}(\xi) = \left(Lu_1^M(t) + Nu_1^M(t) - g(t)\right)|_{t=\frac{\tilde{x}_i}{2}(\xi+1)}, \quad i = 0,1,\cdots,M. \tag{34}$$

By applying numerical integration method given in (17), we can approximate the integral in the right hand of (33) and obtain:

$$u_2(\tilde{x}_i) \simeq u_1^M(\tilde{x}_i) + \frac{\tilde{x}_i}{2}\sum_{l=0}^{p} w_l \lambda\left(\tilde{x}_i, \frac{\tilde{x}_i}{2}(\xi_l+1)\right) F_{1,i}(\xi_l), \quad i = 0,1,\cdots,M. \tag{35}$$

Thus, from (35), (31) and (30), we can obtain the approximation of $u_2(x)$.

Generally, for $n \geq 2$, according to the above method, we can obtain the approximation of $u_n(x)$ as follows:

$$u_n(x) \simeq u_n^M(x) = \sum_{j=0}^{M} \tilde{u}_{nj} T_j\left(\frac{2}{T}x - 1\right), \tag{36}$$

where

$$\tilde{u}_{nj} = \frac{2(-1)^j}{M\tilde{c}_j}\sum_{i=0}^{M} \frac{1}{\tilde{c}_i} u_n(\tilde{x}_i)\cos\left(\frac{\pi ij}{M}\right), \quad j = 0,1,\cdots,M, \tag{37}$$

$$u_n(\tilde{x}_i) \simeq u_{n-1}^M(\tilde{x}_i) + \frac{\tilde{x}_i}{2}\sum_{l=0}^{p} w_l \lambda\left(\tilde{x}_i, \frac{\tilde{x}_i}{2}(\xi_l+1)\right) F_{n-1,i}(\xi_l), \quad i = 0,1,\cdots,M, \tag{38}$$

and

$$F_{n-1,i}(\xi) = \left(Lu_{n-1}^M(t) + Nu_{n-1}^M(t) - g(t)\right)|_{t=\frac{\tilde{x}_i}{2}(\xi+1)}, \quad i = 0,1,\cdots,M. \tag{39}$$

6. The application of H-S-VIM in heat transfer

In order to assess the accuracy of H-S-VIM for solving nonlinear equations, we consider the following example.

6.1. Unsteady nonlinear convective-radiative equation

Consider the cooling of a lumped system [37, 47], with volume V, surface area A, density ρ, specific heat c, emissivity E and initial temperature T_i. At time $t = 0$, the system is exposed to an environment with convective heat transfer with coefficient h and the temperature T_a. The system also loses heat through radiation and the effective sink temperature is T_s. The cooling equation and the initial condition are as follows:

$$c = c_a[1 + \beta(T - T_a)],$$

where β is a constant and c_a is the specific heat at T_a. The cooling equation and the initial condition are as follows:

$$\rho Vc\frac{dT}{dt} + hA(T - T_a) + E\sigma A(T^4 - T_s^4) = 0, \quad T(0) = T_i,$$

which by using

$$u = \frac{T}{T_i}, \quad u_a = \frac{T_a}{T_i}, \quad \tau = \frac{t(hA)}{\rho Vc_a}, \quad \varepsilon_1 = \beta T_i, \quad \varepsilon_2 = \frac{E\sigma T_i^3}{h}, \quad u_s = \frac{T_s}{T_i},$$

we have

$$[1 + \varepsilon_1(u - u_a)]\frac{du}{d\tau} + (u - u_a) + \varepsilon_2(u^4 - u_s^4) = 0, \quad u(0) = 1. \tag{40}$$

For simplicity, we assume $u_a = u_s = 0$. So we have

$$[1 + \varepsilon_1 u]\frac{du}{d\tau} + u + \varepsilon_2 u^4 = 0, \quad u(0) = 1. \tag{41}$$

6.2. VIM solution

In order to solve (41) using VIM, we construct a correction functional as follows:

$$u_{n+1}(\tau) = u_n(\tau) + \int_0^\tau \lambda\{\frac{du_n(t)}{dt} + u_n(t) + \varepsilon_1\tilde{u}_n(t)\frac{d\tilde{u}_n(t)}{dt} + \varepsilon_2\tilde{u}_n^4(t)\}dt. \tag{42}$$

Its stationary conditions can be obtained as:

$$\lambda'(t) - \lambda(t) = 0, 1 + \lambda(t)|_{t=\tau} = 0. \tag{43}$$

The Lagrangian multiplier can therefore be identified as

$$\lambda = -e^{t-\tau}. \tag{44}$$

As a result, we obtain the following iteration formula:

$$u_{n+1}(\tau) = u_n(\tau) - \int_0^\tau e^{t-\tau}\{\frac{du_n(t)}{dt} + u_n(t) + \varepsilon_1 u_n(t)\frac{du_n(t)}{dt} + \varepsilon_2 u_n^4(t)\}dt. \tag{45}$$

We can arbitrarily assign $u_0(\tau) = e^{-\tau}$, because the initial condition is satisfied just by substitution. Now using the iteration formula (45) and $u_0(\tau)$, we can get the first and second iteration results as follows:

$$u_1(\tau) = e^{-\tau} - \frac{1}{3}(-3\,e^{3\tau}\varepsilon_1 + e^{3\tau}\varepsilon_2 + 3\,\varepsilon_1\,e^{2\tau} - \varepsilon_2)e^{-4\tau}, \tag{46}$$

$$u_2(\tau) = e^{-\tau} - \frac{1}{3}\,(-3\,e^{3\,\tau}\varepsilon_1 + e^{3\,\tau}\varepsilon_2 + 3\,\varepsilon_1\,e^{2\,\tau} - \varepsilon_2)e^{-4\,\tau} + e^{-5\,\tau}{\varepsilon_1}^3{\varepsilon_2}^2 - e^{-5\,\tau}\varepsilon_2\,{\varepsilon_1}^4 - \frac{41}{12}\,e^{-5\,\tau}\varepsilon_2\,{\varepsilon_1}^2 + \frac{41}{36}\,e^{-5\,\tau}\varepsilon_1\,{\varepsilon_2}^2 + \cdots. \quad (47)$$

Here, calculating $u_n(\tau)$ for $n \geq 2$ is difficult, because the nonlinear operator $Nu = \varepsilon_1 u\frac{du}{d\tau} + \varepsilon_2 u^4$ is ill-conditioned such that the integration becomes very complicated. As n, the number of iterations is increased, also the number of terms in the approximate solutions increases. This increase is so rapid that the integration becomes both complicated and time consuming.

