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The operational matrix of fractional integration for shifted Legendre polynomials

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Abstract

In this article we implement an operational matrix of fractional integration for Legendre polynomials. We proposed an algorithm to obtain an approximation solution for fractional differential equations, described in Riemann-Liouville sense, based on shifted Legendre polynomials. This method was applied to solve linear multi-order fractional differential equation with initial conditions, and the exact solutions obtained for some illustrated examples. Numerical results reveal that this method gives ideal approximation for linear multi-order fractional differential equations.

Keywords*:* Fractional-order differential equation; operational matrix; shifted Legendre polynomials; Riemann-Liouville fractional integral operator

1. Introduction

Many problems in various fields can be successfully modeled by fractional differential equations, such as theoretical physics, biology, viscoelasticity, electrochemistry and other physical processes. In the last decade, fractional differential equation has attracted the attention of mathematicians, physicists and engineers [1, 2]. Therefore, the accurate methods of solving fractional differential equations (FDEs) are a challenging research these days. There are several analytic methods such as Adomian decomposition method [3], variational iteration method [4] and homotopy perturbation method [5]. There are also many numerical methods introduced for solving FDEs in literature. Podlubny introduced a numerical method for arbitrary order derivative based on the relationship between the Grünwald-Letnikov and Riemann-Liouville derivative [2]. Diethelm et al. has presented predictor-corrector method for numerical solution [6] and also Erjaee et al. have shown good results in numerical method [7]. Recently solving FDEs using orthogonal polynomials have also received considerable attention. Using this method reduces the differential equation to a system of algebraic equations. The operational matrix of fractional derivative has been determined for some type of orthogonal polynomials such as Chebyshev polynomials [8] and Legendre polynomials [9]. Paraskevopoulos has suggested the operational matrix of integration by using these polynomials as a basis in ODEs [10, 11]. Recently, Bhrawya and Alofi [12] derived the operational matrix of fractional integration for shifted Chebyshev polynomials.

In the present article, we extend the application of Legendre polynomials for solving FDEs along the line of Riemann-Liouville. For this purpose we first write the FDEs in the integral form. Then we convert this integral equation to an algebraic equation system by using the Legendre polynomials similar to the operational matrix of fractional integration. Now, by solving the resultant algebraic equations, we obtain an approximation analytical solution for the FDEs.

The article is organized as follows. We start by introducing some necessary definitions and preliminaries for fractional calculus and Legendre polynomials. In Section 3, we state and prove the main result of this article, which gives a matrix form for fractional integration. In Section 4, we apply our method for solving linear multi-order FDEs. In Section 5, we illustrate several examples and we come up with a conclusion in Section 6.

2. Preliminaries and notations

There are several definitions of fractional derivative and integral such as Caputo, Grünwald-Letnikov and Riemann-Liouville. These definitions are not necessarily equivalent in different sense [2]. Here, we state fractional differential operator in the Riemann-Liouville sense.

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Definition 1. The Riemann-Liouville fractional integral operator of order $\geq \mathbf{0}$, of a function $f(x)$, is defined as

$$I^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)dt.\ \alpha>0, x>0, \quad (1)$$
$$I^0 f(x)=f(x).$$

Properties of the operator I^{α} can be found in [11], we just mention the following property

$$I^{\alpha}x^{\gamma}=\frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}x^{\alpha+\gamma} \quad (2)$$

The Riemann-Liouville fractional of order α will be denoted by $D^{\alpha}$and defined by

$$D^{\alpha}f(x)=\frac{d^m}{dx^m}\left(I^{m-\alpha}f(x)\right), \quad (3)$$

where $m-1<\alpha\leq m, m\in N$ and m is the smallest integer order greater than α.

Lemma 1. if $m-1<\alpha\leq m, m\in N$, then

$$D^{\alpha}I^{\alpha}f(x)=f(x),$$
$$I^{\alpha}D^{\alpha}f(x)=f(x)-\sum_{i=0}^{m-1}f^{(i)}(0^{+})\frac{x^i}{i!},\ x>0. \quad (4)$$