6.3. H-S-VIM solution

Now we apply hybrid spectral-variational iteration method for (41). We rewrite equation (41) in the form

$$(1+\varepsilon_1)(\frac{du}{d\tau} + u) - \varepsilon_1(\frac{du}{d\tau} + u) + \varepsilon_1 u\frac{du}{d\tau} + \varepsilon_2 u^4 = 0, \quad u(0) = 1. \quad (48)$$

In order to solve (48) using H-S-VIM, we construct a correction functional, as follows:

$$u_{n+1}(\tau) = u_n(\tau) + \int_0^\tau \lambda\{(1+\varepsilon_1)(\frac{du_n(t)}{dt} + u_n(t)) - \varepsilon_1(\frac{d\tilde{u}_n(t)}{dt} + \tilde{u}_n(t)) + \varepsilon_1\tilde{u}_n(t)\frac{d\tilde{u}_n(t)}{dt} + \varepsilon_2\tilde{u}_n^4(t)\}dt. \quad (49)$$

Its stationary conditions can be obtained as follows:

$$\lambda'(t) - \lambda(t) = 0, 1 + (1+\varepsilon_1)\lambda(t)|_{t=\tau} = 0. \quad (50)$$

The Lagrangian multiplier can therefore be identified as

$$\lambda = \lambda(\tau, t) = \frac{-e^{t-\tau}}{1+\varepsilon_1}. \quad (51)$$

As a result, we obtain the following iteration formula:

$$u_{n+1}(\tau) = u_n(\tau) - \int_0^\tau \frac{e^{t-\tau}}{1+\varepsilon_1}\{\frac{du_n(t)}{dt} + u_n(t) + \varepsilon_1 u_n(t)\frac{du_n(t)}{dt} + \varepsilon_2 u_n^4(t)\}dt. \quad (52)$$

According to subsection 6.2, we assume $u_0(\tau) = e^{-\tau}$. Here, we introduce H-S-VIM with $T = 1, M = 3$ and $p = 15$. By using H-S-VIM, we can get the following results:

$$u_1(\tau) \simeq u_1^M(\tau) = A_0^1(\varepsilon_1, \varepsilon_2) + A_1^1(\varepsilon_1, \varepsilon_2)\tau + A_2^1(\varepsilon_1, \varepsilon_2)\tau^2 + A_3^1(\varepsilon_1, \varepsilon_2)\tau^3, \quad (53)$$

where

$$A_0^1(\varepsilon_1, \varepsilon_2) = 1,$$

$$A_1^1(\varepsilon_1, \varepsilon_2) = -\frac{0.9946917670 + 0.04861741257\,\varepsilon_1 + 0.8366849481\,\varepsilon_2}{1+\varepsilon_1},$$

$$A_2^1(\varepsilon_1, \varepsilon_2) = -\frac{-0.4652490645 + 0.6675430350\,\varepsilon_1 - 1.300159144\,\varepsilon_2}{1+\varepsilon_1},$$

$$A_3^1(\varepsilon_1, \varepsilon_2) = \frac{-0.1026778560 + 0.3165839374\,\varepsilon_1 - 0.5799953807\,\varepsilon_2}{1+\varepsilon_1},$$

and

$$u_2(\tau) \simeq u_2^M(\tau) = A_0^2(\varepsilon_1, \varepsilon_2) + A_1^2(\varepsilon_1, \varepsilon_2)\tau + A_2^2(\varepsilon_1, \varepsilon_2)\tau^2 + A_3^2(\varepsilon_1, \varepsilon_2)\tau^3, \quad (54)$$

where

$$A_0^2(\varepsilon_1, \varepsilon_2) = \frac{1}{(1+\varepsilon_1)^5}({\varepsilon_1}^5 + 5.000000001\,{\varepsilon_1}^4 + 10.0\,{\varepsilon_1}^3 + 10.0\,{\varepsilon_1}^2 + 5.000000001\,\varepsilon_1 + 1.0),$$

$$A_1^2(\varepsilon_1, \varepsilon_2) = \frac{1}{(1+\varepsilon_1)^5}(-4.027047634\,\varepsilon_1 - 0.8375389178\,\varepsilon_2 + \cdots - 1.107504415\,{\varepsilon_1}^4\varepsilon_2),$$

$$A_2^2(\varepsilon_1, \varepsilon_2) = \frac{1}{(1+\varepsilon_1)^5}(1.192434444\,\varepsilon_1 + 1.302323512\,\varepsilon_2 + \cdots + 0.03028760613\,\varepsilon_1{\varepsilon_2}^4),$$

$$A_3^2(\varepsilon_1, \varepsilon_2) = -\frac{1}{(1+\varepsilon_1)^5}(0.1026778663 + 0.09346488597\varepsilon_1 + \cdots + 0.08728867399\,{\varepsilon_1}^4\varepsilon_2).$$