Shifted Legendre polynomials

Let $P_i(x); x\in(0,1)$ be the shifted Legendre polynomials. Then $P_i(x)$ can be obtained as follows:

$$P_{i+1}(x)=\frac{(2i+1)(2x-1)}{(i+1)}P_i(x)-\frac{i}{i+1}P_{i-1}(x), i=1,2,..., \quad (5)$$

where $P_0(x)=1$ and $P_1(x)=2x-1$. The analytic form of the *shifted Legendre polynomial* $P_i(x)$ of degree i given by

$$P_i(x)=\sum_{k=0}^{i}(-1)^{i+k}\frac{(i+k)!x^k}{(i-k)(k!)^2}. \quad (6)$$

where $P_i(0)=(-1)^i$ and $P_i(1)=1$. The orthogonality condition is

$$\int_0^1 P_i(x)P_j(x)\,dx=\begin{cases}\frac{1}{2i+1} & i=j,\\ 0 & i\neq j.\end{cases} \quad (7)$$

A function y(x), square integrable in [0, 1], may be expressed in terms of shifted Legendre polynomials as

$$y(x)=\sum_{i=0}^{\infty}c_iP_i(x),$$

where the coefficients c_i are given by

$$c_i=(2i+1)\int_0^1 y(x)P_i(x)dx,\quad i=1,2,... \quad (8)$$

In practice, only the first $(N+1)$ terms shifted Legendre polynomials are considered. Therefore $y(x)$ can be written in the form

$$y_N(x)\simeq\sum_{i=0}^{N}c_iP_i(x)=C^T\Phi(x), \quad (9)$$

where the shifted Legendre coefficient vector C and the shifted Legendre vector $\Phi(x)$ are given by

$$C^T=[c_0,c_1,...,c_N],$$
$$\Phi(x)=[P_0,P_1,...,P_N]^T. \quad (10)$$

If we define the v times repeated integration of Legendre vector $\Phi(x)$ by $I^v\Phi(x)$, (see [11]), then

$$I^v\Phi(x)\simeq A^v\Phi(x), \quad (11)$$

where v is an integer value and A^v is the operational matrix of integration of $\Phi(x)$. More details are presented in [11].

3. Operational Matrix of Fractional Integration

In this section we generalized the shifted Legendre operational matrix (SLOM) of integration (11) for fractional calculus.

Theorem 3. Let $\Phi(x)$ be a shifted Legendre polynomial then

$$I^v\Phi(x)\simeq A^v\Phi(x), \quad (12)$$

where A^v is the $(N+1)\times(N+1)$ operational matrix of integration of order v in the Riemann-Liouville sense and is defined as follows:

$$A^v=\begin{pmatrix}\sum_{k=0}^{0}\xi_{0,0,k} & \sum_{k=0}^{0}\xi_{0,1,k} & \cdots & \sum_{k=0}^{0}\xi_{0,N,k}\\ \sum_{k=0}^{1}\xi_{1,0,k} & \sum_{k=0}^{1}\xi_{1,1,k} & \cdots & \sum_{k=0}^{1}\xi_{1,N,k}\\ \vdots & \vdots & \cdots & \vdots\\ \sum_{k=0}^{i}\xi_{i,0,k} & \sum_{k=0}^{i}\xi_{i,1,k} & \cdots & \sum_{k=0}^{i}\xi_{i,N,k}\\ \vdots & \vdots & \cdots & \vdots\\ \sum_{k=0}^{N}\xi_{N,0,k} & \sum_{k=0}^{N}\xi_{N,1,k} & \cdots & \sum_{k=0}^{N}\xi_{N,N,k}\end{pmatrix}, \quad (13)$$

where

$$\xi_{i,j,k}=(2j+1)\sum_{l=0}^{j}\frac{(-1)^{i+j+k+l}(i+k)!(l+j)!}{(i-k)!k!(k+\alpha+1)(j-l)!(l!)^2(k+l+\alpha+1)}. \quad (14)$$

Proof: Having the analytic form of the shifted Legendre polynomials (6) and using Eqs. (1) and (2) gets

$$I^{\alpha}P_i(x)=\sum_{k=0}^{i}\frac{(-1)^{i+k}(i+k)!}{(i-k)!(k!)^2}I^{\alpha}\left(x^k\right)$$
$$=\sum_{k=0}^{i}\frac{(-1)^{i+k}(i+k)!}{(i-k)!k!}\frac{x^{\alpha+k}}{\Gamma(k+\alpha+1)}, \quad (15)$$
$$i=0,1,...,N.$$

Now, approximate $x^{\alpha+k}$ by $(N+1)$ terms of shifted Legendre series yields

$$x^{\alpha+k}=\sum_{j=0}^{N}a_{k,j}P_j(x), \quad (16)$$

where

$$a_{k,j} = (2j+1)\int_0^1 x^{\alpha+k} P_j(x)dx$$
$$= (2j+1)\sum_{l=0}^{j} \frac{(-1)^{i+l}(j+l)!}{(j-l)!\,(l!)^2}\int_0^1 x^{\alpha+k}\,dx$$
$$= (2j+1)\sum_{l=0}^{j} \frac{(-1)^{i+l}(j+l)!}{(j-l)!(l!)^2(k+l+\alpha+1)}. \quad (17)$$

Now, by employing Eqs. (15)-(17) we obtain

$$I^\alpha P_i(x) \simeq \sum_{k=0}^{i}\sum_{j=0}^{N} \frac{(-1)^{i+k}(i+k)!}{(i-k)!\,k!\,\Gamma(k+\alpha+1)} a_{k,j} P_j(x)$$
$$= \sum_{j=0}^{N}\left(\sum_{k=0}^{i} \xi_{i,j,k}\right) P_j(x), \quad i = 0,1,\dots,N, \quad (18)$$

where $\xi_{i,j,k}$ is given in Eq. (14). Writing the last equation in a vector form gives

$$I^\alpha P_i(x) \simeq \left[\sum_{k=0}^{i} \xi_{i,0,k}, \sum_{k=0}^{i} \xi_{i,1,k}, \dots, \sum_{k=0}^{i} \xi_{i,N,k}\right] \Phi(x),$$
$$i = 0,1,\dots,N, \quad (19)$$

which produces the desired result.

4. Application of the operational matrix in fractional integral

Now, we are ready to apply the SLOM method to the fractional integration. Here, we apply the method to a multi-order fractional differential equation in the Riemann-Liouville sense. So, let α to be the highest fractional order of FDE. Then by employing properties of fractional integral we can write the FDE as an integral equation of order α. Now by using operational matrix we approximate the resultant integral equation.

Consider the following linear multi-order FDE

$$D^\alpha y(x) = \sum_{i=1}^{k} a_i D^{\beta_i} y(x) + a_{k+1} I^\alpha y(x) = g(x), \quad (20)$$

with initial conditions

$$y^{(i)}(0) = d_i, \qquad i = 0,1,\cdots,n-1, \quad (21)$$

where a_j $(j = 1,\cdots,k+1)$ are real constant coefficients and also $n-1 < \alpha \le n, 0 < \beta_1 < \beta_2 < \cdots < \beta_k < \alpha$ and D^α denote the Riemann-Liouville fractional derivative of order α. For the existence, uniqueness and continuous dependence of the solution to the problem, see [13]. To solve problem (20) and (21) we apply the Riemann-Liouville integral of order α on (20) and using Lemma.1 to get

$$y(x) - \sum_{i=0}^{n-1} y^{(i)}(0^+)\frac{x^i}{i!} =$$
$$\sum_{j=1}^{k} a_j I^{\alpha-\beta_j}\left(y(x) - \sum_{i=0}^{n_j-1} y^{(i)}(0^+)\frac{x^i}{i!}\right) +$$
$$a_{k+1} I^\alpha y(x) + I^\alpha g(x), \quad (22)$$

$$y^{(i)}(0) = d_i, \qquad i = 0,1,\cdots,n-1,$$

where $n_j - 1 < \alpha \le n_j,\ n_j \in N$. Hence

$$y(x) = \sum_{j=1}^{k} a_j I^{\alpha-\beta_j} y(x) + a_{k+1} I^\alpha y(x) + f(x), \quad (23)$$
$$y^{(i)}(0) = d_i, \qquad i = 0,1,\cdots,n-1,$$

where

$$f(x) = I^\alpha g(x) + \sum_{i=0}^{n-1} d_i \frac{x^i}{i!} + \sum_{j=1}^{k} a_j I^{\alpha-\beta_j}\left(\sum_{i=0}^{n_j-1} d_i \frac{x^i}{i!}\right).$$

Now, we approximate $y(x)$ and $f(x)$ by the shifted Legendre polynomials as

$$y_N(x) \simeq \sum_{i=0}^{N} c_i P_i(x) = C^T \Phi(x), \quad (24)$$

$$f(x) \simeq \sum_{i=0}^{N} f_i P_i(x) = F^T \Phi(x), \quad (25)$$

where vector $\mathbf{F}^\mathbf{T} = [\mathbf{f_0}, \mathbf{f_1}, \dots, \mathbf{f_N}]$ is known and $\mathbf{C}^\mathbf{T} = [\mathbf{c_0}, \mathbf{c_1}, \dots, \mathbf{c_N}]$ is an unknown vector. Using Theorem 1 an approximation solution of (24) can be written as

$$I^\alpha y_N(x) = C^T I^\alpha \Phi(x) = C^T A^\alpha \Phi(x), \quad (26)$$

where A^α is defined in theorem 1. Employing Eqs. (24)-(26) the residual $R_N(x)$ for Eq. (23) can be written as

$$R_N(x) = \left(C^T - C^T \sum_{i=1}^{k} a_i A^{(\alpha-\beta_i)} - a_{k+1} C^T A^{(\alpha)} - F^T\right)\Phi(x). \quad (27)$$