For $\varepsilon_1 = 2$ and $\varepsilon_2 = 3$, we can obtain

$$u_1(\tau) \simeq u_1^M(\tau) = 0.9999999998 - 1.200660480\,\tau + 1.010213477\,\tau^2 - 0.4031653750\,\tau^3,$$

$$u_2(\tau) \simeq u_2^M(\tau) = 0.9999999998 - 1.193833843\,\tau + 1.037714238\,\tau^2 - 0.4174625226\,\tau^3,$$

$$u_3(\tau) \simeq u_3^M(\tau) = 0.9999999997 - 1.195895532\,\tau + 1.045569435\,\tau^2 - 0.4171926368\,\tau^3.$$

6.4. Comparison discussions

In this section, the current results are compared with the standard variational iteration method (VIM), homotopy perturbation method (HPM), homotopy analysis method (HAM), differential transformation method (DTM) and the exact solution in order to verify the accuracy of the proposed method. The exact solution of (41) is obtained in the following form [33]:

$$\frac{1}{3}\ln\left(\frac{1+\varepsilon_2 u^3}{(1+\varepsilon_2)u^3}\right) + \frac{1}{3}\frac{\varepsilon_1}{\varepsilon_2^{\frac{1}{3}}}\left[\frac{1}{2}\ln\left(\frac{(1+\varepsilon_2 u^{\frac{1}{3}})(1+\varepsilon_2 u^3)}{(1+\varepsilon_2)(1+\varepsilon_2^{\frac{1}{3}}u)^3}\right) + \sqrt{3}\left(\arctan\left(\frac{2\varepsilon_2^{\frac{1}{3}}-1}{\sqrt{3}}\right) - \arctan\left(\frac{2\varepsilon_2^{\frac{1}{3}}u-1}{\sqrt{3}}\right)\right)\right] = \tau.$$

All the computations associated with the method have been performed by a personal computer having the Intel Pentium 5, 2.2 GHz processor and using Maple 13 with 32 digits precision. The calculations presented in this section adopt a value of $p = 15$ for H-S-VIM (See subsection 4.2). We denote the square residual error of approximate solutions $u(\tau)$ for (41), in the following form:

$$\Delta_n = \int_0^T (\mathrm{RES}(\tau))^2 d\tau, \qquad (55)$$

where $\mathrm{RES}(\tau) = [1+\varepsilon_1 u(\tau)]\frac{du(\tau)}{d\tau} + u(\tau) + \varepsilon_2 u(\tau)^4$.

Figure 1 shows the temperature distribution by using H-S-VIM ($n = 15$ and $M = 20$) and the exact solution for cooling of a lumped system with variable specific heat coefficient on the interval [0,10]. The comparisons between the approximate solutions by using VIM and H-S-VIM with $M = 10$ on the interval [0,1] for $\varepsilon_1 = 0.8, \varepsilon_2 = 0.5$ and $\varepsilon_1 = 3, \varepsilon_2 = 2$ are given in Tables 1 and 2, respectively .

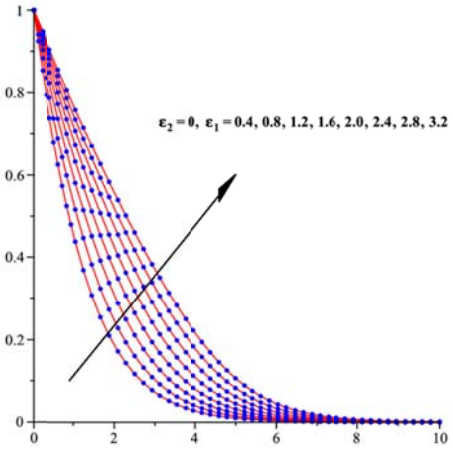


Fig. 1. Comparison of results of H-S-VIM (bold circle) and exact solution (solid line) for (41)

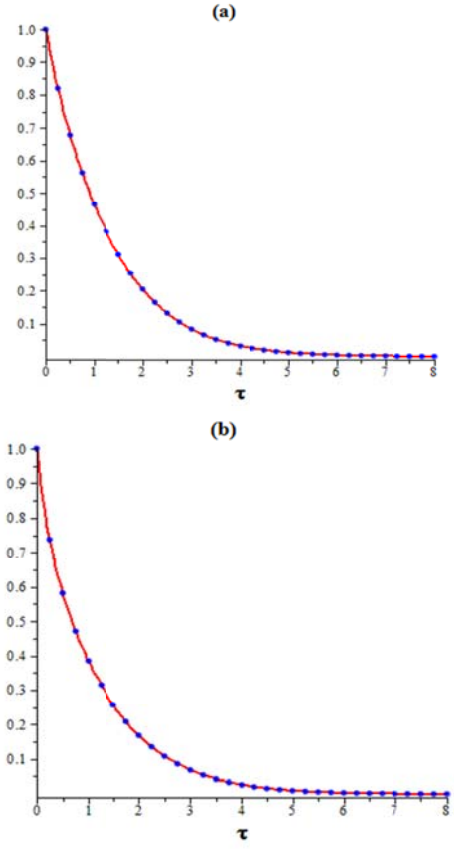
Table 1. Comparisons of Δ_n for $\varepsilon_1 = 0.8, \varepsilon_2 = 0.5$

n	Δ_n(VIM)	CPU time (s)	Δ_n(H − S − VIM)	CPU time (s)
1	0.0182983760	0.843	0.0030789439	1.563
2	0.0090279662	2.344	0.0002215198	1.608
3	Fail	− − −	0.0000159048	1.688
4	Fail	− − −	0.0000011213	1.765
5	Fail	− − −	0.0000000776	1.923

Table 2. Comparisons of Δ_n for $\varepsilon_1 = 3, \varepsilon_2 = 2$

n	Δ_n(VIM)	CPU time (s)	Δ_n(H − S − VIM)	CPU time (s)
1	6.0658113076	0.829	0.09651901015	1.500
2	15.783171494	1.766	0.01415309332	1.547
3	Fail	− − −	0.0018761813	1.687
4	Fail	− − −	0.0002358425	1.844
5	Fail	− − −	0.0000289583	1.907

As shown in Fig. 2, for all values of the small parameters, ε_1 and ε_2, the difference between H-S-VIM with $n = 15, M = 20$ and the exact solution on the interval [0,8] are negligible.



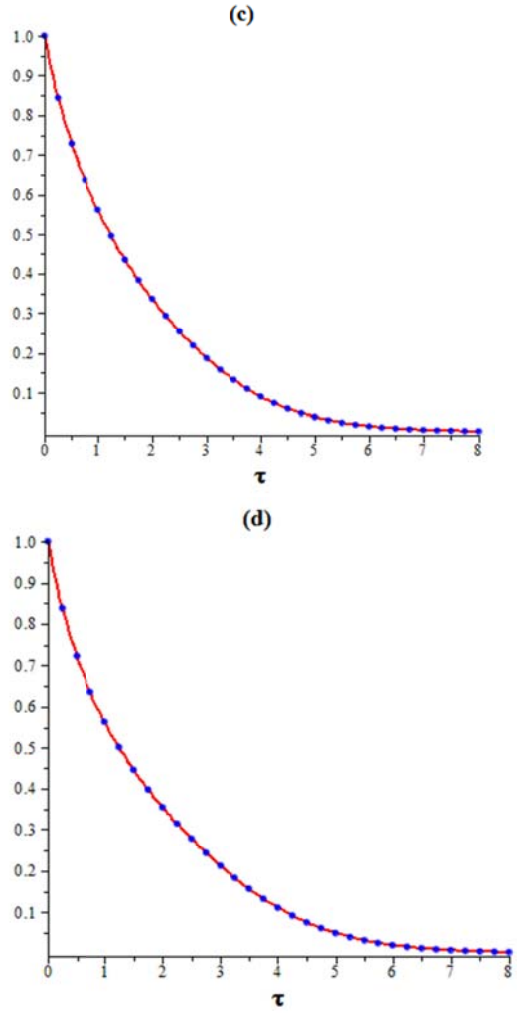


Fig. 2. Comparison of results of H-S-VIM (bold circle) and exact solution (solid line), (a) $\varepsilon_1 = 0.8, \varepsilon_2 = 0.5$, (b) $\varepsilon_1 = 1, \varepsilon_2 = 2$, (c) $\varepsilon_1 = 3, \varepsilon_2 = 2$ and (d) $\varepsilon_1 = 4, \varepsilon_2 = 3$

Comparisons are made between 5th-order of HPM [48], 5th-order of HAM [37] for $h = -0.8$, 5th-order of H-S-VIM with $M = 20$ and exact solution, for some values of $\varepsilon_1, \varepsilon_2$ on the interval [0,1] are plotted in Fig. 3. Since the results of HPM can be obtained as a special case of HAM when $h = -1$, from Figs. 3, it is evident that HPM loses its validity for relatively large values of ε_1 and ε_2.

Table 3. Comparisons of Δ_n of 5th-order solutions of different approaches for ε_1=1

Method	$\varepsilon_2 = 1$	$\varepsilon_2 = 2$	$\varepsilon_2 = 3$
H-S-VIM	0.00000719625	0.00000215180	0.00000328251
MHAM for h=-0.5 [49]	0.00000645794	0.00000225316	0.00185417178
HAM for h=-0.8 [37]	0.00001294614	0.06332372453	4.39671710926
HAM for h=-0.9 [37]	0.00026131005	0.45943918392	270.979958125
HPM (HAM for h=-1) [48]	0.00278084263	2.08587335265	358312.798216

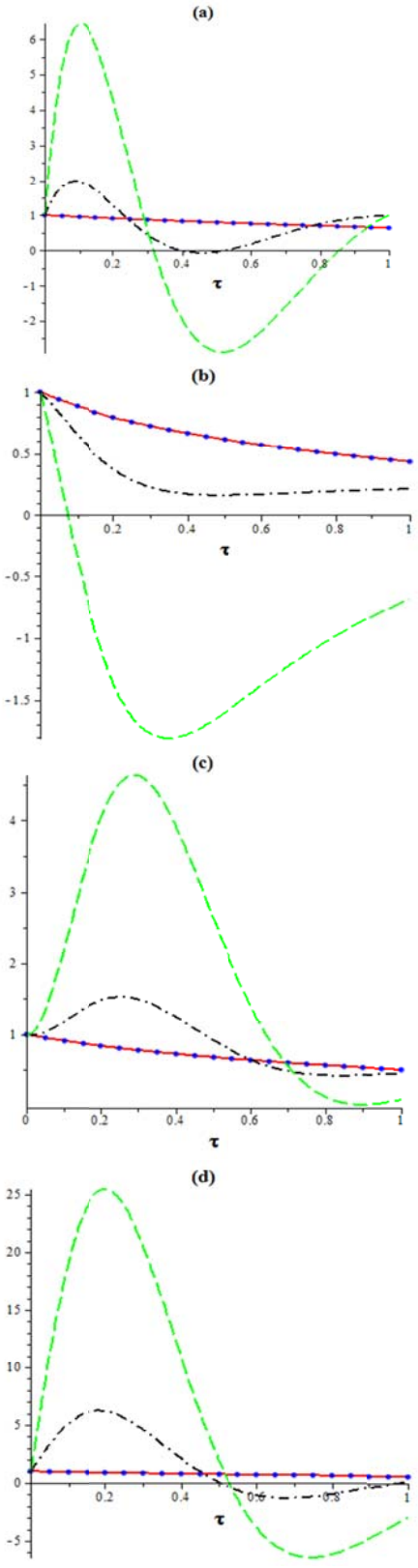


Fig. 3. 5th-order of HPM (dash), 5th-order of HAM for $h = -0.8$ (dash dot), 5th-order of H-S-VIM (bold circle) and exact solution (solid line),(a) $\varepsilon_1 = 3, \varepsilon_2 = 1$, (b) $\varepsilon_1 = 2, \varepsilon_2 = 3$, (c) $\varepsilon_1 = 3, \varepsilon_2 = 3$ and (d) $\varepsilon_1 = 4, \varepsilon_2 = 3$

From Tables 3-5, it seems that H-S-VIM with $M = 20$ gives better approximations than the HPM [48], HAM [37] for $h = -0.8, h = -0.9$ and MHAM [49] on the interval $[0, 6]$.