As in a typical tau method (see [14]), we generate $N - n + 1$ linear algebraic equations by applying

$$\langle R_N(x), P_j(x)\rangle = \int_0^1 R_N(x) P_j(x) dx, \quad j = 0,1,\cdots,N-n. \quad (28)$$

For determining c_i we need $n-1$ equations. Also, by substituting Eq. (24) in Eq. (21) we obtain

$$y^{(i)}(0) = C^T \Phi^{(i)}(x) = \sum_{i=0}^{N} c_i P_i^{(i)}(0) = d_i,$$
$$i = 0,1,\cdots,n-1, \quad (29)$$

where $P_i^{(n)}(0)$ is defined as follows

$$P_i^{(n)}(0) = \frac{(-1)^{i+n}(i+n)!}{(i-n)!\,n!}, \quad n \le i. \quad (30)$$

Now, by solving this set of linear equation and determining the unknown coefficient of vector $C, y(x)$ as in Eq. (24) can be calculated.

5. Error analysis

In this section an error analysis is given for our method. It is well-known that shifted Legendre polynomials $P_i(x)$ form a complete $L^2(\Omega)$

orthogonal set [14], where $\Omega = [-1, 1]$. We recall that $H^m(\Omega)$ is the Sobolev space of all function $u(t)$ on Ω such that $u(t)$ and all it's weak derivatives up to order m are in $L^2(\Omega)$ and define $\|.\|_{H^m(\Omega)}$ as

$$\|.\|_{H^m(\Omega)} = (\sum_{k=0}^{m} \|u^{(k)}(t)\|^2_{L^2(\Omega)})^{\frac{1}{2}}.$$

The semi-norm also denoted by

$$|.|^2_{H^{m;N}(\Omega)} = \sum_{i=min(m,N)}^{N} \|u^{(i)}(t)\|^2_{L^2(\Omega)}.$$

Note that whenever $N \geq m-1$, one has

$$|u|_{H^{m;N}(\Omega)} = \|u^{(m)}\|_{L^2(\Omega)} = |u|_{H^m(\Omega)}.$$

Now, suppose $u_N = \sum_{k=0}^{N} \hat{u}_k P_k$ is the truncated Legendre approximation of a function $u \in H^m(\Omega)$, Then, as has been proved in [14], the truncation error is

$$\|u - u_N\|_{H^{m;N}(\Omega)} \leq CN^{-m}|u|_{H^{m;N}(\Omega)}, \qquad (31)$$

where C is a positive constant depending on m. For our proposed method consider system (23) and let $y(x)$ be the exact solution of this system. The truncation error $e_N(x) = y(x) - y_N(x)$, where $y_N(x) = \sum_{k=0}^{N} c_k P_k(x)$ is the truncated Legendre series of y, can be estimated as follows:

$$e_N(x) = \sum_{i=1}^{k} a_i I^{\alpha-\beta_i} e_N(x) + a_{k+1} I^{\alpha} e_N(x) + e_f(x),$$

where $e_f(x) = f(x) - f_N(x)$. Taking the norm, yields

$$\begin{aligned}
\|e_N(x)\|_{H^m(\Omega)} &\leq \sum_{i=1}^{k} a_i I^{\alpha-\beta_i} \|e_N(x)\|_{H^m(\Omega)} \\
&\quad + a_{k+1} I^{\alpha} \|e_N(x)\|_{H^m(\Omega)} \\
&\quad + \|e_N(x)\|_{H^m(\Omega)} \\
&\leq \sum_{i=1}^{k} a_i \frac{1}{\Gamma(\alpha-\beta_i)} \int_0^x \|(x-\tau)^{\alpha-\beta_i-1} e_N(\tau)\|_{H^m(\Omega)} d\tau \\
&\quad + a_{k+1} \frac{1}{\Gamma(\alpha)} \int_0^x \|(x-\tau)^{\alpha-1} e_N(\tau)\|_{H^m(\Omega)} d\tau \\
&\quad + \|e_f(x)\|_{H^m(\Omega)} \\
&\leq M \int_0^x \|e_N(\tau)\|_{H^m(\Omega)} d\tau + \|e_f(x)\|_{H^m(\Omega)}
\end{aligned}$$

where M is a constant independent of x. Using (31) for sufficiently large N, the error $e_f(x)$ is bounded for $x \in \Omega$

$$\|e_N(x)\|_{H^m(\Omega)} \leq M \int_0^x \|e_N(\tau)\|_{H^m(\Omega)} d\tau + K.$$