Table 4. Comparisons of Δ_n of 5th-order solutions of different approaches for $\varepsilon_1 = 2$

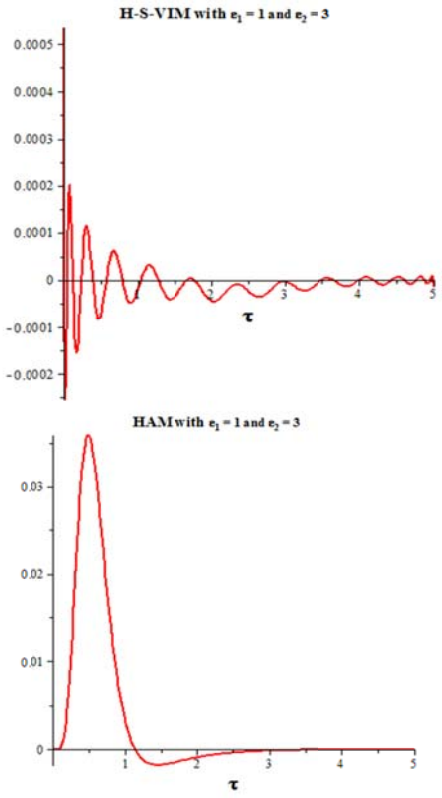
Method	$\varepsilon_2 = 1$	$\varepsilon_2 = 2$	$\varepsilon_2 = 3$
H-S-VIM	0.0011535394	0.00041717183	0.000209991 08
MHAM for h=-0.5 [49]	0.0016937936	0.00043080927	0.00018897928
HAM for h=-0.8 [37]	1.7740609725	0.29707799102	7.73726355555
HAM for h=-0.9 [37]	13.500883632	2.52627773025	24.9031191524
HPM (HAM for h=-1) [48]	86.150467589	18.0667149197	284.907280951

Table 5. Comparisons of Δ_n of 5th-order solutions of different approaches for $\varepsilon_1 = 3$

Method	$\varepsilon_2 = 1$	$\varepsilon_2 = 2$	$\varepsilon_2 = 3$
H-S-VIM	0.0117655542	0.004971082412	0.00268939756
MHAM for h=-0.5 [49]	0.2251073087	0.007103244505	0.00301374746
HAM for h=-0.8 [37]	741.25718288	527.1191196254	140.804634925
HAM for h=-0.9 [37]	9333.2489875	15082.47186259	3983.73926254
HPM (HAM for h=-1) [48]	329295.21567	14603605.62548	314587.045985

Also, Fig. 4 show the 'residual error' for 10-th order approximation H-S-VIM with $M = 25$ and HAM for $h = \frac{-1}{1+\varepsilon_1}$ [37] on the interval $[0, 5]$ and clearly indicate that the H-S-VIM gives rapid convergence. Here the 'residual error' is defined as follows:

$$ResidualError \approx [1 + \varepsilon_1 u(\tau)]\frac{du(\tau)}{d\tau} + u(\tau) + \varepsilon_2 u(\tau)^4.$$



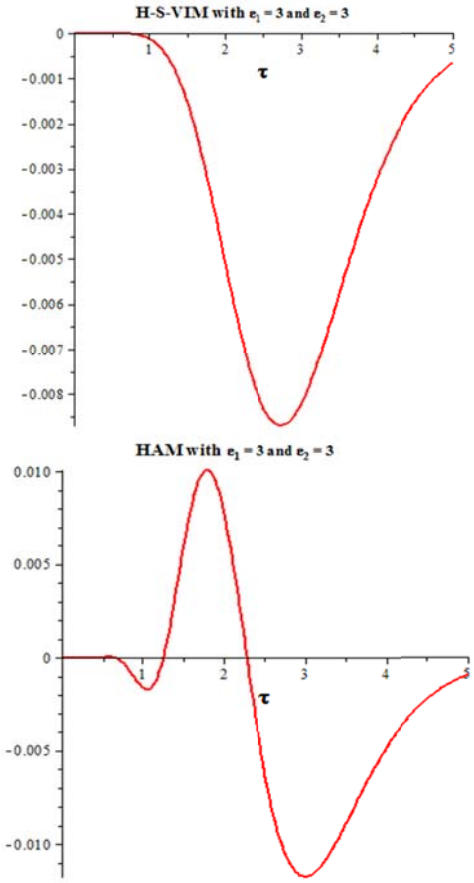


Fig. 4. Comparison between residual error of H-S-VIM and HAM for $h = \frac{-1}{1+\varepsilon_1}$.

In Table 6, we compared the absolute errors of VIM [50], HPM [50], DTM [39] and 4-th order of H-S-VIM with $M = 10$ on the interval $[0,1]$ at $\tau = 0.2$ for (41). In Table 7, we assume that $\varepsilon_1 = 0$ and then compared H-S-VIM ($n = 5$ and $M = 10$) with the study of Ganji et al. [48] and Yaghoobi et al. [39] on the interval $[0,1]$ at $\tau = 0.5$.

Table 6. The absolute errors of VIM, HPM, DTM and H-S-VIM at $\tau = 0.2$ for (41)

ε_1	ε_2	VIM (%)[50]	HPM (%)[50]	DTM (%) [39]	H-S-VIM (%)
0.4	0.4	0.1281	0.1247	0.001888	6.921×10^{-8}
0.4	0.6	0.2934	0.3064	0.005815	0.000001586
0.4	0.8	0.8751	0.9745	0.013804	0.000005403
0.8	0.4	2.5181	1.6447	0.000418	0.00002768
0.8	0.6	1.7748	1.6447	0.001349	0.000008088
0.8	0.8	0.8101	0.7835	0.003305	0.000002448

Table 7. The results of VIM, HPM, DTM and H-S-VIM their absolute errors at $\varepsilon_1 = 0$ and $\tau = 0.5$

ε_2	VIM (%) [48]	HPM (%) [48]	DTM (%) [39]	H-S-VIM (%)
0.0	1.64872×10^{-10}	1.64872×10^{-10}	0.0000002748	1.263×10^{-11}
0.1	4.28955×10^{-5}	7.80444×10^{-5}	0.0000090257	2.126×10^{-10}
0.2	0.000318544	0.000602619	0.0000263434	1.124×10^{-8}
0.3	0.000999783	0.001966880	0.0000550763	1.036×10^{-7}
0.4	0.002207405	0.004516592	0.0000984091	4.800×10^{-7}
0.5	0.004021512	0.008559256	0.0001598850	0.000001529
0.6	0.006490236	0.014371001	0.0002434237	0.000003845
0.7	0.009636553	0.022201898	0.0003533413	0.000008228
0.8	0.013463676	0.032280117	0.0004943625	0.00001567
0.9	0.017959388	0.044815230	0.0006716365	0.00002731
1.0	0.023099547	0.060000859	0.0008907421	0.00004444

Clearly, $u'(0) = -\frac{1+\varepsilon_2}{1+\varepsilon_1}$ and $u''(0) = \frac{(1+\varepsilon_2)(1+3\,\varepsilon_1\varepsilon_2+4\,\varepsilon_2)}{(1+\varepsilon_1)^3}$, the H-S-VIM solution $(n = 2, M = 20)$ with this exact one to show the accuracy for large values of ε_1 and ε_2, (see Fig. 5) when $\varepsilon_2 = 2\varepsilon$ and $\varepsilon_1 = \varepsilon$. Also, Fig. 6 shows the absolute errors of $u'(0)$ and $u''(0)$ with respect to ε_1 and ε_2 by using H-S-VIM with $n = 2$ and $M = 20$ on the interval $[0,1]$ when $(\varepsilon_1, \varepsilon_2) \in [0,10] \times [0,10]$.

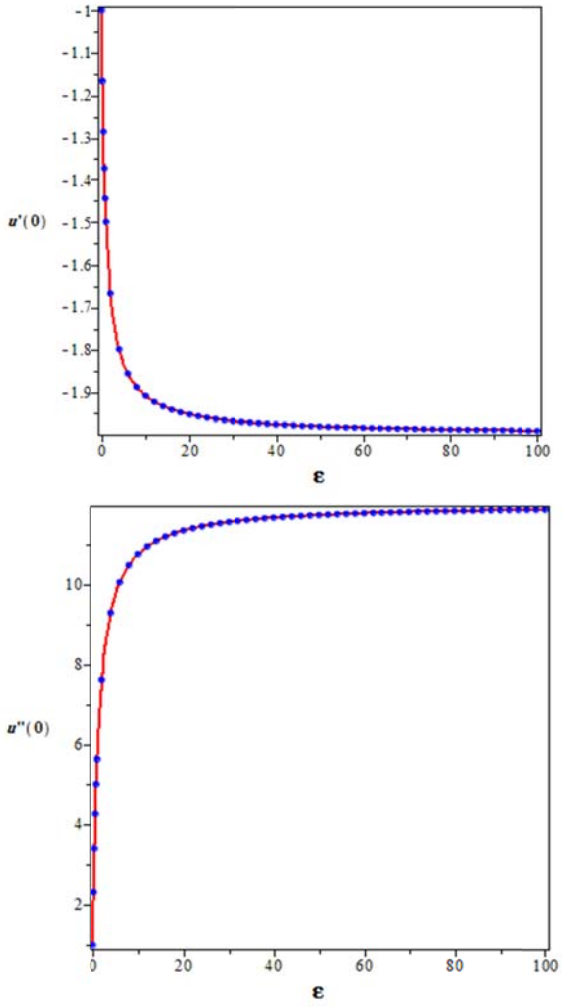


Fig. 5. The graph of $u'(0)$ and $u''(0)$ with respect to ε, 2^{th}-order of H-S-VIM (bold circle) and exact values (solid line)

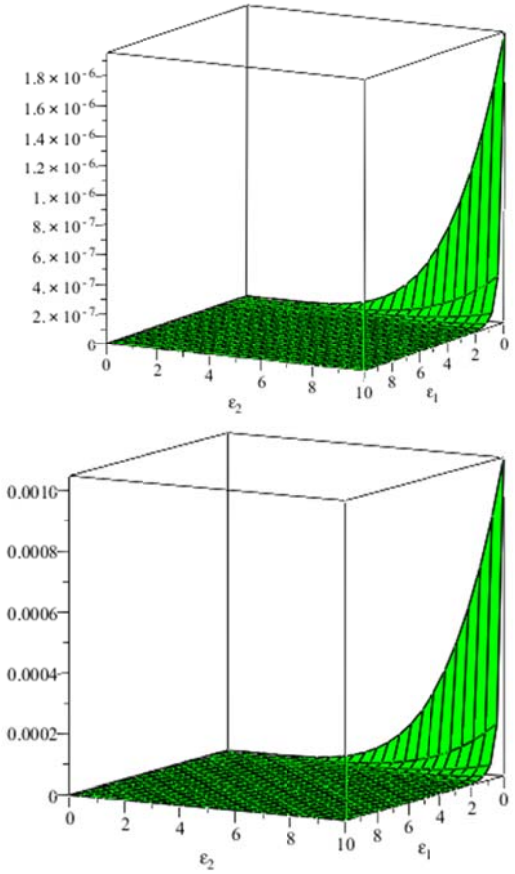


Fig. 6. The absolute errors of $u'(0)$ and $u''(0)$ by using H-S-VIM with $n = 2$ and $M = 20$

7. Conclusion

In this study, we successfully proposed a novel hybrid spectral-variational iteration method (H-S-VIM) for solving nonlinear equations arising in heat transfer. By analyzing and comparing the results obtained and procedures used in H-S-VIM and VIM, we observe that the new approach overcomes the difficulty arising in calculating complicated and time consuming integrals. Moreover, it was shown that for this kind of problem, H-S-VIM is better than HPM, VIM, HAM and DTM because by increasing small parameters of ε_1 and ε_2 the error of H-S-VIM is less than previous solutions in comparison with exact solution. These results shows us the validity and great potential of the H-S-VIM for nonlinear models in mathematical and physics with high accuracy.