Applying Gronwall's Lemma, we get

$$\|e_N(x)\|_{H^m(\Omega)} \leq Ke^{Mx}, \qquad \forall x \in \Omega.$$

6. Illustrative examples

To give a clear overview of this method, we present some illustrative examples.

Example 1. As the first example, we consider the following initial value problem,

$$D^{\frac{3}{2}} y(x) + y(x) = x^2 + 4\sqrt{\frac{x}{\pi}}, \qquad y(0) = 0, \qquad (32)$$

whose exact solution is given by $y(x) = x^2$.

By applying the technique described in Section 4.1 with N = 2, we may write the approximate solution as

$$y(x) = \sum_{i=0}^{2} c_i P_i(x) = C^T \phi(x),$$

$$f(x) \simeq \sum_{i=0}^{2} g_i P_i(x) = F^T \phi(x), \qquad (33)$$

From Theorem 3, we have

$$A^{3/2} = \frac{8}{\sqrt{\pi}} \begin{pmatrix} \frac{1}{15} & \frac{3}{35} & \frac{1}{63} \\ \frac{-1}{35} & \frac{-1}{45} & \frac{1}{77} \\ \frac{1}{315} & \frac{-3}{385} & \frac{-1}{117} \end{pmatrix}, \quad F = \begin{pmatrix} f_0 \\ f_1 \\ f_2 \end{pmatrix}. \qquad (34)$$

Therefore, using Eq.(28) we obtain

$$c_0 + \frac{8}{\sqrt{\pi}}\left(\frac{c_0}{15} - \frac{c_1}{35} + \frac{c_2}{315}\right) - f_0 = 0, \qquad (35)$$

$$c_1 + \frac{8}{\sqrt{\pi}}\left(\frac{3c_0}{35} - \frac{c_1}{45} - \frac{3c_2}{385}\right) - f_1 = 0. \qquad (36)$$

Now, by applying Eq. (29), for the initial condition we obtain

$$c_0 - c_1 + c_2 = 0. \qquad (37)$$

Finally, by solving Eqs. (35)- (37) we obtain

$$c_0 = \frac{1}{3}, \quad c_1 = \frac{1}{2}, \quad c_2 = \frac{1}{6}.$$

Hence we can write

$$y(x) = \sum_{i=0}^{2} c_i P_i(x) = x^2,$$

which is the exact solution.

Table 1. Absolute error for $\boldsymbol{\alpha = 0.85}$ in Example 2

x	N=2	N=4	N=6
0.1	1.9×10^{-2}	7.7×10^{-4}	2.5×10^{-3}
0.2	9.5×10^{-3}	5.2×10^{-3}	7.3×10^{-4}
0.3	4.2×10^{-3}	2.4×10^{-3}	2.1×10^{-3}
0.4	1.6×10^{-2}	2.5×10^{-3}	3.5×10^{-4}
0.5	2.3×10^{-2}	5.2×10^{-3}	2.2×10^{-3}
0.6	2.4×10^{-2}	3.7×10^{-3}	7.0×10^{-4}
0.7	1.6×10^{-2}	1.4×10^{-3}	2.1×10^{-3}
0.8	4.9×10^{-4}	6.2×10^{-3}	1.3×10^{-3}
0.9	2.6×10^{-2}	3.7×10^{-3}	3.0×10^{-3}
1.0	6.3×10^{-2}	1.6×10^{-2}	7.9×10^{-3}

Example 2. We consider the following initial value problem,

$$D^{\alpha}y(x) + y(x) = 0, \quad 0 < \alpha \le 2, \qquad y(0) = 1, \quad y'(0) = 0. \tag{38}$$

The second initial condition is for $\alpha > 1$ only. The exact solution of this problem is as follows [13]:

$$y(x) = \sum_{k=0}^{\infty} \frac{(-x^{\alpha})^k}{\Gamma(\alpha k + 1)},$$

By applying the technique described in Section 4 and solving this problem, the absolute error for $\alpha = 0.85$ and N=2, 4 and 6 are shown in Table 1. In Table 1, we see that a good approximation solution can be achieved by taking a few terms of shifted Legendre polynomials. These results are similar to those counterparts resulted in [9]. The numerical results for y(x) with N =4 and $\alpha = 0.75, 0.85, 0.95$ and 1 are illustrated in Fig. 1.

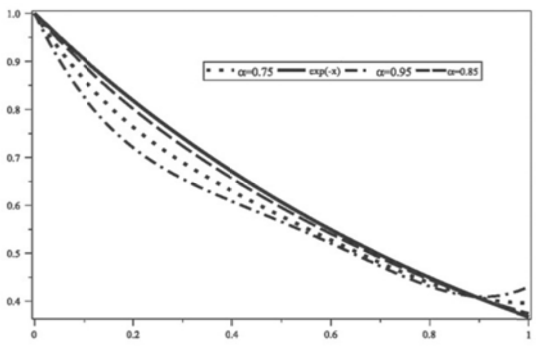

Fig. 1. Comparison of y(x) for different α and N=4 in Example 2

For $\alpha = 1$, the exact solution is given by $y(x) = exp(-x)$. Note that as α approaches 1, the numerical solution converges to the analytical solution $y(x) = exp(-x)$. That is in the limit, the solution of the fractional differential equations approaches to the solution of ordinary differential equations as $\alpha \longrightarrow 0$. Moreover, we present results for $\alpha > 1$. Fig. 2 shows the numerical results for $y(x)$ for $N = 4$ and $\alpha = 1.5, 1.75, 1.95$ and 2. For $\alpha = 2$, the exact solution is given as $y(x) = cos(x)$. Similar to the previous case, from Fig. 2, we see that when α approaches 2, the numerical solution converges to the solution of ordinary differential equation.

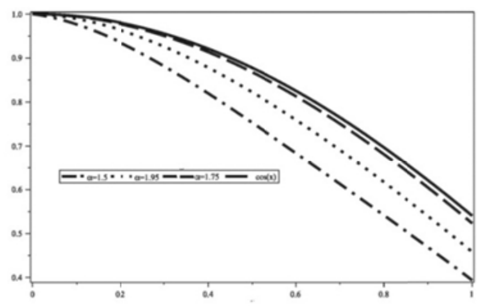

Fig. 2. Comparison of y(x) for N=4 with $\alpha = 1.5, 1.75, 1.95$ in Example 2

Example 3. Consider the equation

$$D_t^2 y(x) - aD_t^{\alpha} y(x) - by(x) = 8, \quad x > 0, \quad 0 < \alpha < 2, \tag{39}$$

with the initial condition

$$y(0) = 0, \qquad y'(0) = 0.$$

Suppose in special case $a = b = -1$, using the method described in Example 1 with $N = 4$ the approximate solution for $\alpha = 0.5$ and 1.5 can be obtained. Numerical results are given in Table 2 by comparison with the exact solution. The exact solution refers to the closed form series solution presented in [15]. Table 2 shows a good numerical approximation solution with the exact solution.

Table 2. Numerical results with comparison to exact solution Example 3

	$\alpha = 0.5$		$\alpha = 1.5$	
x	Our method	Exact sol.	Our method	Exact sol.
0.0	0.000000	0.000000	0.000000	0.000000
0.1	0.039995	0.039750	0.032615	0.125221
0.2	0.157331	0.157036	0.124678	0.033507
0.3	0.347274	0.347370	0.268427	0.267609
0.4	0.604066	0.604695	0.457321	0.455435
0.5	0.920928	0.921768	0.686033	0.684335
0.6	1.290060	1.290457	0.950455	0.950393
0.7	1.702643	1.702008	1.247710	1.249959
0.8	2.148833	2.147287	1.576113	1.579557
0.9	2.617766	2.617001	1.935220	1.935832
1.0	3.097559	3.101906	2.325795	2.315526

7. Conclusion

We derived a general formula for the Legendre operational matrix of fractional integration. We have used this method to approximate the numerical solution of a class of fractional differential equations in Riemann-Liouville sense. Our approach was based on the properties of shifted Legendre polynomials. As we have seen in the illustrated examples, this method obtains a good accuracy of the solution. Moreover, only a small number of shifted Legendre polynomials are needed to obtain a satisfactory result.

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