This paper discusses using Chebyshev polynomials interpolation to approximate functions. Further research can be initiated based on applying other orthogonal polynomials such as Legendre Polynomials and other interpolation such as Fourier interpolation, rational interpolation and spline interpolation.

In the general case, according to the variational iteration method, for the nonlinear differential equation

$$\mathrm{Lu} + \mathrm{Nu} = 0, \quad (56)$$

where L and N are linear and nonlinear operators respectively, the following algorithms can be constructed [51, 52].

1) Variational iteration algorithm-I

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda\{Lu_n(t) + Nu_n(t)\}dt. \quad (57)$$

2) Variational iteration algorithm-II

$$u_{n+1}(x) = u_0(x) + \int_0^x \lambda Nu_n(t)dt. \quad (58)$$

3) Variational iteration algorithm-III

$$u_{n+2}(x) = u_{n+1}(x) + \int_0^x \lambda\{Nu_{n+1}(t) - Nu_n(t)\}dt. \quad (59)$$

In this paper we used the variational iteration algorithm-I. Recently the variational iteration algorithm-II has gained much attention [51, 53, 54], while the variational iteration algorithm-III is rarely used. It is suggested that applying our method into these algorithms (algorithm-II and algorithm-III) can be the subject of further research work.

Nomenclature

A	Area, m^2
c	specific heat, J/kgK
c_a	specific heat at temperature T_a, J/kgK
E	surface emissivity, W
h	Coefficient of natural convection, W/m^2K
L	Linear operator
N	Nonlinear operator
T	Temperature, K
T_a	Environment temperature, K
T_i	Initial temperature, K
T_s	Effective sink temperature, K
V	Volume, m^3Greek symbols
β	constant, volumetric thermal expansion coefficient, $1/K$
ε	small parameter
ρ	mass density, kg/m^3
σ	Stefan–Boltzmann constant
τ	dimensionless temporal coordinate, s
λ	Lagrangian multiplier Subscripts
a	air
i	initial
n	order of approximation
s	surface

References

[1] Drazin, P. G. & Jonson, R.S. (1993). *Soliton: An Introduction*. Cambridge, New York.

[2] Whitham, G. B. (1974). *Linear and Nonlinear Waves*. New York, Wiley.

[3] Debnath, L. (1994). *Nonlinear Water Waves*. Boston, Academic Press.

[4] Adomian, G. (1994). *Solving Frontier Problem of Physics: The Decomposition Method*. Boston, MA, Kluwer Academic Publishers.

[5] He, J. H. (1997). A new approach to nonlinear partial differential equations. *Commun. Nonlinear. Sci. Numer. Simulat.,* 230-235.

[6] He, J. H. (1999). Variational iteration method-a kind of non-linear analytical technique: some examples. *Int. J. Nonlinear. Mech.,* 699-708.

[7] He, J. H. (2003). Homotopy perturbation method: A new nonlinear analytical technique. *Appl. Math. Comput*., 73-79.

[8] He, J. H. (2004). The homotopy perturbation method for nonlinear oscillators with discontinuities. *Appli. Math. Comput.,* 287-292.

[9] Liao, S. J. (1992). *On the proposed homotopy analysis technique for nonlinear problems and its applications*. PhD dissertation, Shanghai Jio Tong University.

[10] Liao, S. J. (2003). *Beyond perturbation: introduction to homotopy analysis method*. Boca Raton: Chapman and Hall/CRC Press.

[11] Malfliet, W. (1992). Solitary wave solutions of nonlinear wave equations. *American Journal of Physics*., 650-654.

[12] Wazwaz, A. M. (2004). The tanh method for traveling wave solutions of nonlinear equations. *Appl. Math. Comput.*, 713-723.

[13] He, J. H. (1999). Variational iteration method-a kind of non-linear analytical technique: some examples. *Int. J. Nonlinear. Mech.*, 699-708.

[14] Inokuti, M., Sekine, H. & Mura, T. (1978). General use of the Lagrange multiplier in nonlinear mathematical physics. In: Nemat-Nasser S,editor. Variational method in the mechanics of solids. *Oxford: Pergamon Press*. 156-162.

[15] He, J. H. (1997). Variational iteration method for delay differential equations. *Commun. Nonlinear. Sci. Numer. Simulat.,* 235-66.

[16]] He, J. H. (2000). Variational iteration method for autonomous ordinary differential systems. *Appl. Math. Comput*., 115-123.

[17] Abdou, M. A. & Soliman, A. A. (2005). Variational iteration method for solving Burger's and coupled Burger's equations. *J. Comput. Appl. Math*., 245-51.

[18] Wang, S. Q. & He, J. H. (2007). Variational iteration method for solving integro-differential equations. *Phys. Lett. A.*, 188-191.

[19] Momani, S. & Abuasad, S. (2006). Application of He's variational iteration method to Helmholtz equation. *Chaos, Solitons and Fractals*., 119-123.

[20] Abdou, M. A. & Soliman, A. A. (2005). New applications of variational iteration method. *Physica D*., 1-8.

[21] Abdou, M. A. (2007). On the variational iteration method. *Phys. Lett. A*., 61-68.

[22] Moghimi, M. & Hejazi, F. S. A. (2007). Variational iteration method for solving generalized Burger-Fisher and Burger equations. *Choas, Solitons and Fractals*., 1756-1761.

[23] Momani, S., Abuasad, S. & Odibat, Z. (2006). Variational iteration method for solving nonlinear boundary value problems. *Appl. Math. Comput.,* 1351-1358.

[24] Tatari, M. & Dehghan, M. (2007). On the convergence of He's variational iteration method. *J. Comput. Appl. Math.,* 121-128.
[25] Tatari, M. & Dehghan, M. (2007). Solution of problems in calculus of variations via He's variational iteration method. *Phys. Lett. A*., 401-406.
[26] Wazwaz, A. M. (2007). A comparison between the variational iteration method and Adomian decomposition method. *J. Comput. Appl. Math*., 129-136.
[27] Wazwaz, A. M. (2009). The variational iteration method for analytic treatment for linear and nonlinear ODEs. *Appl. Math. Comput.,* 120-134.
[28] Abassy, T. A., El-Tawil, M. A. & El Zoheiry, H. (2007). Solving nonlinear partial differential equations using the modified variational iteration-Padé technique. *J. Comput. Appl. Math.,* 73-91.
[29] Abbasbandy, S. (2007). A new application of He's variational iteration method for quadratic Riccati differential equation by using Adomian's polynomials. *J. Comput. Appl. Math.,* 59-63.
[30] Bildik, N. & Konuralp, A. (2006).The use of variational iteration method, differential transform method and adomian decomposition method for solving different types of nonlinear partial differential equations. *Internat. J. Non-Linear Sci. Numer. Simul.*, 65-70.
[31] Abassy, T. A., El-Tawil, M. A. & El Zoheiry, H. (2007). Toward a modified variational iteration method. *J. Comput. Appl. Math.,* 137-147.
[32] Heydari, M. & Loghmani, G. B. (2010). Approximate solution to boundary value preoblems by the modified VIM. *IJST-Trans. A,* 161-167.
[33] Aziz, A. & Na, T. Y. (1984). *Perturbation Method in Heat Transfer*, NewYork, Hemisphere.
[34] Ganji, D. D. (2006). The application of He's homotopy perturbation method to nonlinear equations arising in heat transfer. *Phys. Lett. A*., 337-341.
[35] Tari, H., Ganji, D. D. & Babazadeh, H. (2007). The application of He's variational iteration method to nonlinear equations arising in heat transfer. *Phys. Lett. A.,* 213-217.
[36] Abbasbandy, S. (2006). The application of homotopy analysis method to nonlinear equations arising in heat transfer. *Phys. Lett. A.,* 109-113.
[37] Abbasbandy, S. (2007). Homotopy analysis method for heat radiation equations. *Int. J. Heat. Mass. Transfer.,* 380-387.
[38] Marinca, V. & Herisanu, N. (2008). Application of Optimal Homotopy Asymptotic Method for solving nonlinear equations arising in heat transfer. *Int. J. Heat. Mass. Transfer*., 710-715.
[39] Yaghoobi, H. & Torabi, M. (2011). The application of differential transformation method to nonlinear equations arising in heat transfer. *Int. J. Heat. Mass. Transfer*., 815-820.
[40] Finlayson, B. (1972). *The Method of Weighted Residuals and Variational Principles*, New York. Academic Press.
[41] Canuto, C., Hussaini, M., Quarteroni, A. & Zang, T. (1988). *Spectral Methods in Fluid Dynamics*, Berlin, Springer.
[42] Gottlieb, D. & Orszag, S. (1977). *Numerical Analysis of Spectral Methods, Theory and Applications*. SIAM, Philadelphia, PA.
[43] Snyder, M. A. (1966). *Chebyshev methods in numerical approximation*. Englewood Cliffs. N. J: Prentice-Hall Inc.
[44] Mason, J. C. & Handscomb, D. C. (2003). *Chebyshev Polynomials*. Boca Raton, CRC Press.
[45] Clenshaw, C. W. & Curtis, A. R. (1960). A method for numerical integration on an automatic computer. *Numer. Math*, 197-205.
[46] Kui-Fang, C. (1996). Strictly positive definite functions. *J. Approx. Theory*.,148-158.
[47] Ganji, D. D. & Rajabi, A. (2006). Assessment of homotopy-perturbation and perturbation methods in heat radiation equations. *Int. Commun. Heat. Mass. Transf.,* 391-400.
[48] Ganji, D. D. & Sadighi, A. (2007). Application of homotopy-perturbation and variational iteration methods to nonlinear heat transfer and porous media equations. *J. Comput. Appl. Math*, 24-34.
[49] Biazar, J. & Ghanbari, B. (2012). HAM solution of some initial value problems arising in heat radiation equations. *Journal of King Saud University Science*.,161-165.
[50] Khaleghi, H., Ganji, D. D. & Sadighi, A. (2007). Application of variational iteration and homotopy-perturbation methods to nonlinear heat transfer equations with variable coefficients. *Numerical. Heat. Transfer. Part A*., 25-42.
[51] He, J. H. (2012) . Notes on the optimal variational iteration method. *Appl. Math. Lett,* Doi:10.1016/j.aml.2012.01.004.
[52] He, J. H., Wu, G. C. & Austin, F. (2010). The variational iteration method which should be followed. *Nonlinear Science Letters A*., 1-30.
[53] Wu, B. Y. & Li, X. Y. (2011). Second-order two-point boundary value problems using the variational iteration algorithm-II. *Int. J. Comput. Math*., 1201-1207.
[54] Faraz, N., Khan, Y. & Austin, F. (2010). An alternative approach to differential-difference equations using the variational iteration method. *Z. Naturforsch*., 1055-1059.